Efficient Design of Waveforms for Robust Pulse Amplitude Modulation using Mean Square Error Criteria

Timothy N. Davidson*
Department of Electrical and Computer Engineering,
McMaster University,
Hamilton, Ontario, L8S 4K1, Canada.

ABSTRACT

The design of a pulse shaping filter which provides maximal robustness to an unknown frequency-selective channel is formulated as a convex optimization problem from which an optimal filter can be efficiently obtained. Robustness is measured by the worst-case mean square error of the data estimate over a class of deterministically bounded channels, and the optimization is subject to a constraint on the bandwidth of the filter. The design technique allows efficient exploration of design trade-offs between bandwidth, performance in an ideal channel and robustness to unknown channel distortion. It is used to design chip waveforms with superior performance to the waveform specified in a recent standard for digital mobile telephony.

1 INTRODUCTION

In digital communications, waveform coding is often performed by linear pulse amplitude modulation (PAM) of translated versions of a given waveform [1]. The choice of waveform critically impacts many system performance criteria, including spectral efficiency and mitigation of expected channel imperfections. In applications in which an accurate channel model is available, there are several established techniques by which a waveform can be designed [1]. However, in some wireless applications the transmission environment may undergo substantial variations and it might not be possible to obtain an accurate channel model. In that case, one ought to design waveforms which provide robust performance in the presence of channel uncertainty. In this paper, an efficient technique for the design of such robust waveforms is presented. The technique is targeted at applications in which waveform coding is performed by a baseband digital signal processor, and hence waveform design can be reduced to the design of a multi-rate finite impulse response (FIR) filter. This framework ensures that the designed waveform can be easily implemented.

In this paper, the performance of the PAM scheme is measured in terms of the mean square error (MSE) of the data estimate, and robustness is measured by the worst-case MSE over a deterministically bounded class of channels. Our objective is to find a filter which minimizes this worst-case MSE, subject to a constraint on the MSE in an ideal channel and a constraint on the spectral occupation of the transmitted signal. Unfortunately, this objective and these constraints are non-convex functions of the filter coefficients, and hence direct design of an optimal filter may be complicated by the presence of local minima. Inspired by recent work [2] on the design of self-orthogonal ('root-Nyquist') filters for distortionless channels (and preceding work on the design of standard FIR filters [3]), we will show that the design problem can be transformed into a convex cone programme [4, 5] in the autocorrelation sequence of the filter, from which a globally optimal filter can be efficiently obtained. Using this technique the design tradeoffs between bandwidth, performance in an ideal channel, and robustness to channel uncertainties can be efficiently evaluated. These trade-offs are particularly important in applications in which the channel may vary (slowly, with respect to the pulse duration) but for which equalization is deemed to be too expensive. In a design example we will obtain a chip waveform with improved performance over that specified in a recent standard for digital mobile telephony.

*Email: davidson@ece.mcmaster.ca

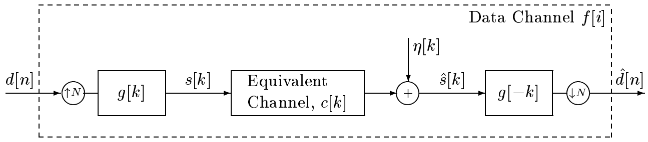


Figure 1: Discrete-time model of baseband PAM.

2 PULSE AMPLITUDE MODULATION

Consider the discrete-time baseband PAM scheme illustrated in Fig. 1. If the baseband equivalent channel does not vary significantly (in time) over the duration of the waveform, the error in the received data estimate $\hat{d}[n]$ is

$$\hat{d}[n] - d[n] = \sum_i \big(f[i] - \delta[i]\big) d[n-i] + \sum_k g[k - Nn]\eta[k],$$

where $f[i] = \sum_k c[k] r_g[k - Ni]$, $r_g[m] = \sum_k g[k]g[k+m]$ is the autocorrelation sequence of $g[k]$, and $\eta[k]$ models the additive noise. For white data with zero mean and energy E per symbol, and white noise with variance N_0, the MSE is

$$\begin{aligned}\text{MSE} &= E\sum_i \big(f[i]-\delta[i]\big)^2 + N_0 r_g[0] \\ &= E\sum_i \Big(r_g[Ni]-\delta[i]+\sum_\ell c_e[\ell+Ni]r_g[\ell]\Big)^2 + N_0 r_g[0],\end{aligned}$$

where $c_e[k] = c[k]-\delta[k]$ denotes the error channel. With the normalization $r_g[0]=1$, we have that

$$\begin{aligned}\text{MSE} = N_0 &+ E\sum_{i\neq 0} r_g[Ni]^2 + 2E\sum_{i\neq 0} r_g[Ni]\sum_\ell c_e[\ell+Ni]r_g[\ell] \\ &+ E\sum_i\Big(\sum_\ell c_e[\ell+Ni]r_g[\ell]\Big)^2. \qquad (1)\end{aligned}$$

The first term in (1) is the MSE due to the noise, the second is the MSE induced by the filters, and the third and fourth terms capture the additional MSE induced by the distorting channel. If the error channel $c_e[k]$ is known, then an optimal filter can be found by minimizing (1). (Related design problems have been well studied [1].) However, in the present paper, we focus on the case where $c_e[k]$ is not known. In Section 3 we will provide a convex optimization problem whose solution minimizes the worst-case MSE over a class of deterministically bounded error channels.[1] The optimization is subject to a constraint on the the spectral occupation of the transmitted signal (the 'bandwidth'). In this paper, that constraint will be that the power spectrum of $s[k]$, $S_s(e^{j\omega}) \propto \big|G(e^{j\omega})\big|^2$, must satisfy a relative spectral mask. A key observation in our development is that $R_g(e^{j\omega}) = |G(e^{j\omega})|^2$, and hence that $S_s(e^{j\omega})$ is a linear function of $r_g[m]$, whereas it is, in general, a non-convex quadratic function of $g[k]$.

3 DESIGN OF ROBUST WAVEFORMS

Applying the triangle and Cauchy-Schwarz Inequalities to the the right had side of (1) [see Appendix A for the details],

$$\text{MSE} \leq N_0 + 2E\Big(\big(\gamma_g+\lambda_1\tilde{B}_g\big)^2+\lambda_2^2\tilde{B}_g^2\Big). \qquad (2)$$

Here, $\gamma_g^2 = 2\sum_{i\geq 1} r_g[Ni]^2$ is the MSE due to the intersymbol interference in an ideal channel (i.e., the 'self-induced MSE'), and $\tilde{B}_g^2 = 1+B_g^2$, where $B_g^2 = 2\sum_{\ell\geq 0} r_g[\ell]^2$, is the MSE sensitivity coefficient for the error channel. The terms $\lambda_1^2 = \sum_{i\neq 0} C_i^2$ and $\lambda_2^2 = C_0^2$, where $C_i^2 = \sum_{\ell=-L+1}^{L-1} c_e[\ell+Ni]^2$, capture the 'size' of the error channel. If the size and structure of the error channel are known (i.e., if λ_1 and λ_2 are known), then one could choose $g[k]$ to minimize the right hand side of (2).[2] However, if, as is often the case, λ_1 and λ_2 are not known, an appropriate design problem is to minimize the sensitivity coefficient $\tilde{B}_g$ (or equivalently B_g), subject to a upper bound on the self-induced MSE $\gamma_g^2 \leq \varepsilon^2$ and a spectral mask on $S_s(e^{j\omega})$. Unfortunately, B_g^2 and γ_g^2 are quartic polynomials of $g[k]$ and the power spectrum is a non-convex quadratic function of $g[k]$. Therefore, a direct design algorithm may be exposed to the intricacies of local minima.

However, the spectral mask constraint generates linear constraints on $r_g[m]$, and B_g^2 and γ_g^2 are convex quadratic functions of $r_g[m]$. To complete the formulation of this design in terms of $r_g[m]$ instead of $g[k]$, we must add the semi-infinite linear constraint: $R_g(e^{j\omega}) \geq 0$ for all $\omega \in [0,\pi]$, which is a necessary and sufficient condition for $r_g[m]$ to be factorizable in the form $r_g[m] = \sum_k g[k]g[k+m]$, (by the Féjer-Riesz Theorem). In order to avoid the intricacies involved in discretizing this semi-infinite constraint, we can apply the Positive Real Lemma (PRL), which states that satisfaction of the semi-infinite constraint is equivalent the existence of a symmetric matrix $\boldsymbol{P}$ such that the following finite dimensional linear matrix inequality (LMI) holds:

$$\boldsymbol{M}(\boldsymbol{P}) \triangleq \begin{bmatrix} \boldsymbol{P}-\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} & \boldsymbol{c}^T-\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{b} \\ \big(\boldsymbol{c}^T-\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{b}\big)^T & 2d-\boldsymbol{b}^T\boldsymbol{P}\boldsymbol{b}\end{bmatrix} \geq 0, \qquad (3)$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_{L-2} \\ 0 & \boldsymbol{0}\end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix}\boldsymbol{0}\\ 1\end{bmatrix}, \qquad (4a)$$

$$\boldsymbol{c} = \big[r_g[L-1],\ldots,r_g[2],r_g[1]\big], \qquad d = 1/2. \qquad (4b)$$

(See [2, 7, 8] for other applications of the PRL in FIR filter design.). The pulse shaping filter design problem can now be cast as the following convex cone programme:

Problem 1 *Given $\rho_\ell(\omega)$, $\rho_u(\omega)$, ε, N and L, find a filter of length L achieving* $\min\alpha$ *over* $r_g[m]$, $m = 0,1,\ldots,L-1$, $\boldsymbol{P} = \boldsymbol{P}^T$, $W > 0$ *and* $\alpha \geq 0$, *subject to* $r_g[0]=1$, $\gamma_g \leq \varepsilon$, $B_g \leq \alpha$, *the spectral mask*

$$W10^{\rho_\ell(\omega)/10} \leq R_g(e^{j\omega}) \leq W10^{\rho_u(\omega)/10}, \quad \textit{for all } \omega\in[0,\pi], \qquad (5)$$

and to the LMI in (3)*, or show that none exist.*

Problem 1 consists of a linear objective, subject to a linear equality constraint ($r_g[0]=1$), linear inequality constraints (5), two second-order cone [5] constraints ($\gamma_g \leq \varepsilon$ and $B_g \leq \alpha$), and an LMI. Hence, Problem 1 is a convex symmetric cone programme and can be efficiently solved using interior point methods [9]. (SeDuMi [10] is a particularly efficient MATLAB-based tool.) These methods obtain a solution with an 'accuracy' of δ in at most $O(\sqrt{n}\log(1/\delta))$ iterations, where n is the number of variables. (In practice, a solution is often obtained in far fewer iterations than this worst-case complexity would suggest.) Once an optimal $r_g[m]$ has been obtained, an optimal filter can be found by spectral factorization (e.g., [11]).

Problem 1 has an intuitively appealing interpretation in the frequency domain: Using Parseval's Relation,

$$B_g^2 = \int_{-\pi}^{\pi}\big(|G(e^{j\omega})|^2-1\big)^2\, d\omega/(2\pi), \qquad (6)$$

and hence minimizing B_g is equivalent to making $|G(e^{j\omega})|$ as flat as possible (in a mean-square sense).

[1] It has already been shown [7] that the minimization of the worst-case 'peak intersymbol interference' [1] over this class of channels can be formulated as a convex optimization problem. The maximization of the 'eye-flatness' can also be formulated as a convex optimization problem [6]

[2] That problem can be formulated as a convex optimization problem in $r_g[m]$, but we will not do that here.

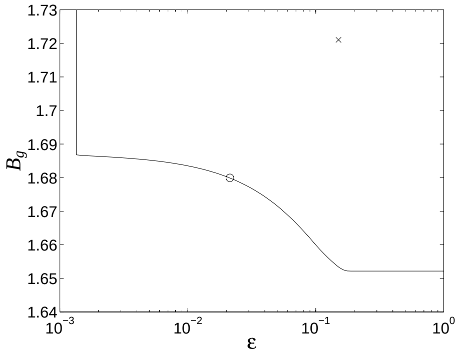


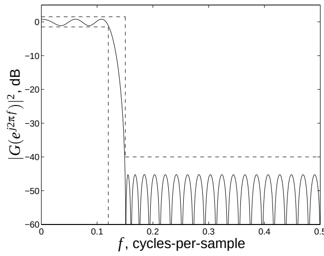
Figure 2: The trade-off between B_g and ε (the bound on γ_g) for the IS95 standard. The '×' and '○' denote the positions achieved by the IS95 filter and the robust filter in Fig. 3(b), respectively.

We now demonstrate the effectiveness of this design technique by designing an improved chip waveform for the IS95 standard for digital mobile telephony [12].

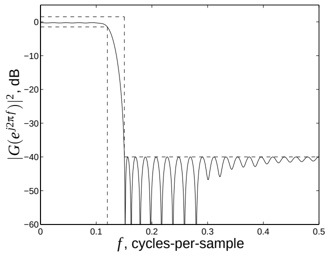
Example 1 The filter specified for the synthesis of the chip waveform in IS95 has $L = 48$ and $N = 4$ and satisfies the spectral mask specified in the standard, but it generates a large self-induced MSE. To determine whether this filter can be improved upon, Problem 1 was solved for various values of ε. (Each instance of Problem 1 was solved, using SeDuMi [10], in about 25 seconds on a 400 MHz PENTIUM II workstation.) In that way we efficiently determined the trade-off between the self-induced MSE and the sensitivity coefficient for the channel-induced MSE shown in Fig. 2. It is clear from that figure that the filter chosen in IS95 is some distance from the optimal filters. The 'floor effect' in Fig. 2 is due to the fact that the low-pass nature of the spectral mask limits the degree of spectral flatness which can be obtained. In addition, by showing that γ_g cannot be made arbitrarily small, we reinforce an earlier result [2] that the shortest self-orthogonal filter which satisfies the IS95 mask has $L = 51$. The spectra of the IS95 filter and a representative optimal filter are plotted in Fig. 3, from which the improved frequency flatness in the passband of the designed filter is apparent.

To demonstrate the performance improvement of the robust filter, we simulated the 'chip error rate' (CER) for the transmission of binary chips over a slowly-varying Rician channel with additive white Gaussian noise and sign detection of the chips. The linear time-invariant 'snap shots' of the channel were of length 41 and hence extend over 10 chips. They were generated with $c[0] = 1$ and the remaining $c[k]$ being independent and Gaussian with zero mean and standard deviation 0.05. (Such channels exhibit a wide variety of frequency selective effects.) The resulting CER curves, averaged over 100,000 channel realizations, are plotted in Fig. 4, from which the improved performance of the robust filter is clear. □

In addition to determining the maximal robustness for a given level of performance in an ideal channel, one might



(a) IS95 filter.



(b) Robust filter.

Figure 3: Relative power spectra (in decibels) of the filters in Example 1, with the spectral mask from IS95.

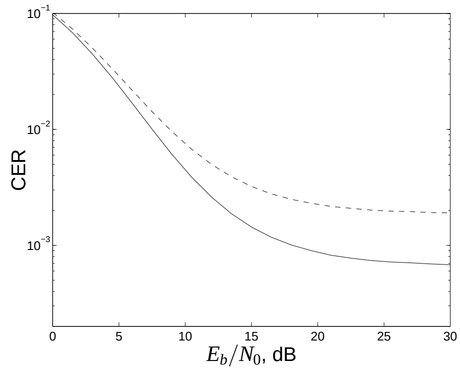


Figure 4: Simulated chip error rates (CER) against signal-to-noise ratio for Example 1. Legend—Dashed: IS95 filter; Solid: robust filter.

also be interested in determining how much 'smaller' the spectral mask can be made if B_g and γ_g are allowed to degrade. For certain mask parameters, this trade-off can be efficiently evaluated using a bisection-based search for the feasibility boundary of a convex cone feasibility problem. (This technique is similar to a technique used in [2] to design self-orthogonal filters.) The feasibility problem which is evaluated at each stage of the bisection search is a modified version of Problem 1 in which α is fixed. We now demonstrate the effectiveness of this technique by evaluating the trade-off between the stopband edge of a spectral mask of the form specified in the IS95 standard and the self-induced MSE.

Example 2 Let f_p and f_s denote the passband and stopband edges, respectively, of the IS95 spectral mask illustrated in Fig. 3. For a given value of ε, the smallest f_s such that there exists a filter which satisfies the mask and has $\gamma_g \leq \varepsilon$ can be efficiently found by a bisection search on $[f_p, 0.5]$ for the feasibility boundary of Problem 1. The resulting trade-offs between the spectral occupation and the self-induced MSE, both with and without the constraint $B_g \leq B_{\mathrm{IS95}}$, are illustrated in Fig. 5. The floor effect in Fig. 5 for large ε is due to the lower bound component of the mask (and the constraint $B_g \leq B_{\mathrm{IS95}}$, if it is present). The level at which the curves flatten for small ε is the smallest f_s which can be achieved by a root-Nyquist filter which satisfies the mask. (This level is greater [2] than the f_s specified in IS95.) Note that the absence of a constraint on B_g allows the power spectrum to

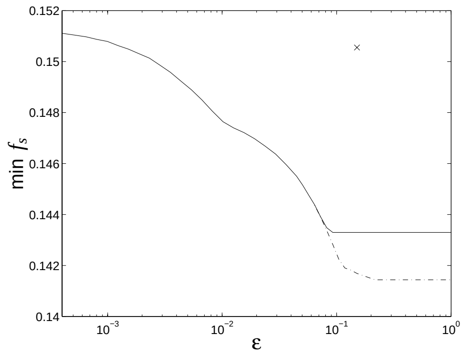


Figure 5: Minimal stopband edge, f_s against ε for the IS95 mask, with (solid) or without (dash-dot) the constraint $B_g \leq B_{\text{IS95}}$. The '×' denotes the position of the IS95 filter.

vary more substantially within the passband mask [c.f. (6)], and hence allows a lower stopband edge to be achieved. □

4 CONCLUSIONS

In this paper, we have shown that the trade-offs between bandwidth, performance in an ideal channel, and robustness in the design of PAM waveforms can be efficiently obtained using convex optimization techniques. Furthermore, the resulting PAM schemes provide lower error rates than those specified in recent standards. Attention was focussed on deterministically bounded models of channel uncertainty, and robustness was measured in a worst-case MSE sense. Although that results in a broadly applicable technique, it may be conservative in terms of average performance if the worst-case environments occur rarely. If an accurate statistical model for the environment is available, an alternative approach would be to minimize the average MSE over that model. Fortunately, that problem can also be formulated as a convex optimization problem, and hence efficiently solved (see [7] for a related problem).

A APPENDIX: DERIVATION OF (2)

Using the notation of Section 3, and applying the Cauchy-Schwarz inequality, we have the following bounds:

$$\Big(\sum_{\ell} c_e[\ell+Ni] r_g[\ell]\Big)^2 \leq \tilde{B}_g^2 C_i^2, \tag{7}$$

$$\Big|\sum_{i\neq 0} r_g[Ni] \sum_{\ell} c_e[\ell+Ni] r_g[\ell]\Big| \leq \gamma_g \Big(\sum_i \Big(c_e[\ell+Ni] r_g[\ell]\Big)^2\Big)^{1/2} \leq \gamma_g \tilde{B}_g \Big(\sum_{i\neq 0} C_i^2\Big)^{1/2} \tag{8}$$

Taking the absolute value of the right hand side of (1), and applying the triangle inequality and (7) and (8),

$$\text{MSE} \leq N_0 + E\big(\gamma_g^2 + 2\gamma_g \tilde{B}_g \lambda_1 + \tilde{B}_g^2(\lambda_1^2 + \lambda_2^2)\big),$$

and hence (2).

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