Lecture 5: Polarization and Related Antenna Parameters

(*Polarization of EM fields – revision. Polarization vector. Antenna polarization. Polarization loss factor and polarization efficiency.*)

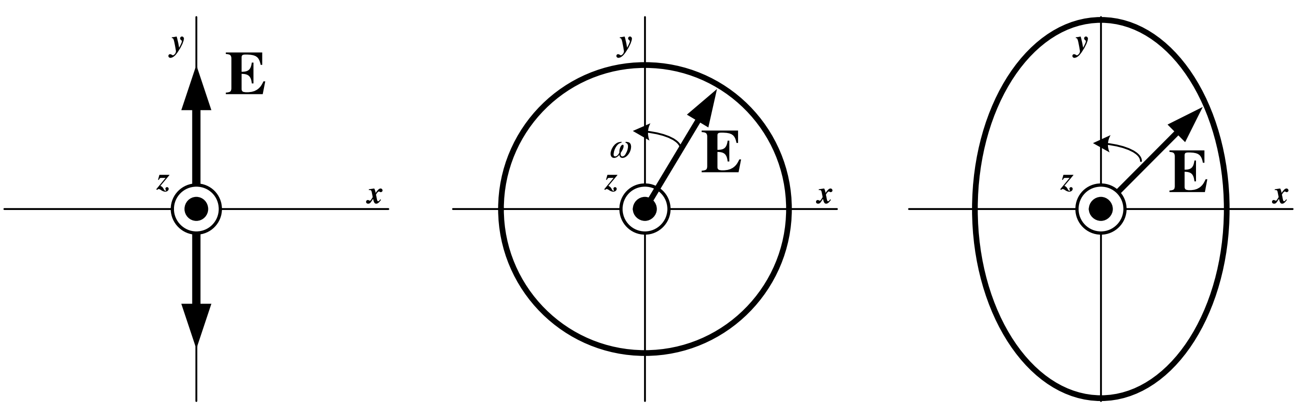
1. Introduction and Definition

The polarization of the EM field describes the orientation of its vectors at a given point and how it varies with time. In other words, it describes the way the direction and magnitude of the field vectors (usually **E**) change in time. Polarization is associated with TEM time-harmonic waves where the **H** vector relates to the **E** vector simply by $\mathbf{H} = \hat{\mathbf{r}} \times \mathbf{E} / \eta$. This is why it suffices to know the polarization of the **E** vector only.

In antenna theory, we are concerned with the polarization of the field in the plane orthogonal to the direction of propagation (the polarization plane)—this is the plane defined by the far-zone vectors **E** and **H**. Remember that the far field is a quasi-TEM field.

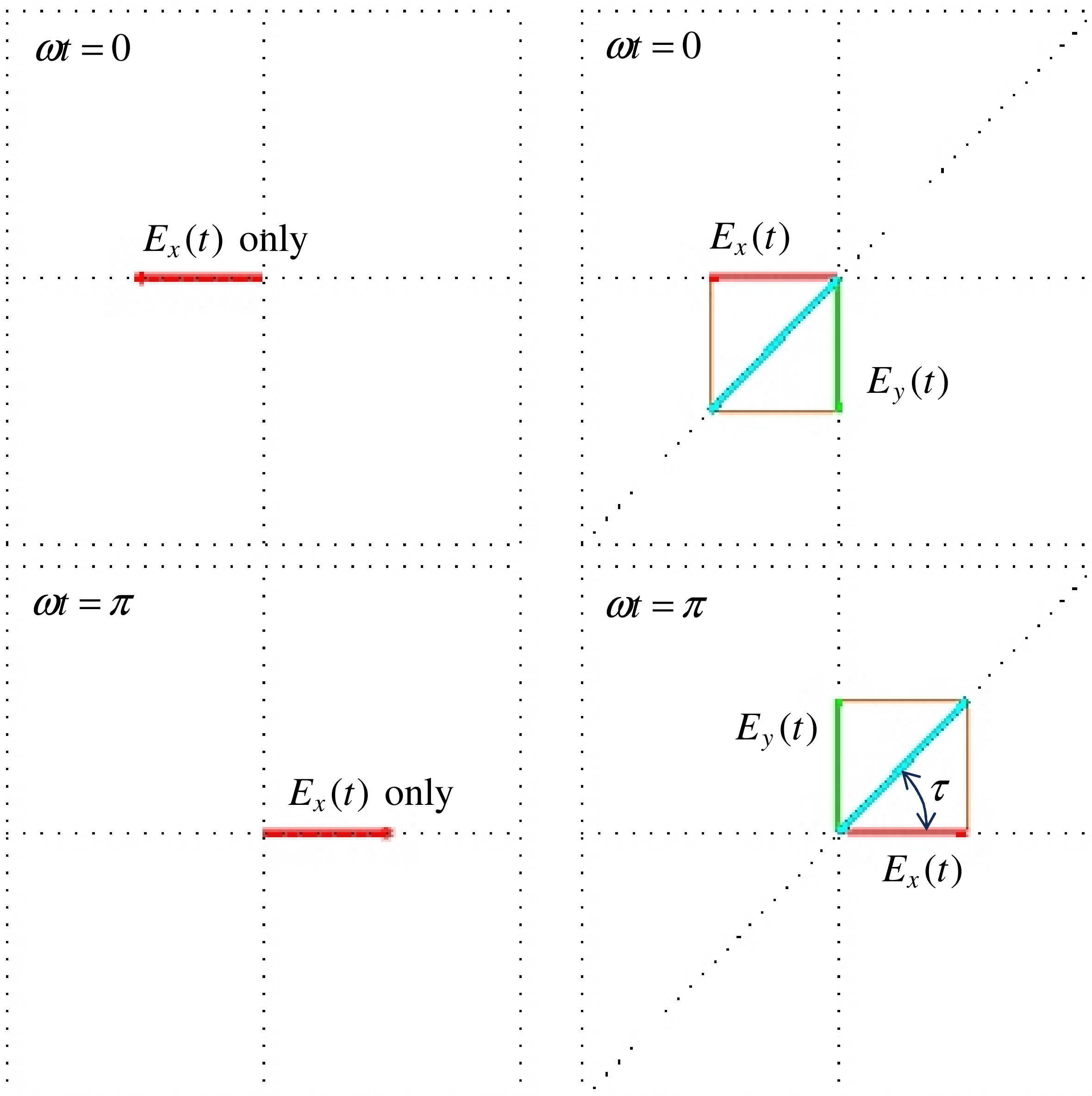
The ***polarization*** is the locus traced by the extremity of the time-varying field vector at a fixed observation point.

According to the shape of the trace, three types of polarization exist for harmonic fields: ***linear***, ***circular*** and ***elliptical***. Any polarization can be represented by two orthogonal linearly polarized fields, ($\hat{\mathbf{x}}E_x, \hat{\mathbf{y}}E_y$) or ($\mathbf{E}_H, \mathbf{E}_V$), which, in general, may have different magnitudes and phases. As far as phases are concerned, what matters is their phase difference $\delta_L = \varphi_y - \varphi_x$.



(a) linear polarization (b) circular polarization (c) elliptical polarization

- If $\delta_L = n\pi$ (n is integer), the field is linearly polarized, i.e., the locus traced by the tip of the $\mathbf{E}$ vector as time flows is a line since
$$\mathbf{E}(t) = (\hat{\mathbf{x}}E_x \pm \hat{\mathbf{y}}E_y)\cos(\omega t). \tag{5.1}$$
The magnitude of the total field is $E_{\mathrm{m}} = \sqrt{E_x^2 + E_y^2}$ whereas the tilt angle of the polarization (relative to the x axis) is $\tau = \pm\arctan(E_y / E_x)$.

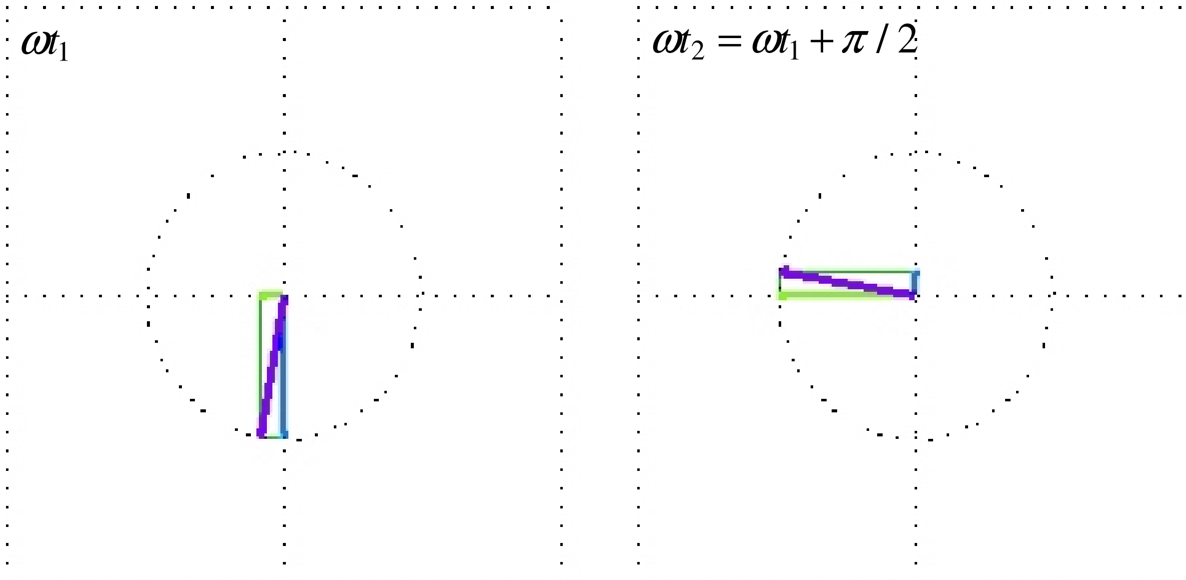


Animation: Linear Polarization, $\delta_L = 0$, $E_x = E_y$

- If $\delta_L = \pi/2$ $(90°)$ and $m_x = m_y$, then the field is circularly polarized.

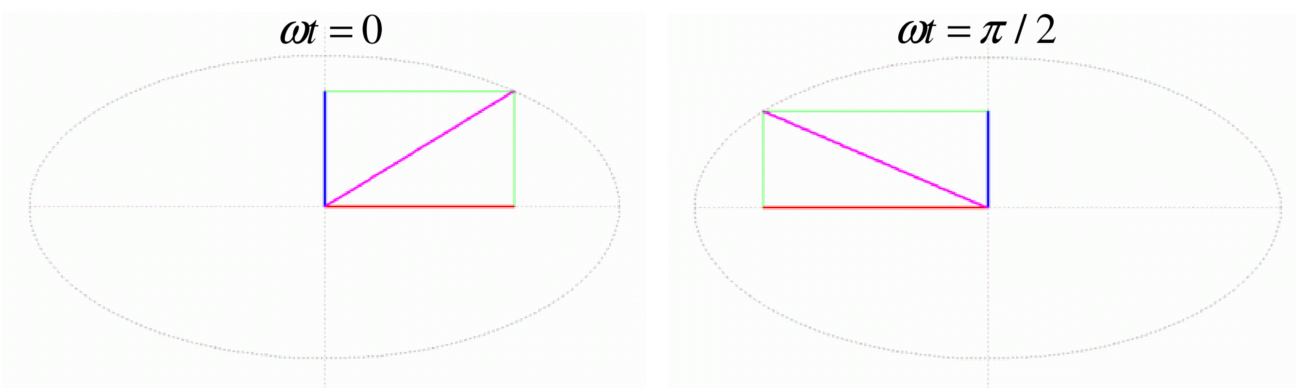
$$\mathbf{E}(t) = m\left[\hat{\mathbf{x}}\cos(\omega t) + \hat{\mathbf{y}}\cos(\omega t \pm \pi/2)\right] = m\left[\hat{\mathbf{x}}\cos(\omega t) \pm \hat{\mathbf{y}}\sin(\omega t)\right] \quad (5.2)$$

Here, m is the magnitude of each field component.



Animation: Clockwise Circular Rotation

- In the most general case, the polarization is elliptical.



Animation: Counter-clockwise Elliptical Rotation

Also, any type of polarization can be represented by a right-hand and a left-hand circularly polarized field components (E_L, E_R), instead of two orthogonal linearly polarized field components. Note that, in a mathematical sense, E_L and E_R are also orthogonal. **[Animation]**

Next, we delve into the mathematical description of the above statements and definitions, and introduce the new concept of polarization vector.

2. Field Polarization in Terms of Two Orthogonal Linearly Polarized Components

The polarization of any field can be represented by the superposition of two orthogonal linearly polarized fields. Assume that a far-field wave propagates along the z-axis. The far-zone field vectors have only transverse components. Then, the set of two orthogonal linearly polarized fields along the x-axis and along the y-axis, is sufficient to represent any TEM$_z$ field. We use this arrangement to introduce the concept of *polarization vector*.

The field (time-dependent or phasor vector) is decomposed into two orthogonal components:

$$\mathbf{e} = \mathbf{e}_x + \mathbf{e}_y \quad \Rightarrow \quad \mathbf{E} = \mathbf{E}_x + \mathbf{E}_y, \tag{5.3}$$

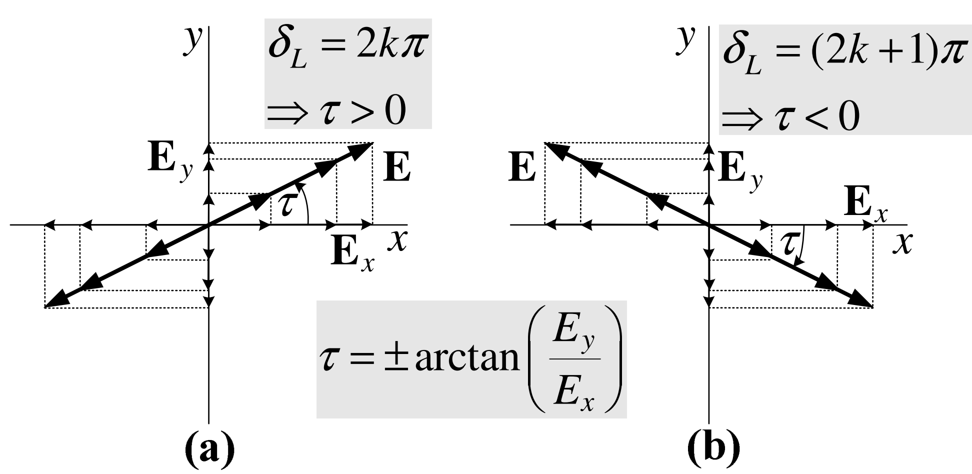
$$\begin{aligned} \mathbf{e}_x &= E_x \cos(\omega t - \beta z)\hat{\mathbf{x}} \\ \mathbf{e}_y &= E_y \cos(\omega t - \beta z + \delta_L)\hat{\mathbf{y}} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{E}_x &= E_x e^{-j\beta z}\hat{\mathbf{x}} \\ \mathbf{E}_y &= E_y e^{-j\beta z} e^{j\delta_L}\hat{\mathbf{y}}. \end{aligned} \tag{5.4}$$

At a fixed position (assume $z = 0$), the equations (5.3) can be written as

$$\begin{aligned} &\mathbf{e}(t) = \hat{\mathbf{x}} \cdot E_x \cos \omega t + \hat{\mathbf{y}} \cdot E_y \cos(\omega t + \delta_L) \\ &\Rightarrow \boxed{\mathbf{E} = \hat{\mathbf{x}} \cdot E_x + \hat{\mathbf{y}} \cdot E_y e^{j\delta_L}} \end{aligned} \tag{5.5}$$

<u>Case 1</u>: Linear polarization: $\delta_L = n\pi, \quad n = 0, 1, 2, \ldots$

$$\begin{aligned} &\mathbf{e}(t) = \hat{\mathbf{x}} \cdot E_x \cos(\omega t) + \hat{\mathbf{y}} \cdot E_y \cos(\omega t \pm n\pi) \\ &\Rightarrow \boxed{\mathbf{E} = \hat{\mathbf{x}} \cdot E_x \pm \hat{\mathbf{y}} \cdot E_y} \end{aligned} \tag{5.6}$$



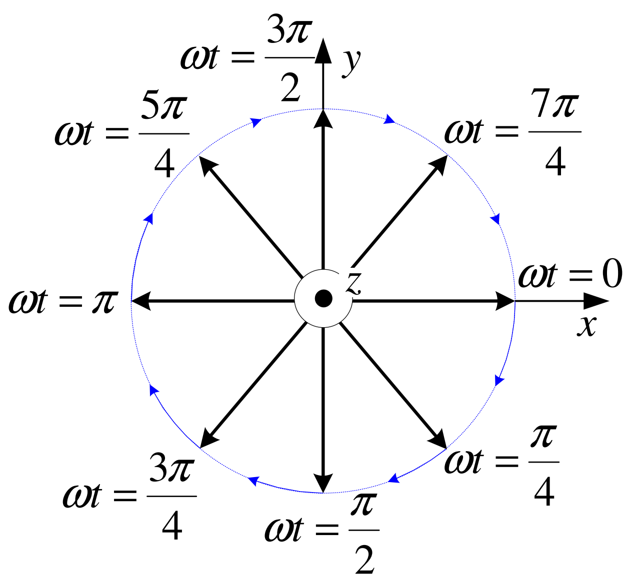
Case 2: Circular polarization:

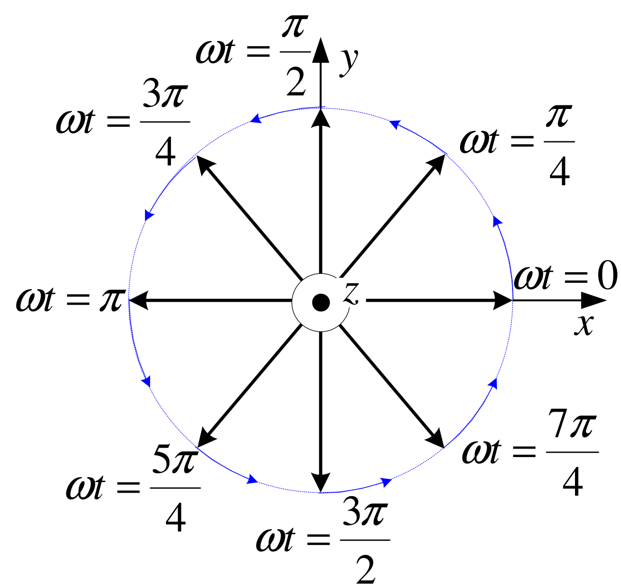
$$E_x = E_y = E_{\mathrm{m}} \quad \text{and} \quad \delta_L = \pm\left(\frac{\pi}{2} + n\pi\right), \quad n = 0,1,2,\ldots$$

$$\mathbf{e}(t) = \hat{\mathbf{x}} E_{\mathrm{m}} \cos(\omega t) + \hat{\mathbf{y}} E_{\mathrm{m}} \cos[\omega t \pm (\pi/2 + n\pi)]$$

$$\Rightarrow \boxed{\mathbf{E} = E_{\mathrm{m}}(\hat{\mathbf{x}} \pm j\hat{\mathbf{y}})} \qquad (5.7)$$

$$\boxed{\begin{array}{l} \mathbf{E} = E_m(\hat{\mathbf{x}} + j\hat{\mathbf{y}}) \\ \delta_L = +\dfrac{\pi}{2} + 2n\pi \end{array}} \qquad \boxed{\begin{array}{l} \mathbf{E} = E_m(\hat{\mathbf{x}} - j\hat{\mathbf{y}}) \\ \delta_L = -\dfrac{\pi}{2} - 2n\pi \end{array}}$$



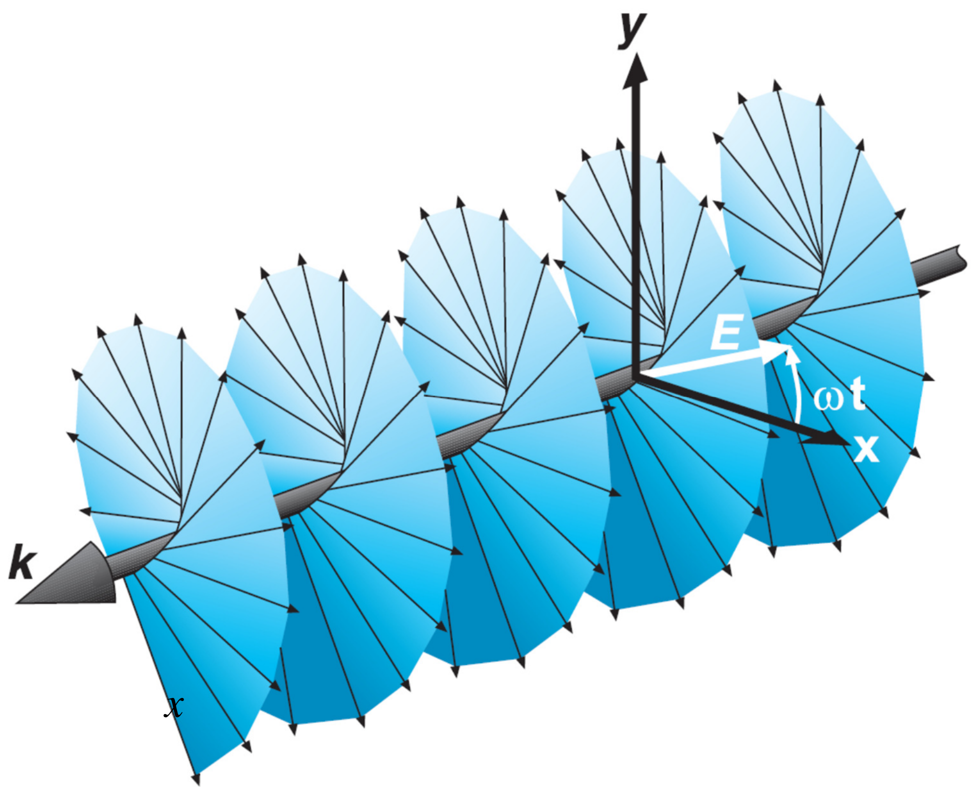


If $+\hat{\mathbf{z}}$ is the direction of propagation: ***counterclockwise (CCW)*** or ***left-hand*** **(LH)** polarization

If $+\hat{\mathbf{z}}$ is the direction of propagation: ***clockwise (CW)*** or ***right-hand*** **(RH)** polarization

Note that the sense of rotation is tied to the direction of propagation. In the example above, if the wave propagates along $-\hat{\mathbf{z}}$, the plot on the left, where $\mathbf{E}(z) = E_{\mathrm{m}}(\hat{\mathbf{x}} + j\hat{\mathbf{y}}) \cdot e^{j\beta z}$, corresponds to a right-hand (RH) wave, while the plot on the right, where $\mathbf{E}(z) = E_{\mathrm{m}}(\hat{\mathbf{x}} - j\hat{\mathbf{y}}) \cdot e^{j\beta z}$, corresponds to a left-hand (LH) wave. *Vice versa*, if the wave propagates along $+\hat{\mathbf{z}}$, then the left plot shows a LH wave, $\mathbf{E}(z) = E_{\mathrm{m}}(\hat{\mathbf{x}} + j\hat{\mathbf{y}}) \cdot e^{-j\beta z}$, whereas the right plot shows a RH wave, $\mathbf{E}(z) = E_{\mathrm{m}}(\hat{\mathbf{x}} - j\hat{\mathbf{y}}) \cdot e^{-j\beta z}$.

A snapshot of the field vector along the axis of propagation is given below for a right-hand circularly polarized (RHCP) wave. Pick an observing position along the axis of propagation (see the plane defined by the x and y axes in the plot below) and imagine that the whole helical trajectory of the tip of the field vector moves along the wave vector $\mathbf{k}$. Are you going to see the vector rotating clockwise or counter-clockwise as you look along $\mathbf{k}$? (Ans.: Clockwise, which is equivalent to RH sense of rotation.)



[Hayt, Buck, Engineering Electromagnetics, 8th ed., p. 399]

<u>Case 3</u>: Elliptic polarization

The tip of the field vector at a given point traces an ellipse as a function of time. This is the most general type of polarization, obtained for any phase difference δ_L and any ratio (E_x / E_y). Mathematically, the linear and the circular polarizations are special cases of the elliptical polarization. In practice, however, the term *elliptical polarization* is used to indicate polarizations *other than linear or circular*.

$$\begin{aligned} &\mathbf{e}(t) = \hat{\mathbf{x}} E_x \cos \omega t + \hat{\mathbf{y}} E_y \cos(\omega t + \delta_L) \\ &\Rightarrow \mathbf{E} = \hat{\mathbf{x}} E_x + \hat{\mathbf{y}} E_y e^{j\delta_L} \end{aligned} \tag{5.8}$$

Show that the trace of the time-dependent vector is an ellipse:

$$e_y(t) = E_y(\cos\omega t \cdot \cos\delta_L - \sin\omega t \cdot \sin\delta_L)$$

$$\cos\omega t = \frac{e_x(t)}{E_x} \text{ and } \sin\omega t = \sqrt{1-\left(\frac{e_x(t)}{E_x}\right)^2}$$

$$\sin^2\delta_L = \left[\frac{e_x(t)}{E_x}\right]^2 - 2\left[\frac{e_x(t)}{E_x}\right]\left[\frac{e_y(t)}{E_y}\right]\cos\delta_L + \left[\frac{e_y(t)}{E_y}\right]^2$$

or (dividing both sides by $\sin^2\delta_L$),

$$\underline{1 = x^2(t) - 2x(t)y(t)\cos\delta_L + y^2(t)}, \tag{5.9}$$

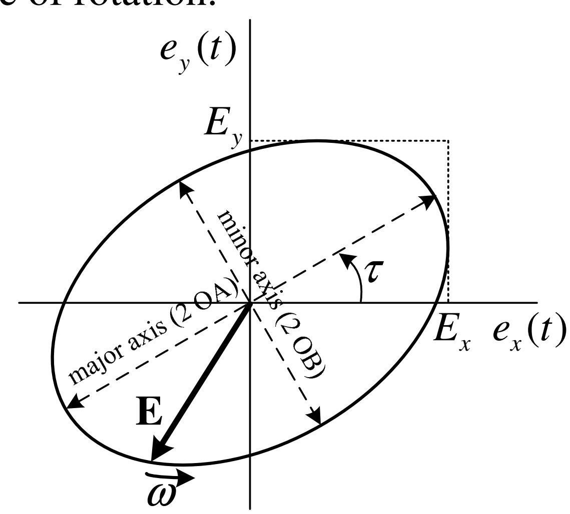
where

$$x(t) = \frac{e_x(t)}{E_x \sin\delta_L} = \frac{\cos\omega t}{\sin\delta_L},$$

$$y(t) = \frac{e_y(t)}{E_y \sin\delta_L} = \frac{\cos(\omega t + \delta_L)}{\sin\delta_L}.$$

Equation (5.9) is the equation of an ellipse centered in the xy plane. It describes the trajectory of a point of coordinates $x(t)$ and $y(t)$ (these are normalized $e_x(t)$ and $e_y(t)$ values) along an ellipse. As time flows, the point moves along the ellipse with an angular frequency ω, i.e., in one period, $T = 2\pi/\omega$, the point completes one elliptical trajectory.

Similarly to the circular polarization, the elliptical polarization can be *right-handed* or *left-handed*, depending on the relation between the direction of propagation and the sense of rotation.



The parameters of the polarization ellipse are given below. Their derivation is given in Appendix I.

a) major axis ($2\times \mathrm{OA}$)

$$\mathrm{OA} = \sqrt{\frac{1}{2}\left[E_x^2 + E_y^2 + \sqrt{E_x^4 + E_y^4 + 2E_x^2 E_y^2 \cos(2\delta_L)}\right]} \tag{5.10}$$

b) minor axis ($2\times \mathrm{OB}$)

$$\mathrm{OB} = \sqrt{\frac{1}{2}\left[E_x^2 + E_y^2 - \sqrt{E_x^4 + E_y^4 + 2E_x^2 E_y^2 \cos(2\delta_L)}\right]} \tag{5.11}$$

c) tilt angle τ

$$\tau = \frac{1}{2}\arctan\underbrace{\left(\frac{2E_x E_y}{E_x^2 - E_y^2}\cos\delta_L\right)}_{A} \pm n\frac{\pi}{2} \tag{5.12}$$

Eq. (5.12) produces an infinite number of angles, $\tau = (\arctan A)/2 \pm n\pi/2$, $n = 1,2,\ldots$. Thus, it gives not only the angle of the major axis of the ellipse with the x axis but also the angle of the minor axis with the x axis. In spherical coordinates, τ is usually specified with respect to the $\hat{\boldsymbol{\theta}}$ direction.

d) axial ratio

$$AR = \frac{\text{major axis}}{\text{minor axis}} = \frac{\mathrm{OA}}{\mathrm{OB}} \tag{5.13}$$

The axial ratio defined in (5.13) attains values between 1 and ∞. The alternative definition of $AR = \mathrm{OB}/\mathrm{OA}$ is also used often. In this case, $0 \le AR \le 1$.

Mathematically speaking, the linear and circular polarizations are special cases of the elliptical polarization:

- If $\delta_L = \pm\left(\frac{\pi}{2} + 2n\pi\right)$ and $E_x = E_y$, then $\mathrm{OA} = \mathrm{OB} = E_x = E_y$, $AR = 1$; the ellipse becomes a circle.
- If $\delta_L = n\pi$, then $\mathrm{OB} = 0$ and $\tau = \pm\arctan(E_y / E_x)$; the ellipse collapses into a line.

3. Field Polarization in Terms of Two Circularly Polarized Components

The representation of a complex vector field as a superposition of circularly polarized components is somewhat less intuitive but it is more useful in the calculation of the polarization ellipse parameters. This time, the total field phasor is represented as the superposition of two circularly polarized waves, one right-handed and the other left-handed. For the case of a wave propagating along $-z$ [see Case 2 and Eq. (5.7)],

$$\mathbf{E} = E_R \underbrace{\frac{(\hat{\mathbf{x}} + j\hat{\mathbf{y}})}{\sqrt{2}}}_{\hat{\mathbf{p}}_R} + E_L \underbrace{\frac{(\hat{\mathbf{x}} - j\hat{\mathbf{y}})}{\sqrt{2}}}_{\hat{\mathbf{p}}_L}. \tag{5.14}$$

Here, E_R and E_L are, in general, complex phasors. Assuming a relative phase difference of $\delta_C = \varphi_R - \varphi_L$, one can write (5.14) as

$$\mathbf{E} = \underbrace{m_R e^{j\delta_C}}_{E_R} (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) / \sqrt{2} + \underbrace{m_L}_{E_L} (\hat{\mathbf{x}} - j\hat{\mathbf{y}}) / \sqrt{2}, \tag{5.15}$$

where m_R and m_L are magnitudes.

The relations between the linear-component and the circular-component representations of the field polarization are easily found as

$$\mathbf{E} = \hat{\mathbf{x}} \underbrace{(E_R + E_L) / \sqrt{2}}_{E_x} + \hat{\mathbf{y}} \underbrace{j(E_R - E_L) / \sqrt{2}}_{E_y} \tag{5.16}$$

$$\Rightarrow \left| \begin{aligned} E_x &= (E_R + E_L)/\sqrt{2} \\ E_y &= j(E_R - E_L) / \sqrt{2} \end{aligned} \right. \tag{5.17}$$

$$\Rightarrow \left| \begin{aligned} E_R &= (E_x - jE_y) / \sqrt{2} \\ E_L &= (E_x + jE_y) / \sqrt{2}. \end{aligned} \right. \tag{5.18}$$

4. Polarization Vector and Polarization Ratio of a Plane Wave

The *polarization vector* is the normalized phasor of the electric field vector. It is a complex-valued vector of unit magnitude, i.e., $\hat{\mathbf{p}}_L \cdot \hat{\mathbf{p}}_L^* = 1$.

$$\boxed{\hat{\mathbf{p}}_L = \frac{\mathbf{E}}{E_\text{m}} = \hat{\mathbf{x}} \frac{E_x}{E_\text{m}} + \hat{\mathbf{y}} \frac{E_y}{E_\text{m}} e^{j\delta_L}, \quad E_\text{m} = \sqrt{E_x^2 + E_y^2}} \tag{5.19}$$

The expression in (5.19) assumes a wave decomposition into linearly polarized (x and y) components, thereby the subscript L. Polarization vector in terms of

RHCP and LHCP components is also used. The polarization vector defined in (5.19) takes the following specific forms in the cases of linearly, circularly and elliptically polarized waves.

<u>Case 1</u>: Linear polarization (the polarization vector is real-valued)

$$\hat{\mathbf{p}} = \hat{\mathbf{x}}\frac{E_x}{E_\mathrm{m}} \pm \hat{\mathbf{y}}\frac{E_y}{E_\mathrm{m}}, \quad E_\mathrm{m} = \sqrt{E_x^2 + E_y^2} \tag{5.20}$$

where E_x and E_y are magnitudes (real-positive).

<u>Case 2</u>: Circular polarization (the polarization vector is complex-valued)

$$\hat{\mathbf{p}}_L = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} \pm j\hat{\mathbf{y}}), \quad E_\mathrm{m} = \sqrt{2}E_x = \sqrt{2}E_y \tag{5.21}$$

The *polarization ratio* is the ratio of the phasors of the two orthogonal polarization components. In general, it is a complex number:

$$\boxed{\tilde{r}_L = r_L e^{\delta_L} = \frac{\tilde{E}_y}{\tilde{E}_x} = \frac{E_y e^{j\delta_L}}{E_x} \text{ or } \tilde{r}_L = \frac{\tilde{E}_V}{\tilde{E}_H}} \tag{5.22}$$

<u>Point of interest</u>: In the case of circular-component representation, the polarization ratio is defined as

$$\boxed{\tilde{r}_C = r_C e^{j\delta_C} = \frac{\tilde{E}_R}{\tilde{E}_L}}. \tag{5.23}$$

The circular polarization ratio $\tilde{r}_C$ is of particular interest since the axial ratio of the polarization ellipse AR can be expressed as

$$\boxed{AR = \left|\frac{r_C + 1}{r_C - 1}\right|}. \tag{5.24}$$

Besides, its tilt angle with respect to the y (vertical) axis is simply

$$\boxed{\tau_V = \delta_C / 2 + n\pi}, \; n = 0, \pm 1, \ldots \,. \tag{5.25}$$

Comparing (5.12) and (5.25) readily shows the relation between the phase difference δ_C of the circular-component representation and the linear polarization ratio $\tilde{r}_L = r_L e^{j\delta_L}$:

$$\delta_C = \arctan\left(\frac{2r_L}{1-r_L^2}\cos\delta_L\right). \tag{5.26}$$

We can calculate the magnitude r_C of the circular polarization ratio from the linear polarization ratio $\tilde{r}_L$ making use of (5.13), (5.24), and (5.26):

$$AR = \left|\frac{r_C+1}{r_C-1}\right| = \sqrt{\frac{1+r_L^2+\sqrt{1+r_L^4+2r_L^2\cos(2\delta_L)}}{1+r_L^2-\sqrt{1+r_L^4+2r_L^2\cos(2\delta_L)}}}. \tag{5.27}$$

Using (5.26) and (5.27) allows for switching between the representation of the wave polarization in terms of linear and circular components.

5. Antenna Polarization

The ***polarization of a transmitting (Tx) antenna*** is the polarization of its radiated wave in the far zone. ***The polarization of a receiving (Rx) antenna*** is the polarization of an incident plane wave, which, for a given power flux density, results in maximum available power at the antenna terminals. Both definitions lead to the same polarization vector, which is the antenna polarization vector $\hat{\mathbf{p}}_a$.

As per the first definition above (Tx mode of operation), the antenna polarization vector $\hat{\mathbf{p}}_a$ is that of the wave it transmits in its own coordinate system (the antenna is at the origin). The second definition (Rx mode of operation) requires that $\hat{\mathbf{p}}_a$ equals the polarization vector of an incident wave $\hat{\mathbf{p}}_w$ such that maximum power is received. This would happen if the dot product of the two unit vectors $\hat{\mathbf{p}}_a$ and $\hat{\mathbf{p}}_w$ attains a maximum value:

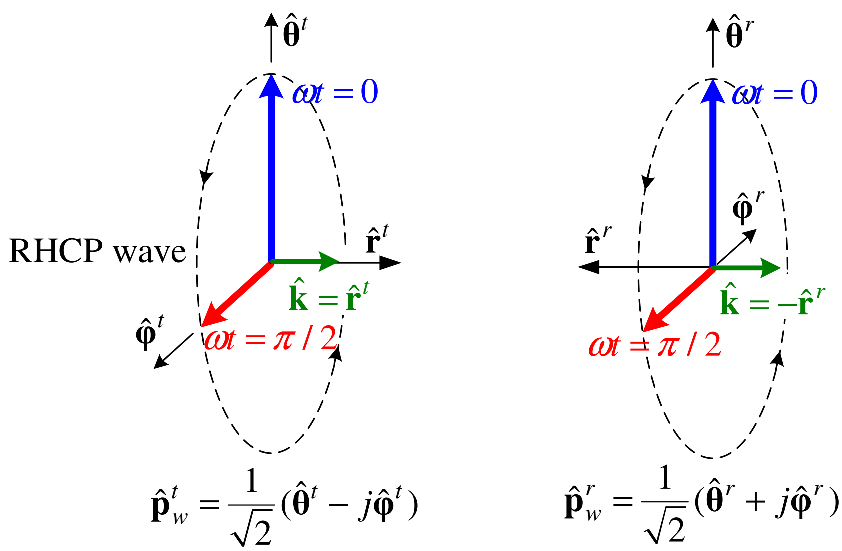
$$\hat{\mathbf{p}}_a \cdot \hat{\mathbf{p}}_w = 1. \tag{5.28}$$

In turn, the condition in (5.28) implies that $\hat{\mathbf{p}}_a$ and $\hat{\mathbf{p}}_w$ must be mutually conjugate, i.e., $\hat{\mathbf{p}}_a = \hat{\mathbf{p}}_w^*$. Next we show that the conjugation of a polarization vector simply implies a change of coordinate system such that if in one coordinate system the wave moves away from the origin, in the other it moves toward the origin.

Consider the polarization vector $\hat{\mathbf{p}}_w^t$ of a wave in the coordinate system of transmission (the wave travels away from the antenna at the origin along $\hat{\mathbf{r}}^t$). Let us now represent this unit vector in the coordinate system of reception, where the wave travels toward the Rx antenna at the origin along $-\hat{\mathbf{r}}^r$. This is illustrated in the figure below with a RHCP wave. The coordinate triplet $(\hat{\mathbf{r}}^t, \hat{\boldsymbol{\theta}}^t, \hat{\boldsymbol{\varphi}}^t)$ represent the coordinate system of the Tx antenna whereas $(\hat{\mathbf{r}}^r, \hat{\boldsymbol{\theta}}^r, \hat{\boldsymbol{\varphi}}^r)$ represents that of the Rx antenna. In antenna analysis, the plane of polarization is usually the plane of $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$ since the far-zone wave travels radially along $\hat{\mathbf{r}}$. Since the Tx and Rx antennas face each other, their coordinate systems are oriented so that $\hat{\mathbf{r}}^r = -\hat{\mathbf{r}}^t$. If we align the axes $\hat{\boldsymbol{\theta}}^t$ and $\hat{\boldsymbol{\theta}}^r$, then $\hat{\boldsymbol{\varphi}}^r = -\hat{\boldsymbol{\varphi}}^t$ must hold. This changes the sign of the respective (2^{nd}) polarization-vector component, which results in its conjugation. Thus, we conclude that ***the transmitted-wave polarization vector must be conjugated when represented in the coordinate system of the Rx antenna***, i.e.,

$$\hat{\mathbf{p}}_w^t = (\hat{\mathbf{p}}_w^r)^* . \tag{5.29}$$

In the context of the Rx antenna polarization-vector definition, we need the incident-wave polarization vector in the Rx coordinate system, $\hat{\mathbf{p}}_w \equiv \hat{\mathbf{p}}_w^r$, to satisfy $\hat{\mathbf{p}}_w = \hat{\mathbf{p}}_a^*$. Thus, $\hat{\mathbf{p}}_a$ is nothing but the polarization vector of the Rx antenna if it were to transmit. This finally proves that both definitions (for the Tx and the Rx mode of antenna operation) are mathematically equivalent.



6. Polarization Loss Factor (Polarization Efficiency)

Generally, the polarization of the Rx antenna is not the same as the polarization of the incident wave. This is called ***polarization mismatch***. The polarization loss factor (PLF) characterizes the loss of EM power due to the polarization mismatch. The PLF is defined so that it attains a value of 1 (or 100%, or 0 dB) if there is no polarization mismatch, i.e., the antenna receives the maximum possible power for the given incident power density of the wave. A PLF equal to 0 ($-\infty$ dB) indicates complete polarization mismatch and inability to capture power from the incident wave. Thus,

$$0 \le \text{PLF} \le 1. \tag{5.30}$$

Note that the polarization loss has nothing to do with dissipation. It can be viewed as a "missed opportunity" to capture as much power from the incident wave as possible. The ***polarization efficiency*** has the same meaning as PLF.

Let us denote the polarization vector of a wave incident upon a Rx antenna as $\hat{\mathbf{p}}_w$. The unit vector $\hat{\mathbf{p}}_w$ is defined in the coordinate system of the Rx antenna, where the Rx antenna is at the origin; thus, $\hat{\mathbf{p}}_w$ describes a wave propagating along $-\hat{\mathbf{r}}$. On the other hand, the polarization vector of the Rx antenna $\hat{\mathbf{p}}_a$ is the polarization vector of its far field if it were to transmit. Thus, $\hat{\mathbf{p}}_a$ describes a wave propagating along $+\hat{\mathbf{r}}$. The PLF is a power-loss quantity; therefore, it is defined as

$$\text{PLF} = |\hat{\mathbf{p}}_w \cdot \hat{\mathbf{p}}_a|^2. \tag{5.31}$$

Maximum PLF of 1 (no polarization loss, maximum received power) is achieved when $\hat{\mathbf{p}}_w^* = \hat{\mathbf{p}}_a$. *Thus, the optimal polarization of the incident wave is the one that matches the polarization of the wave produced by the Rx antenna if it were to replace the Tx one.*

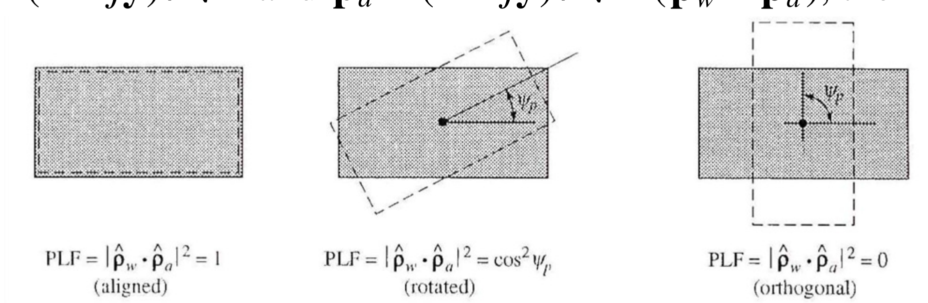
Here are some simple examples:

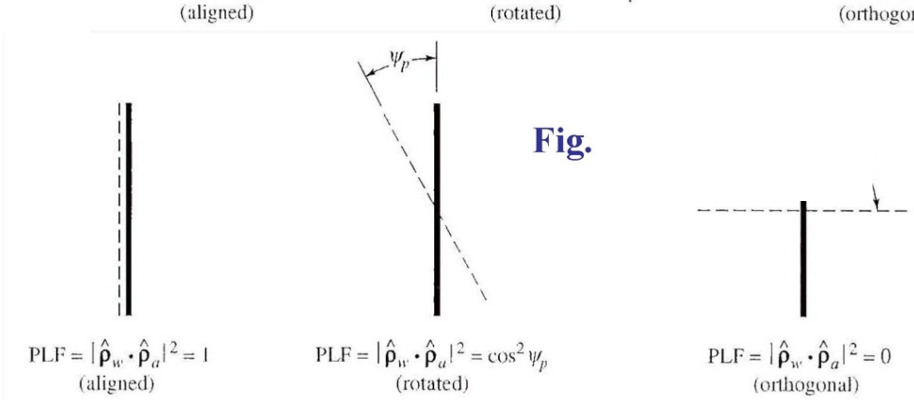
1) if $\hat{\mathbf{p}}_w = \hat{\mathbf{p}}_a^* = \hat{\mathbf{p}}_a = \hat{\mathbf{x}}$, then PLF=1;

2) if $\hat{\mathbf{p}}_w = \hat{\mathbf{x}}$ and $\hat{\mathbf{p}}_a = \hat{\mathbf{p}}_a^* = \hat{\mathbf{y}}$, then PLF=0;

3) if $\hat{\mathbf{p}}_w = \hat{\mathbf{p}}_a = (\hat{\mathbf{x}} + j\hat{\mathbf{y}})/\sqrt{2}$, then PLF=0;

4) if $\hat{\mathbf{p}}_w = (\hat{\mathbf{x}} + j\hat{\mathbf{y}})/\sqrt{2}$ and $\hat{\mathbf{p}}_a = (\hat{\mathbf{x}} - j\hat{\mathbf{y}})/\sqrt{2}$ ($\hat{\mathbf{p}}_w^* = \hat{\mathbf{p}}_a$), then PLF=1.





[Balanis, 2nd ed.]

In a communication link, the PLF has to be expressed by the polarization vectors of the transmitting and receiving antennas, $\hat{\mathbf{p}}_{\mathrm{Tx}}$ and $\hat{\mathbf{p}}_{\mathrm{Rx}}$, respectively. Both of these are defined in the coordinate systems of their respective antennas as the polarization of the transmitted wave. However, these two coordinate systems have their radial unit vectors pointing in opposite directions, i.e., $\hat{\mathbf{r}}_{\mathrm{Rx}} = -\hat{\mathbf{r}}_{\mathrm{Tx}}$ as illustrated in the figure below. Therefore, either $\hat{\mathbf{p}}_{\mathrm{Tx}}$ or $\hat{\mathbf{p}}_{\mathrm{Rx}}$ has to be conjugated when calculating the PLF (it does not matter which one). For example, if the reference coordinate system is that of the Rx antenna, then

$$\mathrm{PLF} = \left| \hat{\mathbf{p}}_{\mathrm{Tx}}^{*} \cdot \hat{\mathbf{p}}_{\mathrm{Rx}} \right|^2 . \tag{5.32}$$

The expression $\mathrm{PLF} = \left| \hat{\mathbf{p}}_{\mathrm{Tx}} \cdot \hat{\mathbf{p}}_{\mathrm{Rx}}^{*} \right|^2$ is also correct.

Examples

Example 5.1. The electric field of a linearly polarized EM wave is

$$\mathbf{E}^{i}(x,y,z) = \hat{\mathbf{x}} E_m(x,y) e^{-j\beta z} .$$

It is incident upon a linearly polarized receiving antenna, which, if in transmitting mode, would generate the field

$$\mathbf{E}_a(x,y,z) = (\hat{\mathbf{x}} + \hat{\mathbf{y}}) \cdot e^{j\beta z} .$$

Find the PLF.

Notice that $\mathbf{E}_a$ propagates along $-z$ in accordance with the requirement that it represents a transmitted wave. Its polarization vector is $\hat{\mathbf{p}}_a = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. On the other hand, the polarization vector of the incident wave is $\hat{\mathbf{p}}^{i} = \hat{\mathbf{x}}$. Both polarization vectors are real-valued. The PLF is then

$$\mathrm{PLF} = \left| \hat{\mathbf{x}} \cdot \frac{1}{\sqrt{2}} (\hat{\mathbf{x}} + \hat{\mathbf{y}}) \right|^2 = \frac{1}{2}$$

$$\mathrm{PLF}_{[\mathrm{dB}]} = 10 \log_{10} 0.5 = -3 \text{ dB}$$

Example 5.2. A transmitting antenna produces a far-zone field, which is RH circularly polarized. This field impinges upon a receiving antenna, whose polarization (in transmitting mode) is also RH circular. Determine the PLF.

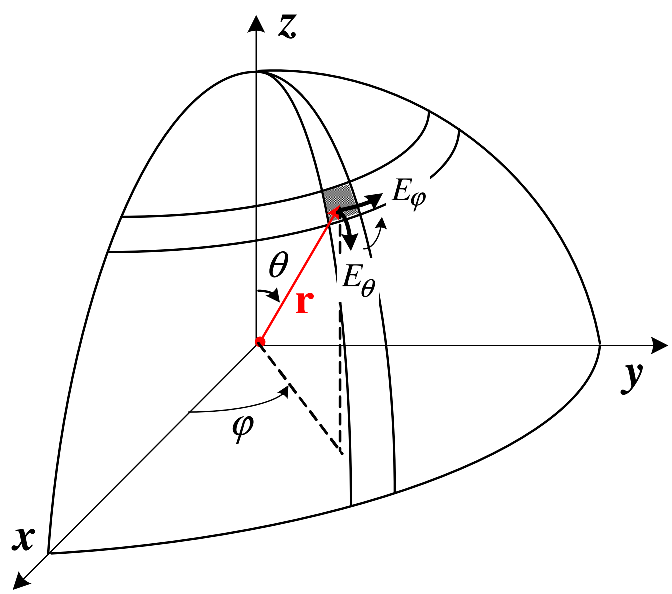
Both antennas (the transmitting one and the receiving one) are RH circularly polarized in transmitting mode. Assume that a transmitting antenna is located at the center of a spherical coordinate system. The far-zone field it would produce is described as

$$\mathbf{E}^{far} = E_m \left[\hat{\boldsymbol{\theta}} \cos \omega t + \hat{\boldsymbol{\varphi}} \cos(\omega t - \pi / 2) \right].$$

This is a RHCP field with respect to the outward radial direction $\hat{\mathbf{r}}$. Its polarization vector is

$$\hat{\mathbf{p}}_{\mathrm{Tx}} = \frac{\hat{\boldsymbol{\theta}} - j\hat{\boldsymbol{\varphi}}}{\sqrt{2}}.$$

This is exactly the polarization vector of the transmitting antenna in its own coordinate system.



Since the receiving antenna is also RHCP, its polarization vector (in its own coordinate system) is

$$\hat{\mathbf{p}}_{\mathrm{Rx}} = (\hat{\boldsymbol{\theta}} - j\hat{\boldsymbol{\varphi}}) / \sqrt{2}.$$

The PLF is calculated as per (5.32):

$$\text{PLF} = |\hat{\mathbf{p}}_{\text{Tx}}^* \cdot \hat{\mathbf{p}}_{\text{Rx}}|^2 = \frac{|(\hat{\boldsymbol{\theta}} + j\hat{\boldsymbol{\varphi}}) \cdot (\hat{\boldsymbol{\theta}} - j\hat{\boldsymbol{\varphi}})|^2}{4} = 1,$$

$$\text{PLF}_{[\text{dB}]} = 10\log_{10} 1 = 0.$$

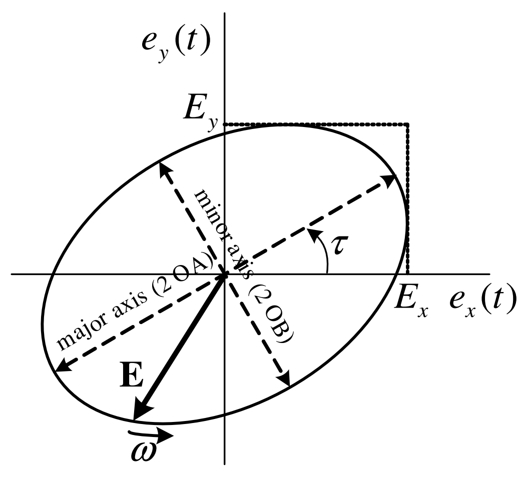
There is no polarization loss. When transmitting with an RHCP antenna in a communication link, it is best to receive with RHCP antenna.

Exercise: Show that an antenna of RH circular polarization (in transmitting mode) cannot receive LH circularly polarized incident wave (or a wave emitted by a left-circularly polarized antenna), i.e., PLF = 0.

Appendix I

Find the tilt angle τ, the length of the major axis OA, and the length of the minor axis OB of the ellipse described by the equation:

$$\sin^2\delta = \left[\frac{e_x(t)}{E_x}\right]^2 - 2\left[\frac{e_x(t)}{E_x}\right]\left[\frac{e_y(t)}{E_y}\right]\cos\delta + \left[\frac{e_y(t)}{E_y}\right]^2. \tag{A-1}$$



Equation (A-1) can be written as

$$a \cdot x^2 - b \cdot xy + c \cdot y^2 = 1, \tag{A-2}$$

where

$x = e_x(t)$ and $y = e_y(t)$ are the coordinates of a point of the ellipse centered in the xy plane;

$$a=\frac{1}{E_x^2\sin^2\delta};$$
$$b=\frac{2\cos\delta}{E_xE_y\sin^2\delta};$$
$$c=\frac{1}{E_y^2\sin^2\delta}.$$

After dividing both sides of (A-2) by (xy), one obtains

$$a\frac{x}{y}-b+c\frac{y}{x}=\frac{1}{xy}. \tag{A-3}$$

Introducing $\xi=\frac{y}{x}=\frac{e_y(t)}{e_x(t)}$, one obtains that

$$x^2=\frac{1}{c\xi^2-b\xi+a}$$

$$\Rightarrow \rho^2(\xi)=x^2+y^2=x^2(1+\xi^2)=\frac{1+\xi^2}{c\xi^2-b\xi+a}. \tag{A-4}$$

Here, ρ is the distance from the center of the coordinate system to the point on the ellipse. We want to know at what values of ξ the maximum and the minimum of ρ occur ($\xi_{\min}$, $\xi_{\max}$). This will produce the tilt angle τ. We also want to know the values of $\rho_{\max}$ (major axis) and $\rho_{\min}$ (minor axis). Then, we have to solve

$$\frac{d(\rho^2)}{d\xi}=0\text{, or}$$

$$\xi_{\mathrm{m}}^2-\frac{2(a-c)}{b}\xi_{\mathrm{m}}-1=0\text{, where } \xi_{\mathrm{m}}\equiv\xi_{\min},\xi_{\max}. \tag{A-5}$$

Solving (A-5) will produce the tilt angle τ, which relates to $\xi_{\max}$ as

$$\xi_{\max}=\tan\tau=\left(y/x\right)_{\max}. \tag{A-6}$$

Substituting (A-6) in (A-5) yields:

$$\left(\frac{\sin\tau}{\cos\tau}\right)^2-2C\left(\frac{\sin\tau}{\cos\tau}\right)-1=0 \tag{A-7}$$

where

$$C = \frac{a-c}{b} = \frac{E_y^2 - E_x^2}{2E_x E_y \cos\delta}.$$

Multiplying both sides of (A-7) by $\cos^2\tau$ and re-arranging results in

$$\underbrace{\cos^2\tau - \sin^2\tau}_{\cos(2\tau)} + \underbrace{2C\sin\tau\cdot\cos\tau}_{C\sin(2\tau)} = 0.$$

Thus, the solution of (A-7) is

$$\tan(2\tau) = -1/C$$

or

$$\tau_1 = \frac{1}{2}\arctan\left(\frac{2E_x E_y \cos\delta}{E_x^2 - E_y^2}\right); \quad \tau_2 = \tau_1 + \frac{\pi}{2}. \tag{A-8}$$

The angles τ_1 and τ_2 are the angles between the major and minor axes with the x axis, respectively. Substituting τ_1 and τ_2 back in ρ (see A-4) yields the expressions for OA and OB.