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Capacity of Optical Intensity Channels With Peak and Average Power Constraints

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***Abstract*— The design and analysis of capacity-approaching input signalling for optical intensity channels are presented. Both peak and average optical power constraints are considered in the analysis. The capacity-achieving distribution for this channel is discrete with a finite number of mass points. In practice, finding this distribution requires solving a complex non-linear optimization at every SNR. In this work, we present a closed form discrete capacity-approaching distribution derived via source entropy maximization. The computation of this distribution is substantially less complex than previous optimization approaches and can be easily computed for different SNRs. The information rates using the derived maxentropic distribution are shown to be negligibly far away from the channel capacity found by non-linear optimization in the SNR range -6 to 6 dB.**

I. Introduction

In optical wireless systems, data are transmitted by modulating the instantaneous intensity of a laser or LED source. As a result, the information bearing signal is restricted to be non-negative. Due to eye safety standards, an average optical power constraint is imposed on the transmitted signal. This constraint is formulated as an average amplitude constraint since intensity modulation is considered. A peak amplitude constraint is also applied due to safety and physical source limitations. In such systems, the noise is modeled as zero-mean, signal-independent, white Gaussian noise [1]. In this work, a closed form expression for a capacity-approaching distribution is developed for wireless optical channels under peak and average optical power constraints to improve the data rates and approach the channel capacity.

Smith [2] developed a procedure to compute the channel capacity and the optimum input distribution for channels with peak and average power constraints through a nonlinear optimization problem. However, analytical expressions for the input distribution are not available, and instead are found via complex optimization routines. Using a similar approach, Shamai [3] showed that a discrete input distribution is a capacity-achieving distribution for Poisson optical channels with bounded-input and power constraints. Later, Shamai and David [4] obtained the same results for quadrature additive Gaussian channels. More attention was directed to these results in the last decades where the discrete input distribution is shown to be a capacity-achieving distribution for many channels [5]–[8]. Subject to bounded-input and average cost constraints, Chan *et al.* [9] showed that the channel capacity is achievable and derived necessary and sufficient conditions for a capacity-achieving input distribution. This procedure was utilized to study signal-dependent optical channels where the capacity-achieving distribution is proven to be discrete with a finite number of mass points. Recently, upper and lower bounds on the channel capacity were developed in [10]. The upper bound is developed based on duality [11], while the lower bound is developed based on entropy power inequality. However, no explicit source distributions were provided and hence no clear insights for communication system design can be drawn. However, in all previous work no analytical close form expression for capacity-achieving nor -approaching distribution has been given.

In this work, an analytical closed form expression for a capacity-approaching input distribution via source entropy maximization is developed under non-negativity, peak and average power constraints. Unlike previous work [12]–[14], a *peak* amplitude constraint is considered explicitly in the analysis. Compared to [10] the information rates of the developed distribution are much tighter to the channel capacity, furthermore, signalling and coding can be designed based on the presented distribution to realize the achievable rates. We present a family of non-uniform discrete source distributions with a finite number of equally spaced amplitude mass points whose mutual information is near the channel capacity. Unlike the capacity-achieving distribution computed via non-linear optimization [2] and varies for each SNR, the developed distribution is fixed over a range of SNRs with fewer number of mass points making implementation simpler.

Section II presents an overview of the channel model and Sec. III presents the development of the proposed discrete distributions. Section IV discusses design considerations and gives numerical results. The paper concludes in Sec. V with directions for future work.

II. System Model

In wireless optical communication links information is modulated as the instantaneous optical intensity of the transmitted signal. We consider optical intensity pulse amplitude modulation (PAM). A discrete time representation for this channel is given by [1]

$$Y = rX + Z \tag{1}$$

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where $0 \leq X \leq A$ is the transmitted optical intensity signal with average optical power constraint $\mathbb{E}\{X\} \leq P$ and peak amplitude A. The constant r represents the combined system losses. Without loss of generality, we set $r = 1$ since it only scales the SNR. The noise, Z, models both thermal noise and ambient light induced shot noise and can be well modeled as zero-mean, signal-independent, Gaussian noise with variance σ^2. According to the Gaussian noise model, the output electrical current, Y can assume negative amplitudes. The optical SNR is defined as P/σ as in previous work [13].

III. Proposed Discrete Distributions

The capacity of the optical wireless intensity channel is defined as the maximum mutual information between channel input and output $\mathbb{I}(X;Y)$ over all possible input distributions, $p(x)$, satisfying the non-negativity, average optical power and peak amplitude constraints, i.e.,

$$C = \max_{p(x)\in\mathcal{P}} \mathbb{I}(X;Y)$$

where

$$\mathcal{P} = \left\{ p(x) : x \in \mathcal{U}, \mathbb{E}\{X\} \leq P, p(x) \geq 0, \int p(x) = 1 \right\},$$

and $\mathcal{U} = [0, A]$. In this work, a capacity-achieving distribution is developed by numerically solving a non-linear optimization problem [2]. Due to the overhead encountered in solving this optimization problem, a novel input distribution based on source entropy maximization is developed. The resulting mutual information is shown to be negligibly close to the channel capacity with the advantage of a substantial reduction in complexity. Unlike previous work, a closed form expression for the input distribution is given.

A. Capacity Achieving Distribution

The capacity-achieving input distribution for additive white Gaussian channels under peak and average power constraints is discrete with a finite number of mass points [2] [9]. Based on this result, consider the subset $\mathcal{Q} \subset \mathcal{P}$ of the discrete input distributions defined as

$$\mathcal{Q} = \left\{ q(x) \in \mathcal{P} : K \in \mathbb{Z}^+,\ q(x) = \sum_{k=0}^{K} a_k \delta(x - x_k) \right\}$$

where $\delta(\cdot)$ is the delta functional, $\mathbb{Z}^+$ is the set of positive integers, $K+1$ is the number of mass points, and a_k and x_k are the amplitudes and positions of the k^{th} mass point respectively. Note that the capacity-achieving distribution belongs to the set Q and can be obtained by solving the non-linear optimization problem,

$$q^*(x) = \arg \max_{q(x)\in\mathcal{Q}} \mathbb{I}(X;Y). \tag{2}$$

In this problem a_k, x_k and K are unknown parameters. Numerical optimization methods can efficiently solve this problem and extract $q^*(x)$.

It was shown in [2] that the number of mass points is monotonically non-decreasing with SNR and a mass point at $x_k = 0$ always exists. For a given SNR, the mutual information is computed for $K = 1$ and repeated for $K = 2, 3, \ldots$ until a stopping criterion is satisfied. Assume that the capacity-achieving distribution has $K^* + 1$ mass points. Solving the optimization problem with $K > K^*$, results in the capacity-achieving input distribution with $K - K^*$ zero amplitude mass point. This fact is used as a stopping criterion for the optimization problem. Therefore, K^* is the minimum K in the optimization problem which results in the maximum mutual information over $\mathcal{Q}$.

B. Capacity-Approaching Distribution: Closed form

For every average optical power, peak amplitude and noise variance, the optimization problem (2) is solved to extract the optimum distribution. In addition to this drawback, the complexity of each run solving the optimization problem is expensive especially when many mass points are considered. In this work, a closed form expression for a capacity-approaching distribution is developed with a substantial reduction in effort required to find the input distribution for different SNR. In addition, an approximate expression for the number of mass points is provided for any SNR.

In order to find a simple expression for a capacity-approaching distribution, two assumptions are applied. In the first step, the input distributions in $\mathcal{Q}$ with $K + 1$ equally spaced mass points are considered through $\mathcal{Q}' \subset \mathcal{Q}$,

$$\mathcal{Q}' = \left\{ q'(x;K) \in \mathcal{Q} : \ell = \frac{A}{K},\ q'(x;K) = \sum_{k=0}^{K} a_k \delta(x - k\ell) \right\}$$

where ℓ is the mass point spacing. In the second step, the discrete input distribution that maximizes the input source entropy $\mathbb{H}(X)$ is considered instead of maximizing the mutual information $\mathbb{I}(X;Y)$. Let $\bar{\mathcal{Q}} \subset \mathcal{Q}'$ denote the set of input distributions that maximize the source entropy for a given K,

$$\bar{\mathcal{Q}} = \left\{ \bar{q}(x;K) \in \mathcal{Q}' : \bar{q}(x;K) = \arg \max_{q'(x;K)\in\mathcal{Q}'} \mathbb{H}(X) \right\} \tag{3}$$

The input distribution $\bar{q}(x;K)$ is obtained by solving the following optimization problem,

$$\bar{q}(x;K) = \arg\max_{a_k} \left(\sum_{k=0}^{K} a_k \log_2 \frac{1}{a_k} \right)$$

$$\text{subject to } \sum_{k=0} a_k = 1,\ a_k \geq 0,\ \sum_{k=0} k\ell a_k \leq P,\ \ell = \frac{A}{K}.$$

Define $\mathcal{J}$ as the Lagrangian associated with the optimization problem as

$$\mathcal{J} = \sum_{k=0}^{K} a_k \log_2 \frac{1}{a_k} - \lambda_1 \left[\sum_{k=0}^{K} a_k - 1 \right] - \lambda_2 \left[\sum_{k=0}^{K} k\ell a_k - P \right].$$

Lemma 3.1: When $A < 2P$ the Lagrange multiplier $\lambda_2 = 0$ and the input distribution is uniform with $a_k = 1/(K+1)$.

Proof: Notice that when $A < 2P$, a uniform distribution over a_k satisfies the average and peak amplitude constraints. Since the uniform distribution is entropy maximizing among all discrete distributions, it must also be the result of the optimization problem. In this case there is slack in the average constraint $\sum_{k=0}^{K} k\ell a_k < P$ and hence $\lambda_2 = 0$. ■

When $A \geq 2P$ the above optimization problem is solved analytically considering all constraints. The optimum input distribution for a given K is

$$\bar{q}(x;K) = \sum_{k=0}^{K} \bar{a}_k \, \delta(x - k\ell) \tag{4}$$

where

$$\bar{a}_k = \frac{t_0^k}{1 + t_0 + t_0^2 + \ldots + t_0^K} \tag{5}$$

and t_0 is a real root of

$$S(t) = \sum_{k=0}^{K} \left(1 - \frac{k}{K}\,\rho\right)\, t^k \tag{6}$$

where $\rho = A/P$ denotes the peak-to-average optical power ratio. For a given K, equation (6) can be solved efficiently to obtain a real root t_0.

Lemma 3.2: When $A \geq 2P$, a real root t_0 exists in $(0, 1]$.

Proof: Notice that $\bar{a}_k \geq 0$ and the denominator in (5) is independent of index k. Thus, $t_0 \geq 0$ since otherwise the sign of $\bar{a}_k$ would alternate. Since $S(0) = 1$ and $S(1) = (K + 1)(1 - \rho/2) \leq 0$ then from the continuity of $S(t)$ there exists at least one real root t_0 in the interval $(0, 1]$. ■

Given A and P, a family of input distributions parameterized by K are obtained as given in (4). In addition, for a given P and σ, the distribution $\bar{q}^*(x;\bar{K})$ that maximizes the mutual information is selected through an exhaustive search within this family where $\bar{K}+1$ is the corresponding number of mass points, i.e.,

$$\bar{q}^*(x;\bar{K}) = \arg \max_{\bar{q}(x)\in\bar{\mathcal{Q}}} \mathbb{I}(X;Y)$$

Clearly, an analytic expression for $\bar{K}$ that maximizes the mutual information is mathematically intractable since the mutual information expression depends on a nested relation between t_0 and K.

In order to avoid this time consuming search for $\bar{K}$, a simple approximation is made from numerical analysis. Given A/P, the SNR, P/σ, at which the mutual information for an input distribution with $K+1$ is greater than that with K is computed numerically with $K = 1$. The normalized mass points spacing

$$\frac{\ell}{\sigma} = \left(\frac{A}{P}\right)\left(\frac{P}{\sigma}\right)\left(\frac{1}{K}\right)$$

is shown in Fig. 1 versus $\rho = A/P$ for different K. As shown, the relation can be approximated by a constant. A simple closed form approximation is given by,

$$\tilde{K} = \left\lfloor \frac{A}{2.5\sigma} \right\rfloor \tag{7}$$

Although this simulation is conducted on a finite domain, it covers the practical values for wireless optical systems. In particular, the peak-to-average power ρ (x-axis) varies from 2 to 8. In addition, the number of mass points in this simulation varies from $K + 1 = 2$ to 8 mass points which is within practical implementation limits. Note, that any selection of $\tilde{K}$ will a valid lower bound on the channel capacity. In Sec. IV, it is shown that the resulting mutual information with this approximation remains close to the channel capacity.

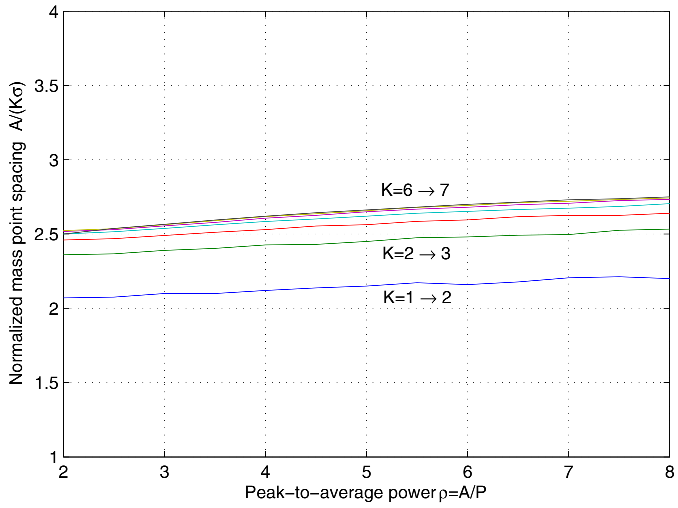

Fig. 1. Normalized spacing ℓ/σ versus peak-to-average optical power A/P.

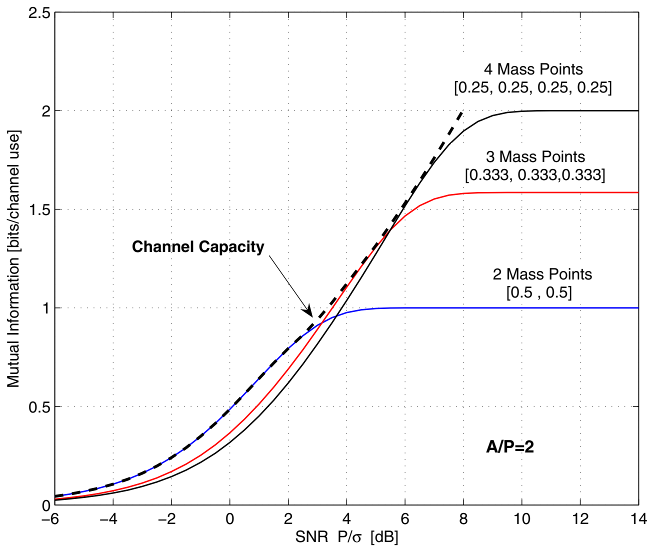

Fig. 2. Channel capacity and mutual information for the proposed input distribution with different numbers of mass points versus SNR for $\rho = 2$.

IV. Results

A. Channel Capacity and Mutual Information

The mutual information of the proposed input distributions and the channel capacity for $\rho = 2$, 4 and 6 are considered.

Fig. 2 shows the mutual information versus SNR at $\rho = 2$ for 2, 3 and 4 mass points. For comparison, the channel capacity is found by solving the non-linear optimization problem in (2) . Clearly, a negligible gap can be noticed between the mutual information and the channel capacity for different SNRs. From *Lemma 3.1*, when $\rho = 2$ the maxentropic input distribution is uniform with $K + 1$ probability mass points as shown in Fig. 2. Notice that when $\rho = 2$ the uniform distribution is a capacity-approaching distribution and the maxentropic distribution is fixed over a range of SNRs.

When $\rho > 2$ and as shown in Fig. 3 and Fig. 4 for $\rho = 4$ and $\rho = 6$ respectively, the capacity-approaching distributions are non-uniform. Also the maxentropic input distributions approach the channel capacity over a wide range of SNRs

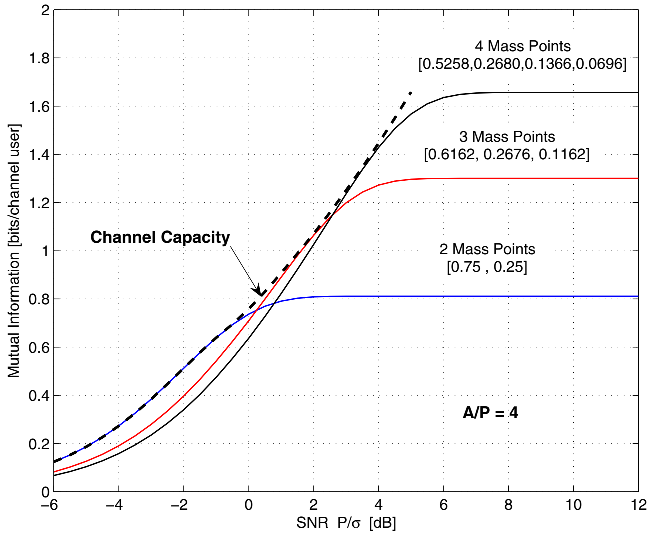

Fig. 3. Channel capacity and mutual information for the proposed input distribution with different number of mass points versus SNR for $\rho = 4$.

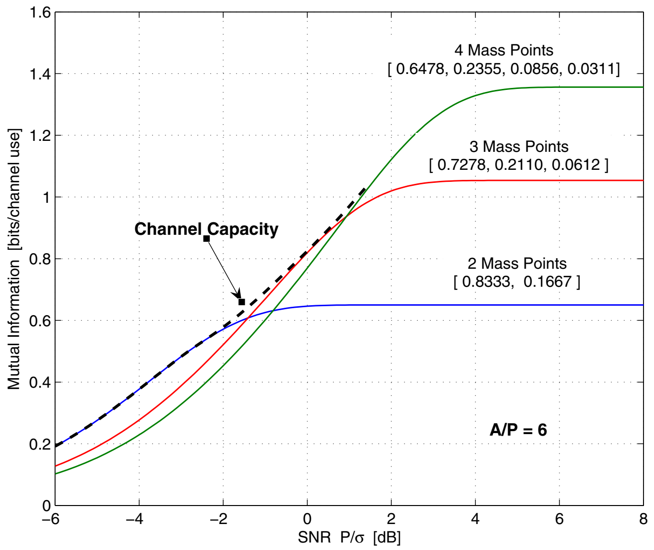

Fig. 4. Channel capacity and mutual information for the proposed input distribution with different number of mass points versus SNR for $\rho = 6$.

for both $\rho = 4$ and 6. Notice that at low SNRs, an input distribution with two mass points, $\bar{q}(x; 1)$, is sufficient to approach the channel capacity where

$$\bar{q}(x; 1) = \frac{\rho - 1}{\rho}\, \delta(x) + \frac{1}{\rho}\, \delta(x - A).$$

Notice that as ρ increases the resulting capacity-approaching distributions become increasing non-uniform with most weight on the zero-amplitude point.

B. Channel Capacity

The capacity of the wireless optical intensity channel versus SNR is shown in Fig. 5 when $\rho = 4$. The optimum distribution is achieved by solving the non-linear optimization problem (2) numerically. The mutual information maximized over the maxentropic input distribution given in (4) is also presented.

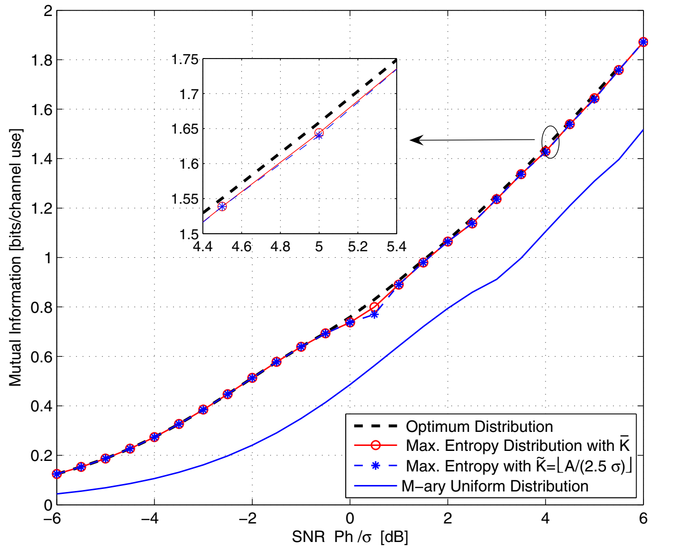

Fig. 5. Channel capacity and mutual information for the proposed maxentropic input distributions with $\bar{K}$ and $\tilde{K}$ and the uniform input distributions for $\rho = 4$.

Clearly, the resulting mutual information is a tight lower bound on channel capacity over a range of SNR. That is the proposed input distribution achieves the gain in terms of data rates offered by the optical channel with a substantial reduction in computing the input distribution.

Figure 5 also plots the mutual information using the proposed source distribution $\bar{q}(x; \tilde{K})$ when the number of mass points is approximated using (7). Notice that the mutual information is also close to the channel capacity and differs negligibly with the case when a search is performed to find $\bar{K}$. Thus, the approximation (7) does not incur a significant penalty in terms of rate. For comparison, the mutual information using an M-ary uniform distribution satisfying the peak and average power constraints is also presented. A remarkable gap between the mutual information and the channel capacity is noticed. As a result, non-uniform signalling is essential for optical intensity channels especially as ρ increases.

C. Input Distribution and Number of Mass Points

Fig. 6 shows the number of mass points in the capacity-achieving source distribution $K^* + 1$ which results from the optimization problem (2). Although K^* is fixed over a range of SNR, the capacity-achieving input distribution varies for every SNR. Fig. 6 also plots the number of mass points, $\bar{K} + 1$, for the proposed maxentropic distribution, $\bar{q}(x; \bar{K})$, that maximizes the mutual information for a given SNR. In this case for a given $\bar{K}$ the input distribution $\bar{q}(x; \bar{K})$ is fixed. Since $\bar{K}$ is fixed over a range of SNR as shown in Fig. 6 then, unlike the capacity-achieving distribution, the maxentropic distribution that maximizes the mutual information $\bar{q}(x; \bar{K})$ is fixed. In addition, the mutual information with input distribution $\bar{q}(x; \tilde{K})$ using $\tilde{K}$ from (7) is presented. Although $\bar{K} \neq \tilde{K}$ for some SNRs, the resulting mutual information shown in Fig. 5 is close and tight to the channel capacity. Notice that both $\bar{K}$ and $\tilde{K}$ are less than or equal to K^* which simplifies

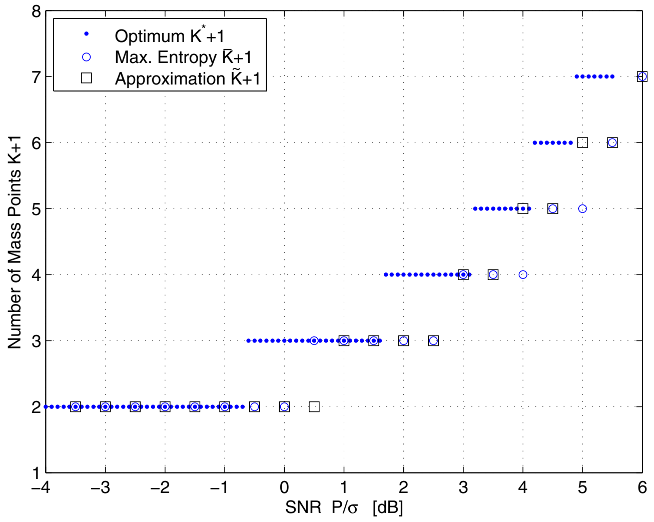

Fig. 6. Number of mass points versus SNR obtained for the proposed input distribution, $\bar{K}$, the approximation, $\tilde{K}$, and the optimum, K^*, for $\rho = 4$.

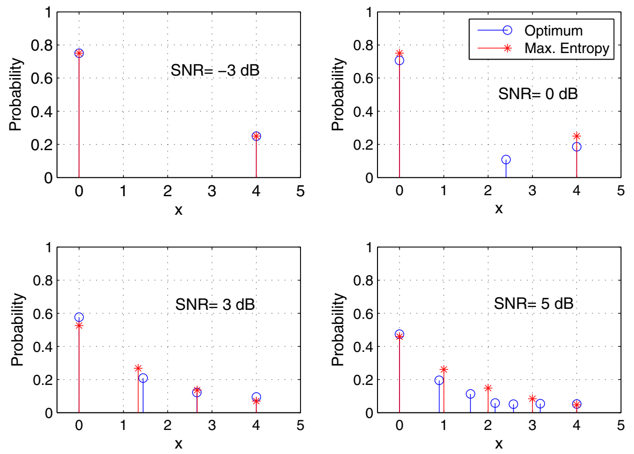

Fig. 7. The optimum and the proposed input distribution for different SNRs with $\rho = 4$.

the implementation of the optical transmitter while sacrificing little in terms of achievable rates.

Fig. 7 shows the optimum, $q^*(x)$, and the maxentropic, $\bar{q}(x; \bar{K})$, input distributions for SNR=$[-3, 0, 3, 5]$ dB at $\rho = 4$. When SNR=-3 dB, an input distribution with two mass points achieves the channel capacity. In this low SNR region the optimum and the maxentropic distributions are close to each other as shown. Notice that the spacing between mass points for the maxentropic input distribution is fixed to $\ell = A/K$ where for the optimum distribution it is a free parameter in the optimization problem. As SNR increases, the number of mass points also increases. When SNR=3 dB, the spacing and the mass point amplitudes for the maxentropic distribution are close to the optimum distribution. Although when SNR=5 dB, a different number of mass points is noticed between the optimum and the maxentropic distributions, the difference in the resulting mutual information for both distributions is negligible as shown from Fig.5

V. Conclusions

The capacity of optical intensity channels with peak and average optical power constraints are considered. The capacity-achieving distribution is found through numerical solution for a non-linear optimization problem. A closed form for a capacity-approaching distribution based on entropy maximization is presented. In addition to the substantial complexity reduction in generating this distribution compared to the optimum distribution, a negligible gap between the resulting mutual information and the channel capacity is noticed. Unlike the capacity-achieving distribution where for each SNR value different input distribution is obtained, the proposed input distribution is fixed over a range of SNRs. For a given peak and average optical power and noise variance, an approximate closed form expression is provided for the number of mass points selections. Over a practical range of SNRs, the number of probability mass points is less than the capacity-achieving distribution, reducing implementation complexity. The derived capacity-approaching distributions serve as a useful tool not only to bound the channel capacity but to guide the development of channel coding and modulation for such optical wireless channels.

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