ELEC ENG 3BB3:

Cellular Bioelectricity

Notes for Lecture 13

Friday, January 31, 2014

5. IMPULSE PROPAGATION

We will look at:

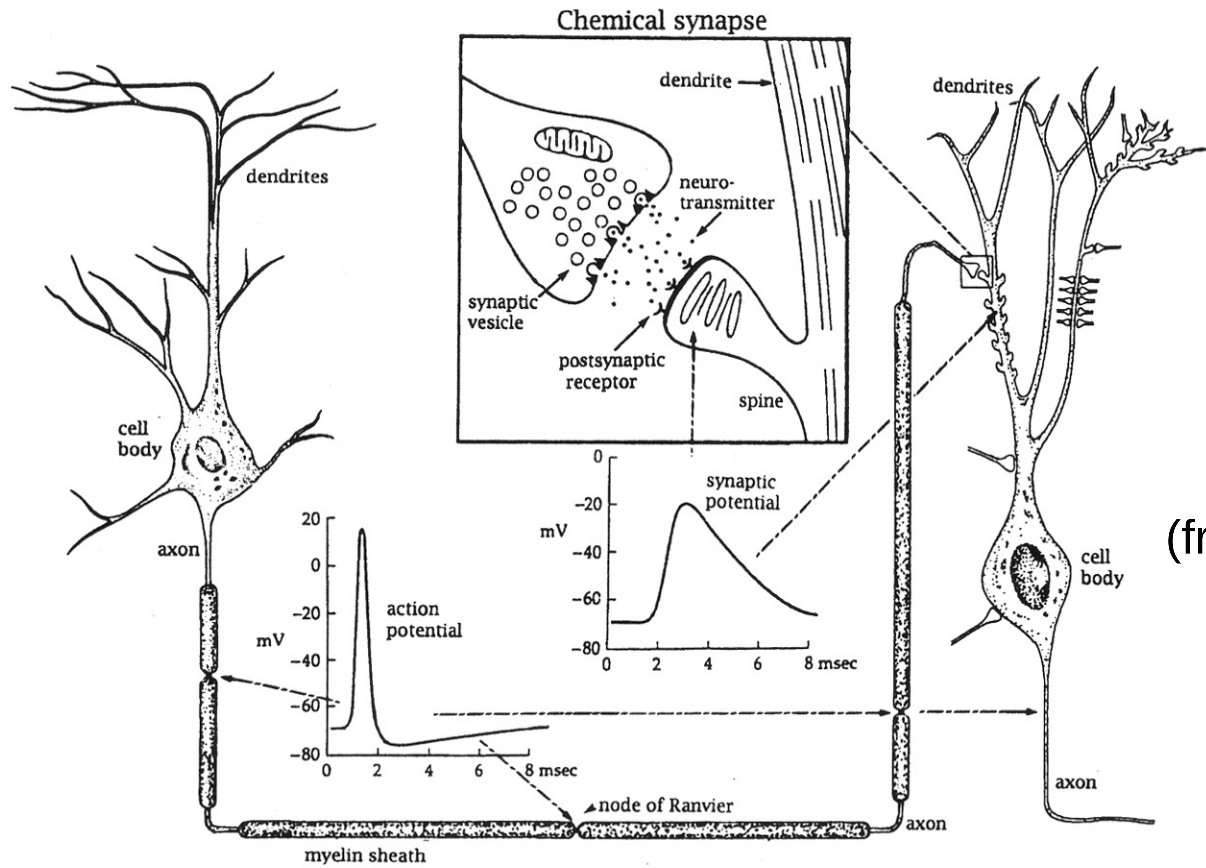
- Core-conductor model
- Cable equations
- Local circuit currents during propagation
- Mathematics of propagating action potentials
- Numerical solutions for propagating action potentials
- Propagation velocity constraint for uniform fiber
- Propagation in myelinated nerve fibers

Impulse propagation:

Up until now, we have only considered the properties of (i) a small patch of membrane or (ii) a space-clamped axon, such that the transmembrane potential is identical for all the membrane.

However, in practice we are often concerned about the *propagation* of transmembrane potential impulses (i.e., waveforms), particularly action potentials, along the length of axons, or muscle fibers.

Impulse propagation (cont.):



(from Johnston and Wu)

Figure 1.2 Neurons convey information by electrical and chemical signals. Electrical signals travel from the cell body of a neuron (left) to its axon terminal in the form of action potentials. Action potentials trigger the secretion of neurotransmitters from synaptic terminals (upper insert). Neurotransmitters bind to postsynaptic receptors and cause electric signals (synaptic potential) in the postsynaptic neuron (right). Synaptic potentials trigger action potentials, which propagate to the axon terminal and trigger secretion of neurotransmitters to the next neuron. (Adapted from Kandel et al. 1991 and from L.L. Iversen, copyright © 1979 by Scientific American, Inc. All rights reserved.)

Core-conductor model:

In the *core-conductor model* we approximate an axon or a segment of a dendrite as a uniform cylinder.

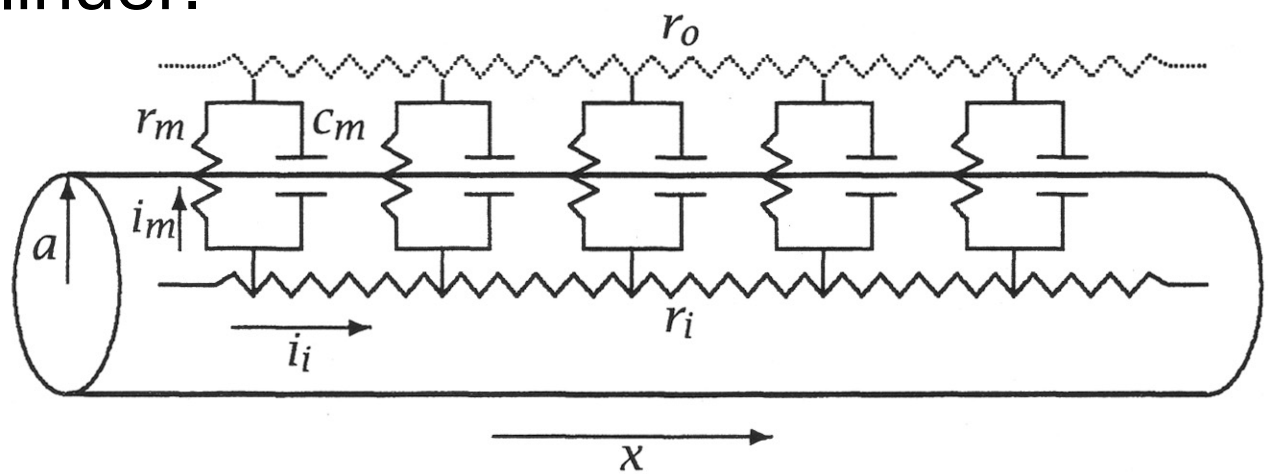


Figure 4.6 Diagram for current flow in a uniform cylinder such as an axon or segment of dendrite.

(from Johnston and Wu)

Each small (cylindrical) segment of membrane is electrically linked (axially) to the next segment by the intra- and extra-cellular electrolytes.

Resistance and capacitance in a cylindrical fiber:

If the *resistivity* of the intracellular electrolyte is R_i (Ω cm), then for a cylindrical fiber of radius a the axial (longitudinal) *resistance per unit length* is:

$$r_i = \frac{R_i}{\pi a^2} \quad \Omega/\mathrm{cm}. \tag{2.55'}$$

(Note the convention that (i) resistivity or specific resistance/capacitance is designated by an uppercase letter and (ii) resistance or capacitance per unit length is designated by a lowercase letter.)

Resistance and capacitance in a cylindrical fiber (cont.):

If R_m (Ω cm^2) and C_m (μF/cm^2) are the *specific resistance* and the *specific capacitance*, respectively, of the membrane, then the *membrane resistance times length* is:

$$r_m = \frac{R_m}{2\pi a} \quad \Omega \text{ cm}, \tag{2.56$'$}$$

and the *membrane capacitance per unit length* is:

$$c_m = C_m 2\pi a \quad \mu\text{F/cm}. \tag{2.57$'$}$$

Core-conductor model (cont.):

If a single fiber described by the core-conductor model lies in a restricted extracellular space, then longitudinal current flow can occur in the extracellular electrolyte and longitudinal variations in the extracellular potential can result.

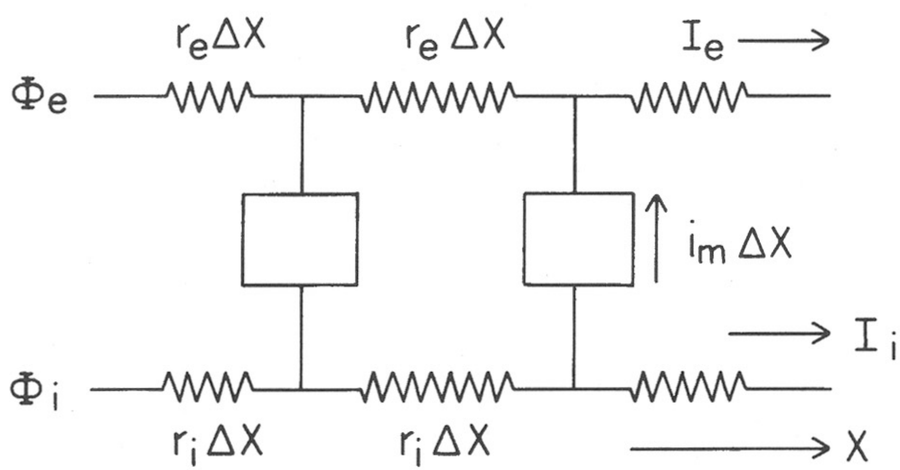


Figure 6.1. The linear core-conductor model for a single fiber lying in a restricted extracellular space. Longitudinal extracellular and intracellular currents are I_e and I_i, while extracellular and intracellular potentials per unit length are designated Φ_e and Φ_i, respectively.

Core-conductor model (cont.):

Under linear (i.e., subthreshold) conditions, each membrane patch of length Δx can be described by a lumped RC circuit.

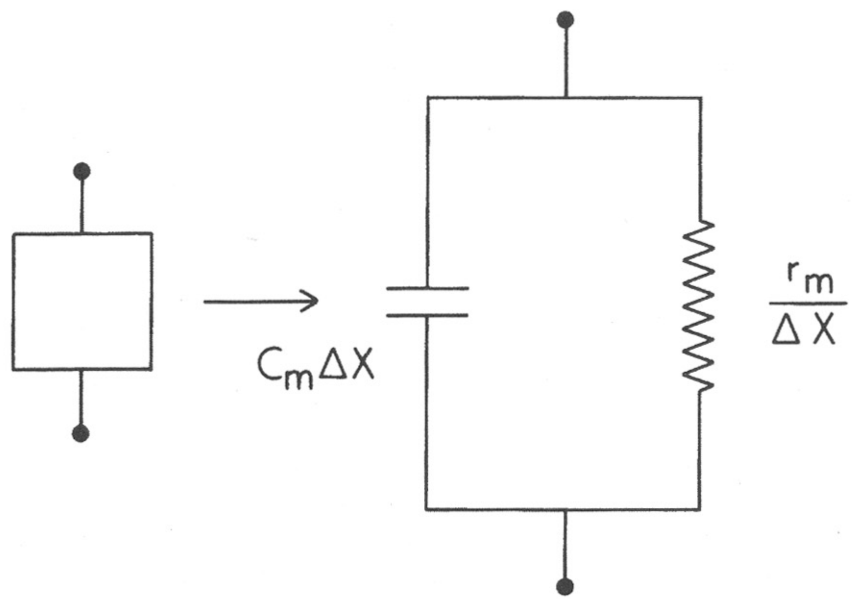


Figure 6.2. Electrical representation of a cylindrical fiber membrane element of length Δx under (linear) subthreshold conditions.

Core-conductor model (cont.):

Under nonlinear (i.e., suprathreshold/ transthreshold) conditions, each membrane patch of length Δx must be described by the HH model.

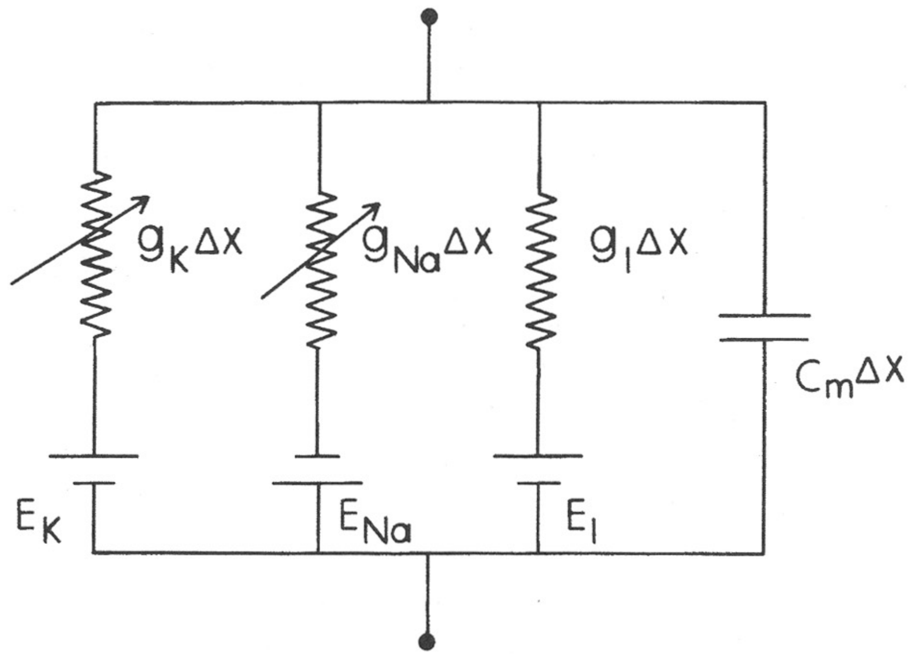


Figure 6.3. Electrical representation of the membrane for a fiber of length Δx under transthreshold conditions. The conductances g_K, g_{Na}, and g_ℓ are found from the Hodgkin–Huxley equations and are converted to units of S/cm for the linear core-conductor model.

Core-conductor model assumptions:

1. The transmembrane and longitudinal currents, as well as the intra- and extra-cellular potentials, are functions only of the axial (longitudinal) coordinate x. That is, we have a one-dimensional cable model.

2. For a fiber with a restricted extracellular space, the extracellular current can only flow in the axial (longitudinal) direction.
   In the case of a larger extracellular space, the resistance of the extracellular electrolyte is assumed to be negligible, i.e., $r_e \approx 0$.

Nerve fiber bundle showing restricted extracellular spaces:

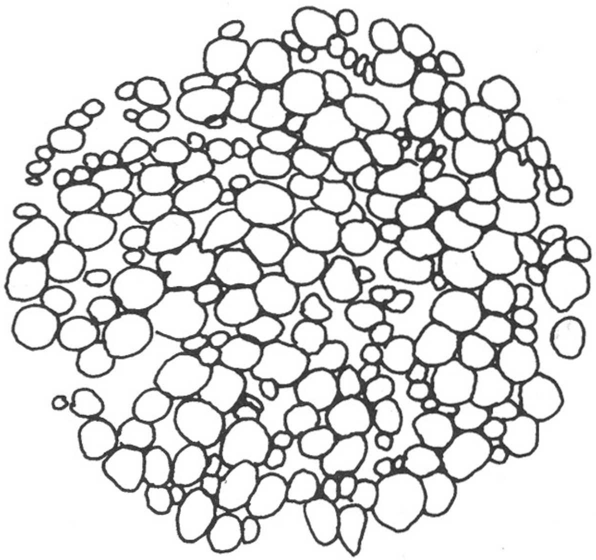


Figure 6.4. Photomicrograph of a transverse section of a cat saphenous nerve fascicle. Few fibers have a circular cross section and some are quite convoluted, but they can be approximated as circular or, better, as elliptical. Except near the periphery the interstitial currents can be expected to be essentially axial. If all fibers are approximately the same and behaving synchronously, and if the total number is N while the total interstitial cross-sectional area is A_e, then each fiber is associated with an interstitial cross-sectional area of A_e/N. Figure 6.1 would then apply to a typical fiber with $r_e = R_e/(A_e/N)$. This figure is from W. Olson, PhD dissertation, University of Michigan, 1985; also W. Olson, X. Wit, and S. L. BeMent, Compound action potential reconstructions and predicted fiber diameter distributions, in *Conduction Velocity Distributions*, L. J. Dorfman, K. L. Cummins, and L. J. Leifer, eds., A.R. Liss, New York, 1981. Reprinted by permission of Wiley-Liss Inc., a subsidiary of John Wiley and Sons.

Core-conductor model assumptions (cont.):

3. The radius of a fiber is typically many times smaller than its length, such that the intracellular current can be assumed only to flow in the axial (longitudinal) direction. The resistance per unit length of the intracellular electrolyte is found via Eqn. (2.55′).
4. For nerve and muscle under passive conditions, the membrane is represented by passive components (shown in Fig. 6.2) with values for r_m and c_m found via Eqns. (2.56′) and (2.57′).
   Under active conditions, the HH model is utilized, as illustrated in Fig. 6.3.

Cable equations:

Ohm's and Kirchhoff's laws can be applied to the core-conductor circuit (shown in Fig. 6.1 of Plonsey and Barr) to obtain the *cable equations* for a uniform cylindrical fiber of arbitrary length.

It is then desirable to evaluate the cable equations in the limit as $\Delta x \rightarrow 0$, such that the equations describe the behaviour of the fiber as a *continuous* function of axial (longitudinal) position, rather than a set of discrete membrane patches.

Cable equations (cont.):

The resulting relationship between the extracellular potential gradient in the axial direction as a function of the axial current is:

$$\frac{\partial \Phi_e}{\partial x} = -I_e r_e, \tag{6.1}$$

and likewise the intracellular potential gradient is:

$$\frac{\partial \Phi_i}{\partial x} = -I_i r_i. \tag{6.2}$$

Cable equations (cont.):

If current leaves the intracellular space by crossing the membrane, then the intracellular current will show an axial *decrease*, while the transmembrane current will be *positive*. This conservation of current is described by:

$$\frac{\partial I_i}{\partial x} = -i_m, \qquad (6.3)$$

where i_m is the transmembrane current per unit length.

(Note that i_m is a linear function of the membrane potential under passive (subthreshold) conditions but is a nonlinear function under active (suprathreshold) conditions.)

Cable equations (cont.):

In contrast, the extracellular current will *increase* axially due to any transmembrane current.

In stating this relationship we will also allow for the possibility that a current may be injected into the extracellular space from polarizing electrodes, giving:

$$\frac{\partial I_e}{\partial x} = i_m + i_p, \qquad (6.4)$$

where i_p is the current per unit length injected from the polarizing electrodes.

Cable equations (cont.):

Example current pathways for extracellular current injection:

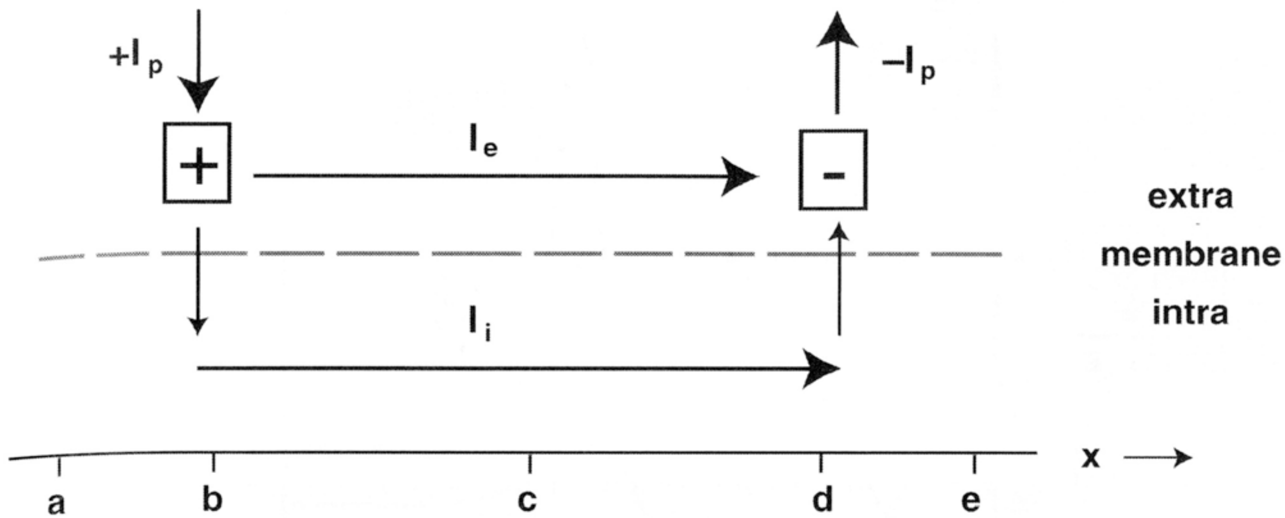


Figure 6.5. Current Pathways Cartoon. A membrane (dotted line) separates the extracellular space (above) from intracellular space (below). Letters a through e at the bottom identify particular x coordinates. Stimulus current I_p enters the extracellular space through an extracellular current source at b, and leaves the extracellular space through an extracellular current sink at d. As drawn, the stimulus current divides into an intracellular component I_i and an extracellular component I_e at b, then returns at d. A negative transmembrane current (negative because inward) exists at b and a positive transmembrane current exists at d. Total current I is zero at a and e but equal to I_p at c. For purposes of discussing an example, it is convenient to assign $I_p = 3$ and $I_i = 1$, so $I_e = 2$. Thus the membrane current is -1 at b and $+1$ at d.