ELEC ENG 3BB3:

Cellular Bioelectricity

Notes for Lecture 19

Friday, February 14, 2014

Lumped monopole source model:

For a typical action potential waveform, the monopole source density is triphasic, with the three phases each extending over a fairly short section of the fiber, as illustrated in the next slide.

Consequently, these three source density regions can be approximated by three lumped monopoles M_1, M_2 & M_3.

This is readily shown by approximating the AP waveform as a triangle (see next slide).

Lumped monopole source model (cont.):

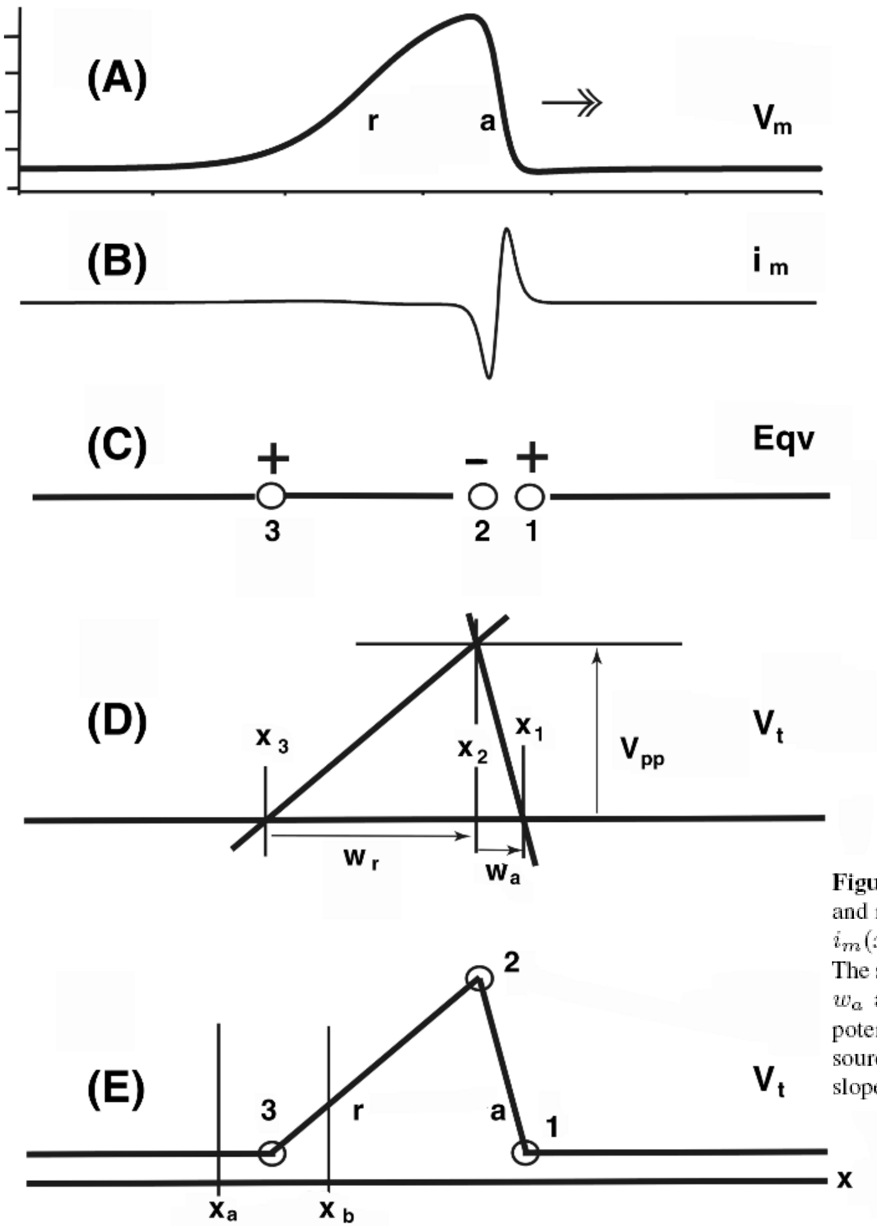


Figure 8.8. Monopole Sources. (**A**) Monophasic action potential $V_m(x)$. The activation and recovery phases are identified with the small letters a and r. (B) Membrane current $i_m(x)$. (C) Lumped equivalent monopole sources. (D) Triangularized action potential. The sides of the triangle have a slope equal to the maximum slope of V_m of panel A. Widths w_a and w_r are for the activation and recovery phases, spatially. (E) Triangular action potential with encircled sites where there is a slope change. Sites 1 to 3 lead to monopole sources M_1 to M_3. The activation phase of V_t has slope $A_{\max}$ and the recovery phase has slope $B_{\max}$.

Lumped monopole source model (cont.):

From Eqn. (8.19), the total monopole strength arising from the distributed monopole densities in any interval x_a to x_b is given by:

$$
\begin{aligned}
M &= \pi a^2 \sigma_i \int_{x_a}^{x_b} \frac{\partial^2 V_m}{\partial x^2} \, \mathrm{d}x \\
&= \pi a^2 \sigma_i \left(\left. \frac{\partial V_m}{\partial x} \right|_{x=x_b} - \left. \frac{\partial V_m}{\partial x} \right|_{x=x_a} \right). \quad (8.21)
\end{aligned}
$$

Lumped monopole source model (cont.):

The *magnitude* of the slope of the activation phase of width w_a and peak potential V_{pp} is:

$$A_{\mathsf{max}} = V_{\mathsf{pp}}/w_a\,, \tag{8.22}$$

and the *magnitude* of the slope of the repolarization phase of width w_r is:

$$B_{\mathsf{max}} = V_{\mathsf{pp}}/w_r\,. \tag{8.22}$$

Lumped monopole source model (cont.):

Consequently:

$$M_1 = \pi a^2 \sigma_i A_{\mathsf{max}}, \tag{8.24}$$

$$M_2 = -\pi a^2 \sigma_i \left(A_{\mathsf{max}} + B_{\mathsf{max}} \right), \tag{8.25}$$

$$M_3 = \pi a^2 \sigma_i B_{\mathsf{max}}. \tag{8.26}$$

These three lumped (i.e., discrete) monopole sources are located at the activation onset, AP peak & repolarization offset and approximate the actual distributed monopole source densities.

Lumped monopole source model (cont.):

The net extracellular field potential generated by these three sources is then:

$$\Phi_p = \frac{1}{4\pi\sigma_e}\left(\frac{M_1}{r_1} + \frac{M_2}{r_2} + \frac{M_3}{r_3}\right) \qquad (8.27)$$

$$= \frac{a^2\sigma_i}{4\sigma_e}\left(\frac{A_{\text{max}}}{r_1} - \frac{A_{\text{max}} + B_{\text{max}}}{r_2} + \frac{B_{\text{max}}}{r_3}\right), \qquad (8.28)$$

where r_1, r_2 & r_3 are the respective distances from the sources to the field point.

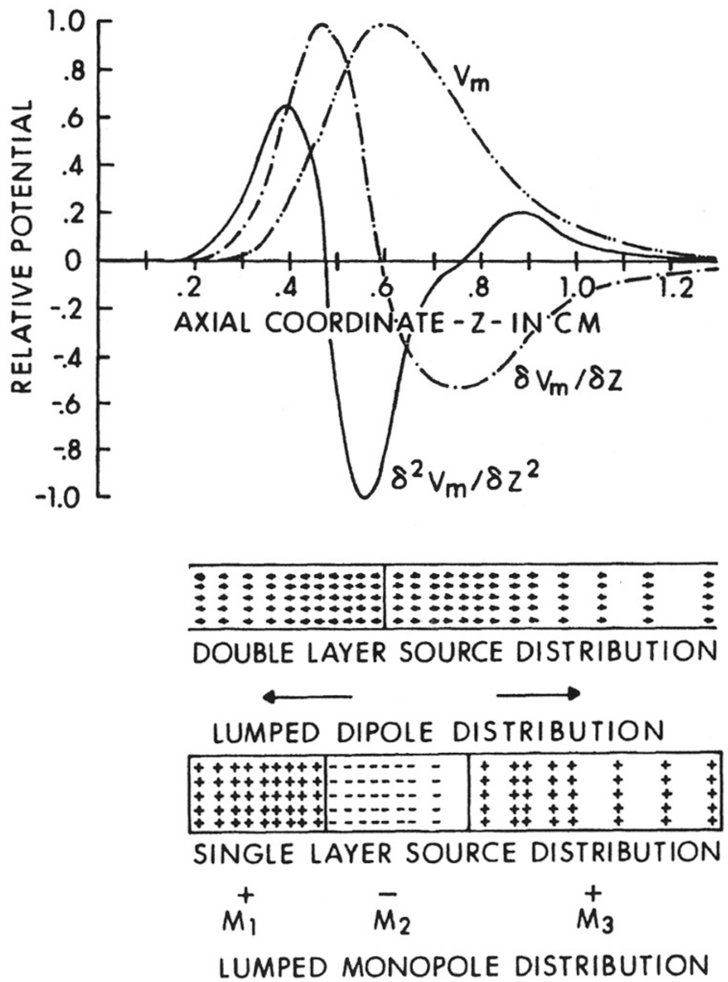
Lumped monopole source model (cont.):

Note:-

1. A lumped monopole source is a good approximation only at a reasonable distance from the fiber.
2. Eqn. (8.21) can be used to determine three lumped monopoles directly from the AP waveform, rather than the triangular approximation. In this case, each monopole is located at the “center of gravity” of the monopole source density that it is approximating.

Lumped fiber source models (cont.):

The spatial waveform of an action potential propagating in the negative x direction, and its first two spatial derivates, with the resultant source densities.



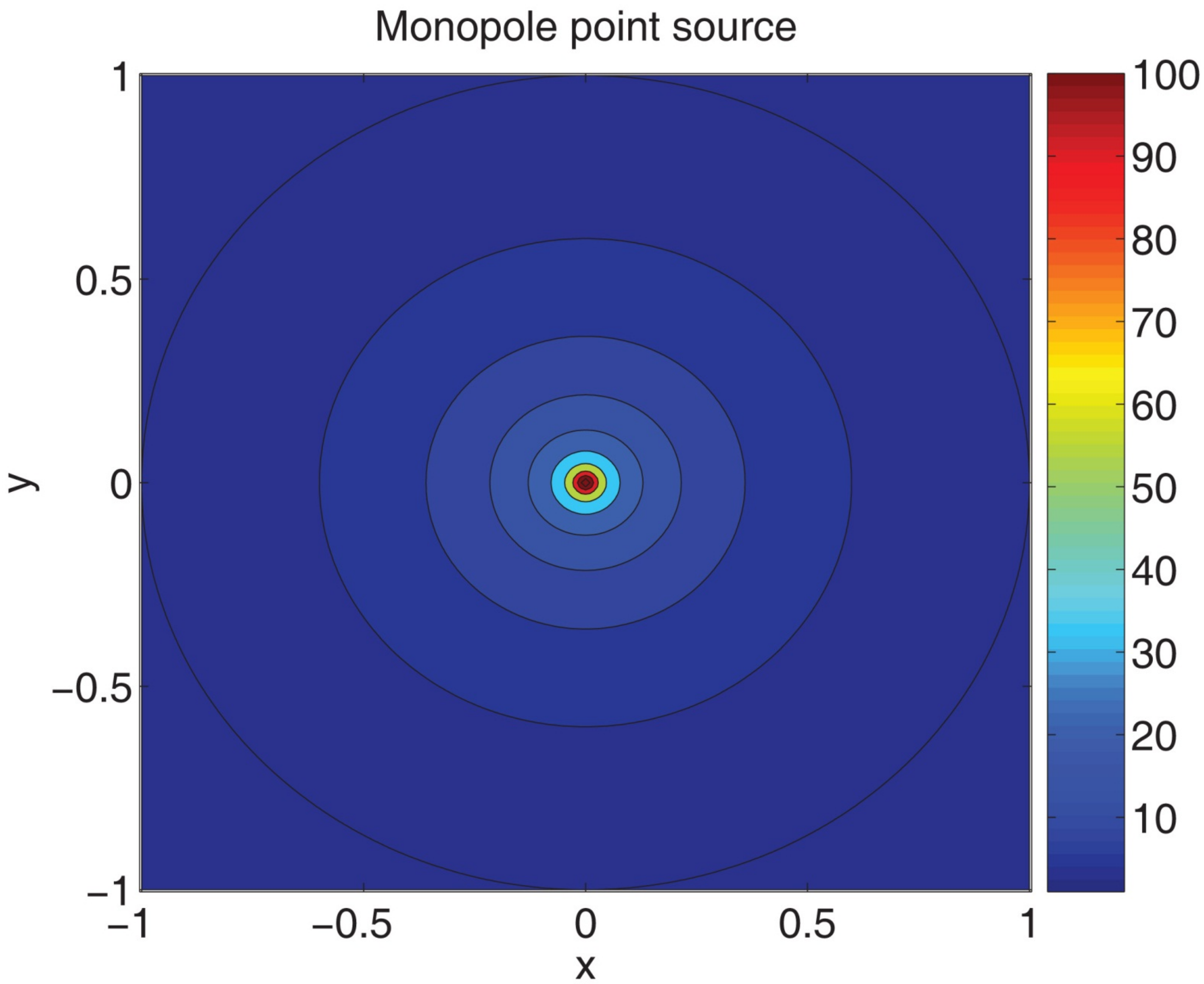
(From Plonsey & Barr, 2nd Edition)

Figure 8.2. Monophasic action potential v_m and its first two spatial derivatives. The corresponding double-layer and single-layer distributed sources are also indicated. Lumped dipole and monopole sources are shown (their locations are approximate). [From R. Plonsey, Action potential sources and their volume conductor fields, *Proc. IEEE* **65**:601–611 (1977). Copyright 1977, IEEE.]

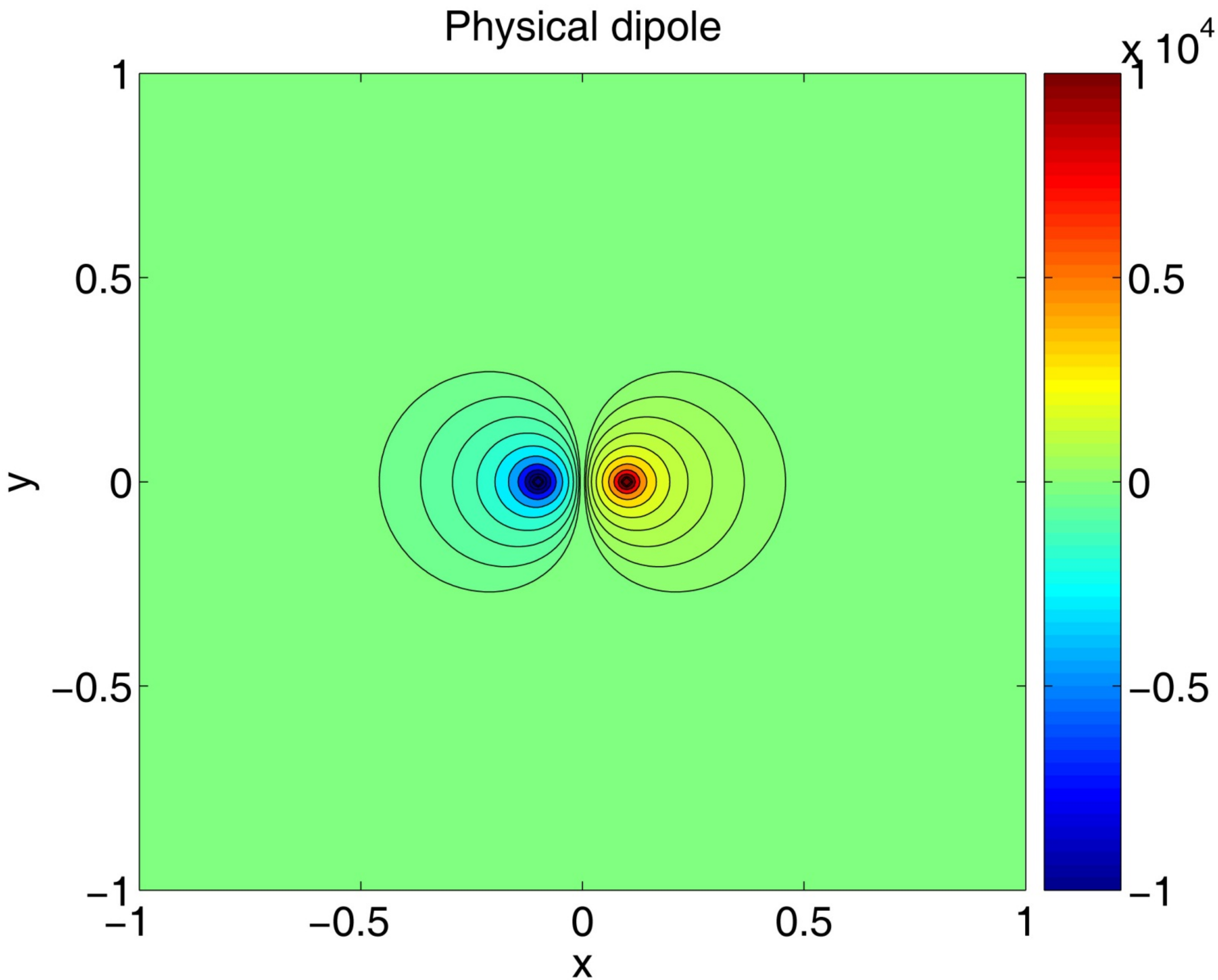
Dipole source definition:

The “tripole” lumped monopole model derived in the last lecture could also be interpreted as a pair of physical dipoles, one with current strength $+\pi a^2\sigma_i A_{max}$ at the onset of activation and $-\pi a^2\sigma_i A_{max}$ at the peak of the AP, and another with current strength $+\pi a^2\sigma_i B_{max}$ at the offset of repolarization and $-\pi a^2\sigma_i B_{max}$ at the peak of the AP.

Dipole source definition (cont.):



Dipole source definition (cont.):



Dipole source definition (cont.):

For a physical dipole with current strengths $+I_0$ and $-I_0$ separated by a distance d, the amount of uncancelled field created by the two monopoles is dependent on the product $p = I_0 d$.

Thus, the physical dipole equation:

$$\Phi_d = -\frac{I_0}{4\pi\sigma}\frac{1}{r_0} + \frac{I_0}{4\pi\sigma}\frac{1}{r_1}\,, \qquad (2.23)$$

is well approximated by the expression:

$$\Phi_d = \frac{I_0}{4\pi\sigma}\frac{\partial\,(1/r)}{\partial d}d\,, \qquad (2.25)$$

for d small compared to r_0 and r_1.

Dipole source definition (cont.):

The directional derivate of (1/r) is the component of the gradient of (1/r) in the direction of d, i.e.:

$$\frac{\partial}{\partial d}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right)\cdot \bar{a}_d\,, \qquad (2.26)$$

where $\bar{a}_d$ is the unit vector in the direction of the displacement $\bar{d}$.

Consequently, Eqn. (2.25) can be reformulated as:

$$\Phi_d = \frac{I_0}{4\pi\sigma}\nabla\left(\frac{1}{r}\right)\cdot \bar{d}\,. \qquad (2.28)$$

Dipole source definition (cont.):

An idealized or “perfect” dipole is the point source equivalent of a physical dipole. It is created by letting d $\rightarrow$ 0 and $I_0 \rightarrow 1$ such that $p = I_0 d$ remains constant and finite. Eqn. (2.28) then becomes:

$$\Phi_d = \frac{1}{4\pi\sigma} \nabla \left(\frac{1}{r} \right) \cdot \bar{p}, \tag{2.29}$$

where the dipole vector *p* = pa_d is typically located at a point midway between the positive and negative physical dipoles sources and oriented in the direction of the positive source.

Dipole source definition (cont.):

The gradient of (1/r) can be shown to equal:

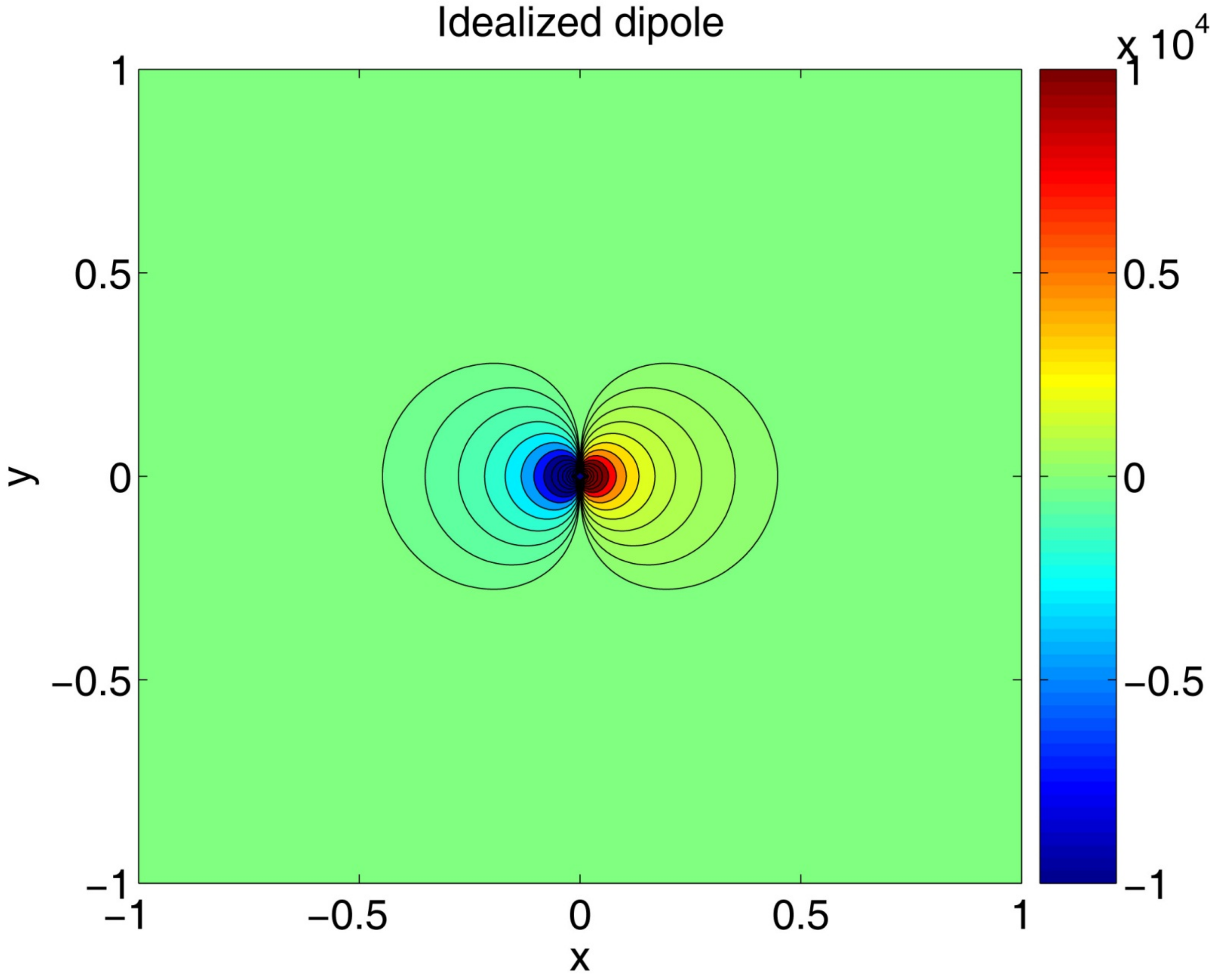
$$\nabla\left(\frac{1}{r}\right) = \frac{\bar{a}_r}{r^2}, \tag{2.32}$$

where $\bar{a}_r$ is the unit vector from the dipole source location to the field point. If μ is the angle between $\bar{a}_r$ and vector p, then:

$$\Phi_d = \frac{\bar{a}_r \cdot \bar{p}}{4\pi\sigma r^2} \tag{2.33}$$

$$= \frac{1}{4\pi\sigma}\frac{p\cos\theta}{r^2}. \tag{2.34}$$

Dipole source definition (cont.):



Single-fiber dipole source model:

Eqn. (8.12) can be rewritten in the form:

$$\Phi_e = \frac{a^2\sigma_i}{4\sigma_e}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(\frac{\partial V_m}{\partial x}\right)\frac{1}{r}\,\mathrm{d}x\,. \qquad (8.32)$$

Integrating by parts gives:

$$\Phi_e = \frac{a^2\sigma_i}{4\sigma_e}\left\{\left[\frac{\partial V_m}{\partial x}\frac{1}{r}\right]\Bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial V_m}{\partial x}\frac{\mathrm{d}(1/r)}{\mathrm{d}x}\,\mathrm{d}x\right\}. \quad (8.34)$$

Single-fiber dipole source model (cont.):

The integrated part of (8.34) is zero if the action potential is not at the ends of the fiber, and it is only necessary to integrate over the region L of the fiber where $\partial V_m/\partial x \neq 0$, giving:

$$\Phi_e = \frac{a^2\sigma_i}{4\sigma_e}\int_L \left[-\frac{\partial V_m}{\partial x}\right]\frac{\mathrm{d}(1/r)}{\mathrm{d}x}\,\mathrm{d}x. \quad (8.35)$$

Again, the directional derivative is:

$$\frac{\mathrm{d}(1/r)}{\mathrm{d}x} = \bar{a}_x \cdot \nabla\left(\frac{1}{r}\right). \quad (8.36)$$

Single-fiber dipole source model (cont.):

Substituting (8.36) into (8.35) gives:

$$\Phi_e = \frac{a^2\sigma_i}{4\sigma_e}\int \left[-\frac{\partial V_m}{\partial x}\,\bar{a}_x\right]\cdot\left[\nabla\left(\frac{1}{r}\right)\right]\mathrm{d}x. \quad (8.37)$$

Eqn. (8.37) can be considered the integral of the **dipole source density**:

$$\bar{\tau}_\ell \equiv -\pi a^2\sigma_i\frac{\partial V_m}{\partial x}\bar{a}_x \approx I_i\bar{a}_x\,, \qquad (8.41)$$

with units of current (current times length per unit length) oriented in the x direction. Note the final approximation in (8.41) is true for $r_e < r_i$.

Single-fiber dipole source model (cont.):

The field potential generated by a dipole source density is then:

$$\Phi_e = \frac{1}{4\pi\sigma_e} \int \bar{\tau}_\ell \cdot \left[\nabla \left(\frac{1}{r} \right) \right] \mathrm{d}x \, . \qquad (8.43)$$

Because the first derivate of a typical AP spatial waveform is biphasic, a propagating AP produces a pair of dipole source regions, each with the dipoles pointing away from the peak of the AP.

Note that these dipole source regions are consistent with the diffuse axial extracellular currents I_e produced by a propagating AP.

Lumped dipole source model:

The two dipole source density regions can be approximated by two lumped dipoles, a dipole D_p pointing in the +ve x direction at the activation phase and a dipole D_n pointing in the ¡ve x direction at the repolarization phase of an AP propagating in the +ve x direction.

Note that these dipoles would be switched if the AP were propagating in the -ve x direction.

Lumped dipole source model (cont.):

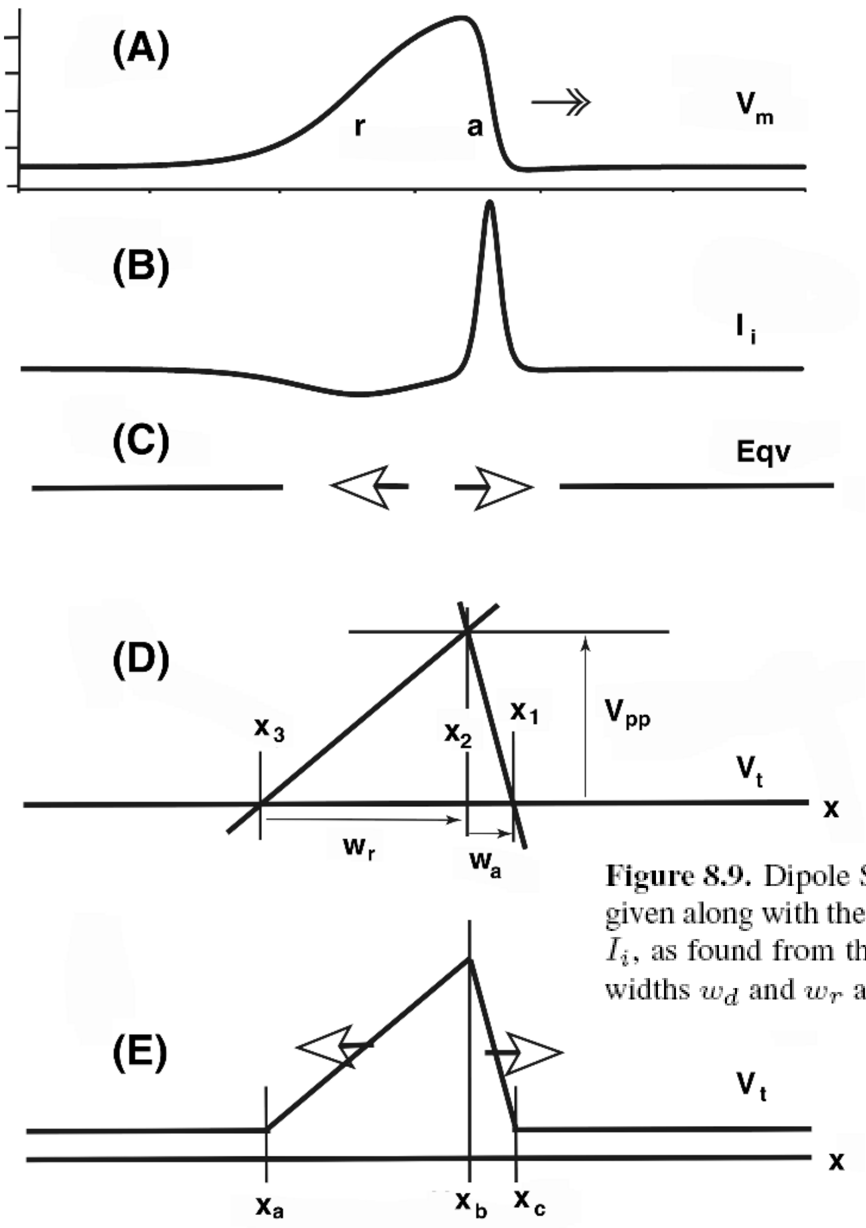


Figure 8.9. Dipole Sources. The action potential $V_m(x)$ and its first spatial derivative are given along with the approximating triangularized action potential $V_t(x)$ and axial current I_i, as found from the spatial derivative of V_m. Depolarization and repolarization spatial widths w_d and w_r are for the approximating triangular action potential.

Lumped dipole source model (cont.):

From Eqn. (8.41), the total dipole strength arising from the distributed dipole densities in any interval x_1 to x_2 is given by:

$$D = -\pi a^2 \sigma_i \int_{x_1}^{x_2} \frac{\partial V_m}{\partial x} \, \mathrm{d}x$$

$$= \pi a^2 \sigma_i \left[V_m(x_1) - V_m(x_2) \right] . \qquad (8.46)$$

Lumped dipole source model (cont.):

Hence, the lumped dipole strength associated with the activation interval x_b to x_c is:

$$\begin{aligned} D_p &= \pi a^2 \sigma_i \left[V_{\text{peak}} - V_{\text{rest}} \right] \\ &= \pi a^2 \sigma_i V_{\text{pp}} \,, \end{aligned} \tag{8.47}$$

and the lumped dipole strength for the repolarization interval x_a to x_b is:

$$\begin{aligned} D_n &= \pi a^2 \sigma_i \left[V_{\text{rest}} - V_{\text{peak}} \right] \\ &= -\pi a^2 \sigma_i V_{\text{pp}} \,. \end{aligned} \tag{8.48}$$

Lumped dipole source model (cont.):

Note:-

1. Each lumped dipole is located at the "center of gravity" of the dipole source density that it approximates.
2. A lumped dipole source is a good approximation only at a reasonable distance from the fiber.
3. Eqn. (8.46) can be used to determine lumped dipoles directly from the AP waveform, rather than the triangular approximation.