Lecture 7: Antenna Noise Temperature and System Signal-to-Noise Ratio

(Noise temperature. Antenna noise temperature. System noise temperature. Minimum detectable temperature. System signal-to-noise ratio.)

1. Noise Temperature of Bright Bodies

The performance of a telecommunication system depends on the signal-to-noise ratio (SNR) at the receiver's input. The electronic circuitry of the RF front end (amplifiers, mixers, etc.) has a significant contribution to the system noise. However, the antenna itself is sometimes a significant source of noise, too. The antenna noise can be divided into two types according to its physical source: noise due to the loss resistance of the antenna and noise, which the antenna picks up from the surrounding environment.

Any object whose temperature is above the absolute zero radiates electromagnetic (EM) energy. Thus, an antenna is surrounded by noise sources, which create noise power at the antenna terminals. Here, we are not concerned with man-made sources of noise, which are the subject of the EM interference (EMI) science. We are also not concerned with intentional sources of EMI (jamming). We are concerned with natural sources of EM noise, which is thermal in nature, such as *sky noise* and *ground noise*.

The concept of antenna noise temperature is critical in understanding how the antenna contributes to the system noise in low-noise receiving systems such as in radioastronomy and radiometry. It is also important in understanding the relation between an object's temperature and the noise power it generates at the receiving antenna terminals. This thermal power is the signal used in passive remote sensing (radiometry) and thermal imaging. The low-noise receiver for thermal-noise signals is the *radiometer*. Typically, the remote object's temperature is measured by comparison with the noise due to background sources and the receiver itself.

Every object (e.g., a resistor R) with a physical temperature above zero (0°K = −273 °C) emits heat energy. The ***noise power per hertz*** p_h (also known as ***noise power spectral density***) is given by Nyquist's relation:[1]

$$p_h = kT_P, \text{ W/Hz} \tag{7.1}$$

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[1] See Appendix for more detailed discussion on thermal noise power.

where T_P is the physical temperature of the object in K (Kelvin degrees) and k is Boltzmann's constant ($k \approx 1.38\times10^{-23}$ J/K).

In the case of a resistor, this is the noise power, which can be measured at the resistor's terminals with a matched load. Thus, a resistor can serve as a noise generator. Often, we assume that heat energy is evenly distributed in the frequency band Δf. Then, the associated thermal noise power within Δf is

$$P_{\mathrm{N}} = kT_P\Delta f \text{ , W.} \tag{7.2}$$

The noise power radiated by the object depends not only on its physical temperature but also on the ability of its surface to let the heat leak out. This radiated heat power (or brightness power P_B) is associated with the so-called ***equivalent temperature*** or ***brightness temperature*** T_B of the body via the power-temperature relation in (7.2):

$$P_B = kT_B\Delta f \text{ , W.} \tag{7.3}$$

In general, the brightness temperature T_B is not the same as the physical temperature of the body T_P. The two temperatures are proportional:

$$T_B = (1-|\Gamma_s|^2)\cdot T_P = \varepsilon T_P, \text{ K} \tag{7.4}$$

where Γ_s is the reflection coefficient at the surface of the body and ε is what is called the ***emissivity*** of the body. The brightness power P_B relates to the thermal-noise (or heat) power P_{N} the same way as T_B relates to T_P, i.e., $P_B = \varepsilon P_{\mathrm{N}}$.

2. Antenna Noise Temperature

The power radiated by a bright body P_B, when intercepted by an antenna, generates noise power P_A at its terminals. The equivalent temperature associated with the received power P_A at the antenna terminals is called the ***antenna temperature*** T_A of the object, where, again, $P_A = kT_A\Delta f$. Here, Δf is a bandwidth, which falls within the antenna bandwidth and is sufficiently narrow to ensure constant noise-power spectral density. We assume first that the antenna is loss-free. We will include these losses later.

To understand how the antenna temperature relates to that of bright bodies, we first need to understand how the antenna "collects" the incident power of bright bodies to form the total received power P_A at the antenna terminals. The power received by an antenna depends on its effective aperture $A_e(\theta,\varphi)$ for an incident wave arriving from the (θ,φ) direction. Assume that the power-flux density from this direction is $W_B(\theta,\varphi)$ (W/m^2). This power-flux density is due to a distant point-like bright body of cross-section dS_B, which subtends the differential solid angle $d\Omega$. Then, the power received from this direction is

$$P_{\text{Rx}}(\theta,\varphi) = A_e(\theta,\varphi) \cdot W_B(\theta,\varphi), \text{ W}. \tag{7.5}$$

Assuming that the bright body at (θ,φ) radiates isotropically a total power of $P_B(\theta,\varphi)$, and expressing the effective area by the directivity, we obtain

$$P_{\text{Rx}}(\theta,\varphi) = \frac{\lambda^2}{4\pi} D(\theta,\varphi) \cdot \frac{P_B(\theta,\varphi)}{4\pi R^2}, \text{ W}. \tag{7.6}$$

Here, R is the distance between the bright body and the antenna. In turn, the directivity is expressed in terms of the antenna solid angle Ω_A as

$$D(\theta,\varphi) = 4\pi \frac{\bar{U}(\theta,\varphi)}{\Omega_A}, \tag{7.7}$$

leading to

$$P_{\text{Rx}}(\theta,\varphi) = \lambda^2 \frac{\bar{U}(\theta,\varphi)}{\Omega_A} \cdot \frac{P_B(\theta,\varphi)}{4\pi R^2}. \tag{7.8}$$

Here, $\bar{U}(\theta,\varphi)$ is the normalized power radiation pattern of the antenna. The distance R between the bright body and the antenna relates to the cross-section of the body dS_B and the solid angle it subtends $d\Omega$ as

$$R^2 = dS_B / d\Omega, \text{ m}^2. \tag{7.9}$$

Substituting (7.9) into (7.8) yields

$$P_{\text{Rx}}(\theta,\varphi) = \lambda^2 \frac{\bar{U}(\theta,\varphi)}{\Omega_A} \frac{P_B(\theta,\varphi)}{4\pi dS_B} d\Omega. \tag{7.10}$$

Next, we view the bright body as a transmitting antenna of gain $G_B = 1$ (isotropic radiator) and we assume that its cross-section dS_B is a good representation of its effective area. Employing the relation between antenna effective area and its gain [see (4.65) in Lecture 4], we make the substitution

$$\frac{\lambda^2}{4\pi dS_B} = \frac{1}{G_B} = 1 \tag{7.11}$$

in (7.10). This leads to

$$P_{\text{Rx}}(\theta,\varphi) = \frac{\bar{U}(\theta,\varphi) P_B(\theta,\varphi)}{\Omega_A} d\Omega. \tag{7.12}$$

In general, an antenna receives thermal noise power simultaneously from all directions. Therefore, the total noise power P_A at its terminals is a superposition of the weighted power contributions from all directions:

$$P_A = \oiint_{4\pi} P_{\text{Rx}}(\theta,\varphi) = \frac{1}{\Omega_A} \oiint_{4\pi} \bar{U}(\theta,\varphi)\, P_B(\theta,\varphi)\, d\Omega . \qquad (7.13)$$

Thus, the antenna normalized power pattern acts as a weighting factor in the superposition integral (7.13). Analogously, the antenna temperature T_A is expressed in terms of the brightness temperature T_B of all bright bodies weighted by the radiation pattern as

$$T_A = \frac{1}{\Omega_A} \oiint_{4\pi} \bar{U}(\theta,\varphi) T_B(\theta,\varphi)\, d\Omega . \qquad (7.14)$$

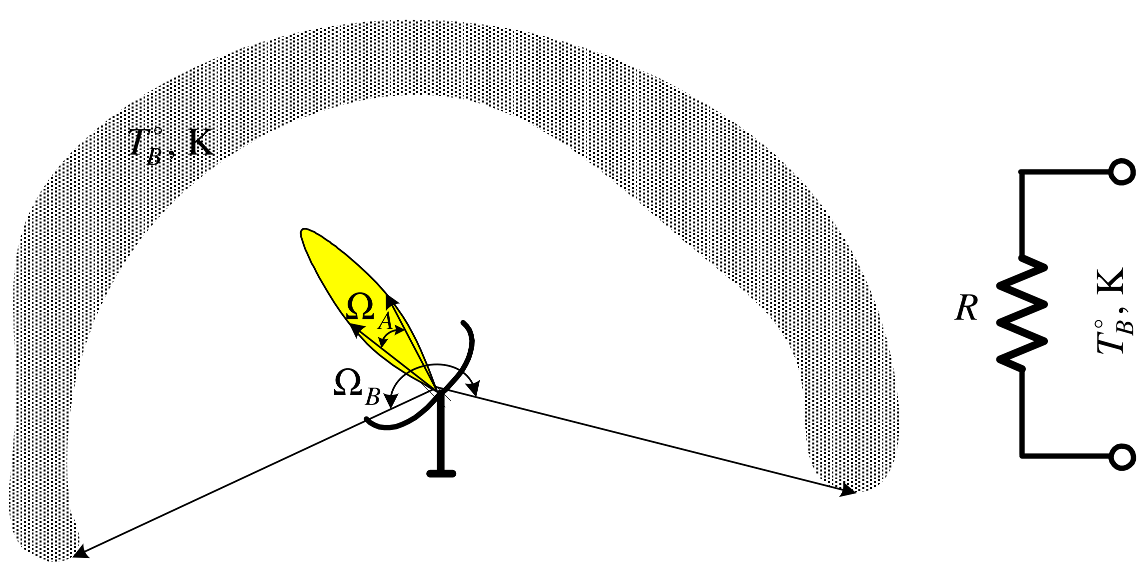
2.1. Antenna noise temperature due to uniform bright background

Let us first assume that the entire antenna pattern "sees" a uniformly "bright" object of brightness temperature $T_B(\theta,\varphi) = T_B = const.$, which surrounds the antenna from all directions (see the illustration below). We again assume that the antenna is loss-free, i.e., it does not generate noise itself. Then, as per (7.14), the noise power measured at its terminals is

$$T_A = \frac{T_B}{\Omega_A} \underbrace{\oiint_{4\pi} \bar{U}(\theta,\varphi)\, d\Omega}_{\Omega_A} = T_B \;\Rightarrow\; T_A = T_B . \qquad (7.15)$$

Equivalently, the antenna noise power is simply:

$$P_A = kT_A\Delta f = kT_B\Delta f \,. \qquad (7.16)$$



2.2. Antenna incremental temperature for large bright bodies

The situation described in Section 2.1 above is of practical importance. When an antenna is pointed right at the night sky, i.e., all of its pattern is occupied by the night sky, its noise temperature is very low: $T_A = 3^\circ$ to 5° K at frequencies between 1 and 10 GHz. This is the microwave noise temperature of the night sky. The higher the elevation angle, the lower the night-sky temperature because of the lower physical temperature of the atmosphere toward zenith. The sky noise depends on the frequency. It depends on the time of the day, too. Closer to the horizon, it is mostly due to the thermal radiation from the Earth's surface and the atmosphere. Closer to the zenith, it is mostly due to cosmic rays from the sun, the moon and other bright sky objects, as well as the deep-space background temperature commonly referred to as the *cosmic microwave background* ($T_{\text{CMB}} \approx 2.725^\circ$ K).[2] The latter is a left-over thermal effect from the very origin of the universe (the *big bang*).

An antenna may also be pointed toward the ground, e.g., when mounted on an airplane or a satellite. The noise temperature of the ground is much higher than that of the night sky because of its higher physical temperature. The ground noise temperature is about 300° K and it varies during the day. The noise temperature at zero elevation (horizon) is about 100° to 150° K.

The discussion above makes it clear that an antenna always receives noise power due to background thermal radiation. Thus, when it is pointed at a single large bright object of interest, in addition to this object's brightness temperature, the antenna temperature includes contributions from the background.

Now, consider a bright body, which is large enough to subtend a solid angle larger than the solid angle of the antenna's main beam, $\Omega_B > \Omega_{\text{MB}}$.[3] To discern this bright body in the thermal-noise background, it has to put out sufficient power to "stand out". To obtain its brightness temperature T_B (assumed constant within the main beam), the antenna temperature is acquired with the beam on and off the target. The difference is the ***antenna incremental temperature*** ΔT_A. In the absence of the bright body, the antenna measures only the background noise:

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[2] C.T. Stelzried, A.J. Freiley, and M.S. Reid, *Low-noise Receiving Systems*. Artech, 2010.

[3] Remember the definition of the main beam solid angle: $\Omega_{\text{MB}} = \int_0^{2\pi}\int_0^{\theta_1} \bar{U}(\theta,\varphi)\sin\theta d\theta d\varphi$, where θ_1 is the first-null beam width. In contrast, the antenna solid angle is $\Omega_A = \int_0^{2\pi}\int_0^{\pi} \bar{U}(\theta,\varphi)\sin\theta d\theta d\varphi$.

$$T_A^{\text{bg}} = \frac{1}{\Omega_A} \oiint_{4\pi} \bar{U}(\theta,\varphi) T_{\text{bg}}(\theta,\varphi)\, d\Omega. \tag{7.17}$$

When the main beam of the antenna is pointed at the bright body, its temperature includes the bright body's contribution ΔT_A:

$$T_A^{\text{bg}} + \Delta T_A = \frac{1}{\Omega_A} \oiint_{4\pi} \bar{U}(\theta,\varphi) T_{\text{bg}}(\theta,\varphi)\, d\Omega + \frac{1}{\Omega_A} \iint_{\Omega_{\text{MB}}} \underbrace{\bar{U}(\theta_0,\varphi_0)}_{1} T_B\, d\Omega. \tag{7.18}$$

Here, (θ_0, φ_0) is the antenna boresight where the pattern value is 1. Thus,

$$\Delta T_A = \frac{\Omega_{\text{MB}}}{\Omega_A} T_B,\ \Omega_B > \Omega_{\text{MB}}. \tag{7.19}$$

High-gain low-sidelobe antennas feature $\Omega_{\text{MB}} \approx \Omega_A$, in which case:

$$\Delta T_A = T_B,\ \Omega_B > \Omega_A. \tag{7.20}$$

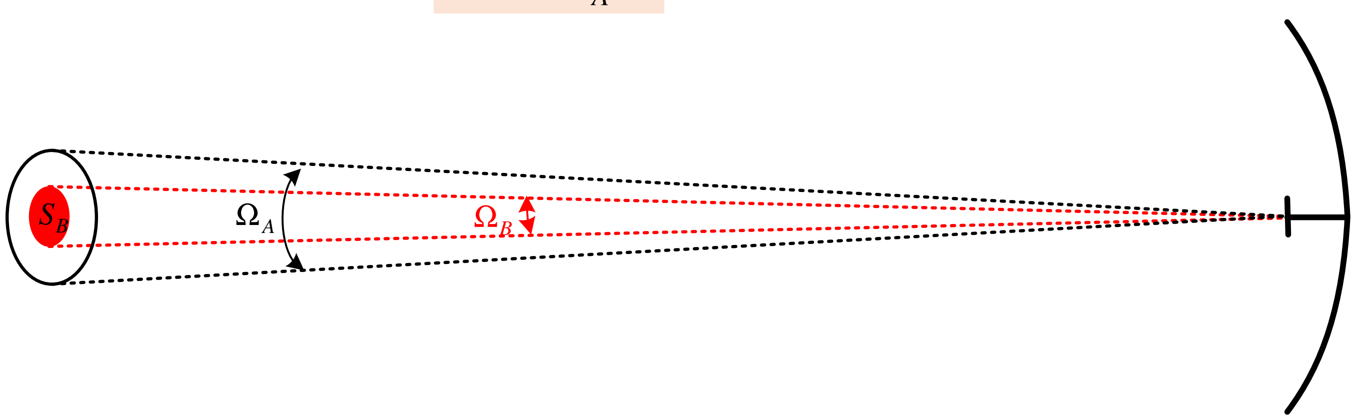
2.3. Antenna incremental temperature for small bright bodies

Now consider a bright object, which subtends a solid angle Ω_B that is smaller than the main-beam solid angle: $\Omega_B < \Omega_{\text{MB}}$ (see illustration below). Again, to separate the power received from the bright body from the background, the incremental antenna temperature ΔT_A is measured with the beam on and off the object. This time, ΔT_A is ***not equal*** to the bright body temperature T_B because:

$$\Delta T_A = \frac{1}{\Omega_A} \iint_{\Omega_B} \underbrace{\bar{U}(\theta_0,\varphi_0)}_{1} T_B\, d\Omega = T_B \frac{\Omega_B}{\Omega_A}. \tag{7.21}$$

Thus, the antenna incremental temperature ΔT_A is proportional to the brightness temperature of the small bright body:

$$\Delta T_A = \frac{\Omega_B}{\Omega_A} T_B \text{ K},\ \Omega_B \ll \Omega_A. \tag{7.22}$$



2.4. Source flux density from noise sources

In radioastronomy and remote sensing, a bright body is also characterized by the ***noise-source flux density S′*** it creates at the antenna aperture. S' is power flux per unit area (the Poynting vector strength) per *hertz*. If the received noise power spectral density is p_h (in W/Hz) and the antenna effective area is A_e, then

$$S' = \frac{p_h}{A_e} = \frac{k\Delta T_A}{A_e},\ \mathrm{Wm^{-2}Hz^{-1}}. \tag{7.23}$$

Note that, for a given radiation source, the received power density p_h at the antenna terminals is proportional to its effective area A_e. Thus, just like T_B, the source flux density is a metric for the source strength, not the antenna used to measure it. In radioastronomy, the unit for the source flux density is *jansky*, where 1 Jy = 10^{-26} $\mathrm{Wm^{-2}Hz^{-1}}$,[4] i.e., $S'_{\mathrm{Jy}} = S'_{\mathrm{Wm^{-2}Hz^{-1}}} \times 10^{26}$.

2.5. Impact of polarization on antenna noise temperature

The antenna temperature T_A is proportional to the noise power P_A it receives. But P_A depends on whether the antenna is polarization matched to the radiation source. All derivations above did not account for the PLF, i.e., they assumed that the antenna and the bright-body source were polarization matched. A thermal-radiation source is typically *unpolarized*, i.e., its polarization is random. Thus, about half of the bright-body noise power cannot be picked up by the antenna, the polarization of which is fixed. For this reason, all relations between ΔT_A and T_B must include a PLF = 0.5. For example, the expression for a small bright body (7.22) should be written as

$$\Delta T_A = \frac{1}{2}\frac{\Omega_B}{\Omega_A} T_B,\ \Omega_B \ll \Omega_A. \tag{7.24}$$

Similarly, in the case of a large bright body, (7.20) becomes

$$\Delta T_A = 0.5 T_B,\ \Omega_B > \Omega_A, \tag{7.25}$$

and the expression for the noise-source flux density (7.23) is:

$$S = \frac{k\Delta T_A}{2A_e},\ \mathrm{Wm^{-2}Hz^{-1}}. \tag{7.26}$$

[4] Karl G. Jansky was the first one to use radio waves for astronomical observations.

2.6. Impact of radiation pattern on antenna noise temperature

The antenna pattern strongly influences the antenna noise temperature. High-gain antennas (such as reflector systems), when pointed at elevation angles close to the zenith at night, have negligible noise level. However, if an antenna has significant side and back lobes, which are pointed toward the ground or the horizon, its noise power is much higher. The worst case for an antenna is when its main beam points towards the ground or the horizon, as is often the case with satellite or airborne antennas that are pointed toward the earth.

The application of (7.14) in the case of high-gain antennas with only a few beams (main, side or back lobes) described by their beam efficiencies is straightforward. For example, consider an idealized normalized antenna power pattern of the form:

$$\bar{U}(\theta,\varphi)=\begin{cases}\bar{U}_{\text{MB}}=1, \text{ within the solid angle } \Omega_{\text{MB}} \text{ (main beam)}\\ \bar{U}_{\text{SL}}, \text{within the solid angle } \Omega_{\text{SL}} \text{ (side lobe)}\\ \bar{U}_{\text{BL}}, \text{within the solid angle } \Omega_{\text{BL}} \text{ (back lobe) .}\end{cases} \tag{7.27}$$

Here, $\bar{U}_{\text{SL}},\bar{U}_{\text{BL}}<1$ are the normalized radiation-pattern values of the side and back lobes. Let the main beam be entirely occupied by a body of brightness temperature $T_{B,\text{M}}$ whereas the side and back lobes "see" bright bodies of temperatures $T_{B,\text{S}}$ and $T_{B,\text{B}}$, respectively. Then, (7.14) is written as

$$T_A=\frac{1}{\Omega_A}\oiint_{4\pi}\bar{U}(\theta,\varphi)T_B(\theta,\varphi)d\Omega=\frac{1}{\Omega_A}\left(T_{B,\text{M}}\Omega_{\text{MB}}+\bar{U}_{\text{S}}T_{B,\text{S}}\Omega_{\text{SL}}+\bar{U}_{\text{B}}T_{B,\text{B}}\Omega_{\text{BL}}\right). \tag{7.28}$$

At the same time, the beam efficiency (BE) of the main beam is

$$BE_{\text{M}}=\frac{\iint_{\Omega_{\text{MB}}}U(\theta,\varphi)d\Omega}{\iint_{4\pi}U(\theta,\varphi)d\Omega}=\frac{\iint_{\Omega_{\text{MB}}}\overbrace{\bar{U}_{\text{MB}}}^{1}d\Omega}{\underbrace{\iint_{4\pi}\bar{U}(\theta,\varphi)d\Omega}_{\Omega_A}}=\frac{\Omega_{\text{MB}}}{\Omega_A}. \tag{7.29}$$

Similarly, the beam efficiency of the side lobe is

$$BE_{\text{S}}=\frac{\iint_{\Omega_{\text{SL}}}\bar{U}_{\text{SL}}d\Omega}{\underbrace{\iint_{4\pi}\bar{U}(\theta,\varphi)d\Omega}_{\Omega_A}}=\bar{U}_{\text{SL}}\frac{\Omega_{\text{SL}}}{\Omega_A}, \tag{7.30}$$

and that of the back lobe is

$$BE_{\mathrm{B}} = \bar{U}_{\mathrm{BL}} \frac{\Omega_{\mathrm{BL}}}{\Omega_A}. \tag{7.31}$$

Thus, we can write (7.28) as

$$T_A = T_{B,\mathrm{M}}\, BE_{\mathrm{MB}} + T_{B,\mathrm{M}}\, BE_{\mathrm{SL}} + T_{B,\mathrm{B}}\, BE_{\mathrm{BL}}. \tag{7.32}$$

The application of (7.32) in the calculation of antenna noise is illustrated in the example below.

Example (modified from Kraus, p. 406): A circular reflector antenna of 500 m^2 effective aperture operating at $\lambda = 20$ cm is directed at the zenith. What is the total antenna temperature assuming the sky temperature close to zenith is equal to 10° K, whereas at the horizon it is 150° K? Take the ground temperature equal to 300° K and assume that one-half of the minor-lobe beam is in the back direction (toward the ground) and one-half is toward the horizon. The main beam efficiency is $BE_{\mathrm{M}} = 0.7$.

Such a large reflector antenna is highly directive and, therefore, its main beam "sees" only the sky around the zenith. The main beam efficiency is 70%. Thus, according to the first term in (7.32) the noise contribution of the main beam is

$$T_A^{MB} = 10 \times 0.7 = 7, \text{ K}. \tag{7.33}$$

This antenna has only one other (minor) lobe of some constant normalized power-pattern value of $\bar{U}_{\mathrm{SL}} = \bar{U}_{\mathrm{BL}} = \bar{U}_{\mathrm{lobe}}$. Its beam efficiency is $BE_{\mathrm{lobe}} = 1 - 0.7 = 0.3$, i.e., under uniform illumination from all directions, it contributes 30% to the overall received power whereas 70% is due to the main beam. We can effectively split this side lobe into two parts: half of the minor lobe points to ground and the other half points toward the horizon. Each of these half-lobes has a beam efficiency of $0.5BE_{\mathrm{lobe}} = 0.15$. Thus, according to the second and third terms in (7.32), the contribution from the half-lobe directed toward ground is

$$T_A^{GBL} = 300 \times 0.15 = 45, \text{ K} \tag{7.34}$$

whereas the contribution from the half-lobe directed toward the horizon is

$$T_A^{HBL} = 150 \times 0.15 = 22.5, \text{ K}. \tag{7.35}$$

The total antenna noise temperature is

$$T_A = T_A^{MB} + T_A^{GBL} + T_A^{HBL} = 74.5 \text{ K}. \tag{7.36}$$

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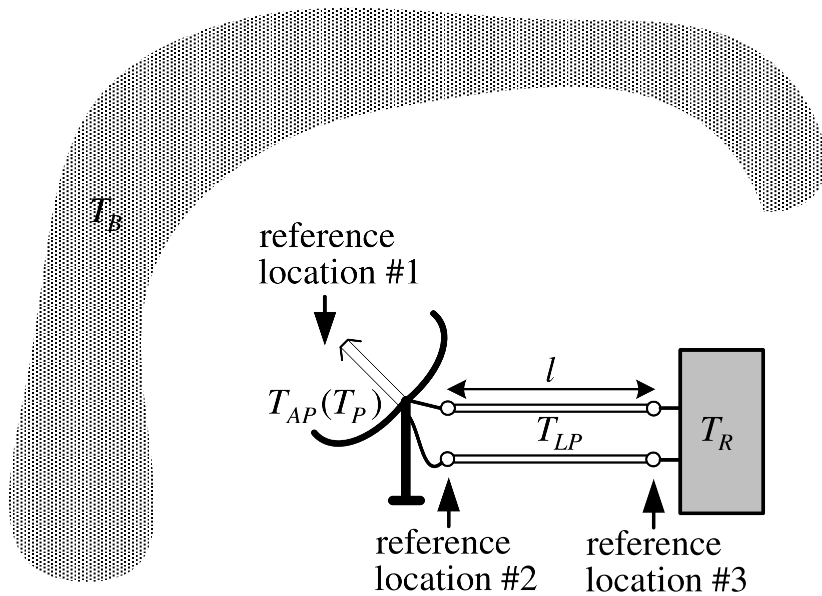
3. System Noise Temperature

The antenna is a part of a receiving system, which consists of several cascaded components: antenna, transmission line (or waveguide) assembly and receiver (see figure below). All these system components, the antenna included, have their contributions to the system noise due to their non-zero temperature and losses. The system noise level is a critical factor in determining its sensitivity and SNR.



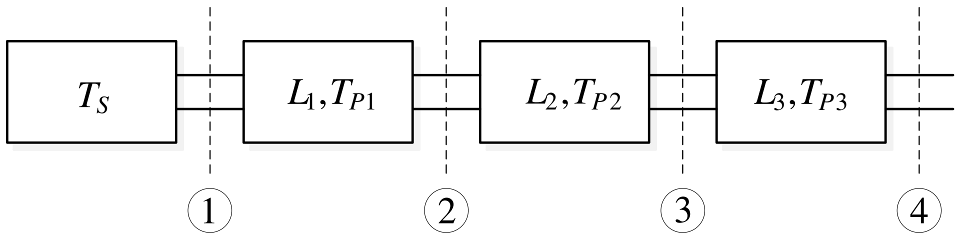
3.1. Noise Analysis of Cascaded Matched Two-port Networks[5]

To understand the noise analysis of the receiver system, we must first review the basics of the noise analysis of cascaded two-port networks. For simplicity, we assume that all networks are impedance matched, which is realistic.

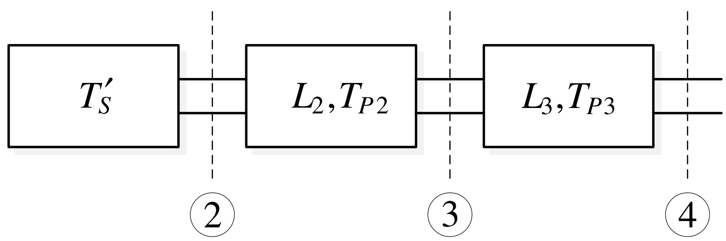
In the figure below (case (a)), a generic cascaded network is shown where the first component on the left is the noise source (e.g., the antenna picking up noise from the sky) with noise temperature T_S. The remaining two-port components are characterized by their physical temperatures T_{Pi} and by their *loss factors* (or *loss ratios*) L_i, $i = 1, 2 \ldots$. The loss factor is the input-to-output noise-power ratio, $L = P_{\text{in}} / P_{\text{ou}}$. In the case of a passive lossy two-port network (such as a waveguide or a transmission line), L is the inverse of the efficiency, i.e., $L = e^{+2\alpha l}$, where α is the attenuation constant of the waveguide or the transmission line and l is its length. The antenna, too, must be viewed as a two-port network if it is lossy,

[5] From T.Y. Otoshi, "Calculation of antenna system noise temperatures at different ports—revisited," *IPN Progress Report*, Aug. 15, 2002.

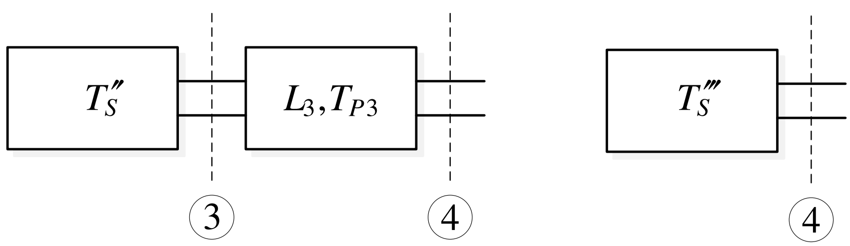
where its “input port” is its aperture receiving the noise signals from the environment and its output port is its connection to the transmission line. In this case, $L = e_A^{-1}$ where e_A is the antenna efficiency.



(a) original network



(b) equivalent source noise temperature at location 2



(c) equivalent source noise temperatures at locations 3 and 4

For a passive two-port component, $e = P_{\text{ou}} / P_{\text{in}} \leq 1$ whereas $L = P_{\text{in}} / P_{\text{ou}} \geq 1$. In noise theory, any two-port for which $L \geq 1$, i.e., it exhibits power loss, is referred to as “attenuator” although this component does not necessarily need to be an attenuator; it could be, for example, the entire antenna-plus-feed assembly. On the other hand, if $L < 1$, the component exhibits gain, and it is referred to as an “amplifier”. In this case, the efficiency is replaced by the gain G, which, just like the efficiency, is the output-to-input power ratio $P_{\text{ou}} / P_{\text{in}}$ but it is greater than 1. As with the efficiency, the relationship $L = G^{-1}$ holds.

Figure (a) above shows a cascaded network of two-ports, each characterized by its loss factor L_i, $i = 1,2,3$, and by its physical temperature T_{Pi}, $i = 1,2,3$. The first component of the cascaded network is a noise source with temperature T_S. Figure (b) shows a network where an equivalent source of temperature T_S' replaces the original source plus its neighboring two-port (L_1, T_{P1}). The equivalent source T_S' at location 2 is

$$T_S' = L_1^{-1} T_S + (1 - L_1^{-1}) T_{P1}. \tag{7.37}$$

From (7.37), it is evident that in addition to the usual "attenuated" term $L_1^{-1} T_S$, there is a contribution due to the physical temperature of the 1st two-port after the noise source. This contribution is referred to as the ***device output equivalent noise temperature***,

$$T_{D1}^{\text{ou}} = (1 - L_1^{-1}) T_{P1}. \tag{7.38}$$

This contribution is entirely determined by the two-port physical temperature and its loss factor, i.e., it does not depend on the source.

To understand where (7.37) comes from, we can re-write it as

$$T_S' = T_{P1} + \frac{T_S - T_{P1}}{L_1}. \tag{7.39}$$

We see from (7.39) that the equivalent-source power at position ②, represented by its equivalent temperature T_S', consists of two terms. The first term, T_{P1}, represents the noise power due to the non-zero physical temperature of the 1st two-port after the noise source. This power travels away toward the noise source and toward the 2nd two-port. On the other hand, the numerator $T_S - T_{P1}$ in the 2nd term of (7.39) represents the noise power at the input of the 1st two-port. This is because T_S is the noise-source power traveling toward its input whereas T_{P1} represents its own thermal noise power traveling toward the noise source.[6] This input power $T_S - T_{P1}$ is attenuated by L_1^{-1} (the efficiency of the 1st two-port) to produce the 2nd contribution in the equivalent noise power at the output of the 1st two-port.

Analogously, the equivalent source noise temperature T_S'' at location ③ is

$$T_S'' = L_2^{-1} T_S' + (1 - L_2^{-1}) T_{P2} \tag{7.40}$$

where

$$T_{D2}^{\text{ou}} = (1 - L_2^{-1}) T_{P2} \tag{7.41}$$

is the 2nd device output equivalent noise temperature.

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[6] Remember the expression $|a|^2 - |b|^2$ for the total power at the input of a microwave network where a and b are the incident and the scattered (outgoing) root-power waves, respectively. $|a|^2$ represents the incoming power whereas $|b|^2$ represents the outgoing power.

We can repeat this step for the network location ④ where we obtain the equivalent source noise temperature T_S'''. In each case, in addition to the "attenuated" source power, we add the respective ***device output equivalent noise temperature***,

$$T_{Di}^{\text{ou}} = (1 - L_i^{-1}) T_{Pi}, \; i = 1, 2, \ldots . \tag{7.42}$$

As an illustration of the general procedure, let us state the equivalent source noise temperature T_S''' at location④:

$$T_S''' = T_S (L_1 L_2 L_3)^{-1} + (1 - L_1^{-1}) T_{P1} (L_2 L_3)^{-1} + (1 - L_2^{-1}) T_{P2} L_3^{-1} + (1 - L_3^{-1}) T_{P3} . \tag{7.43}$$

3.2. Transferring System Noise Temperature along Lossy Networks

The rule of transferring noise temperature from the output port of a lossy two-port to its input port (or *vice versa*) is simple:

$$T_{\text{in}} = L T_{\text{ou}} = T_{\text{ou}} / e , \tag{7.44}$$

where e is the device efficiency (or gain). This rule, while simple, is not immediately obvious. A formal proof can be found in the Appendix of

B.L. Seidel and C.T. Stelzried, "A radiometric method for measuring the insertion loss of radome materials," *IEEE Trans. Microw. Theory Thech.*, vol. MTT-16, No. 9, Sep. 1968, pp. 625–628.

We can now define the ***equivalent noise temperature of a lossy component at its input*** (also known as ***device input equivalent noise temperature***) by substituting T_{Di}^{ou} from (7.42) as T_{ou} in (7.44):

$$T_{Di}^{\text{in}} = L_i T_{Di}^{\text{ou}} = (L_i - 1) T_{Pi}, \; i = 1, 2, \ldots . \tag{7.45}$$

Note that, just like T_{Di}^{ou}, T_{Di}^{in} depends only on the intrinsic device loss factor L_i and its physical temperature T_{Pi}, i.e., it has nothing to do with noise sources attached to the device. It is also worth noting that (7.45) suggests that T_{Di}^{in} could be much larger than the physical temperature T_{Pi} if the device is very lossy, i.e., if $L_i \gg 1$ ($e_i \ll 1$).

Finally, we discuss the physical meaning of the ***device input equivalent noise temperature*** through an alternative way of deriving the relationship in (7.45). We omit the subscript i hereafter. Consider a noise source of temperature T_S at a device input. The source noise power is $kT_S \Delta f$. To find the output noise power of the device, we add the two input contributions – that of the noise source and that due to the device input equivalent noise temperature, and then multiply the result by the device efficiency:

$$P_{\mathrm{N,ou}} = e(kT_S\Delta f + kT_D^{\mathrm{in}}\Delta f)\,. \tag{7.46}$$

To find the relation between the device input equivalent noise temperature T_D^{in} and its physical temperature T_P, we consider the particular case when the temperature of the source T_S is equal to the physical temperature T_P of the device. In this case, the output noise power must be $P_{\mathrm{N,ou}} = kT_P\Delta f$ because the whole system of the lossy device plus the source is at the physical temperature T_P. Substituting $T_S = T_P$ in (7.46) results in

$$P_{\mathrm{N,ou}} = e(kT_P\Delta f + kT_D^{\mathrm{in}}\Delta f) = kT_P\Delta f \tag{7.47}$$

which, when solved for T_D^{in}, produces (7.45). Note that we have not imposed any restrictions on the actual values of T_S and T_P but have only required that T_D^{in} depends solely on T_P (i.e., it is independent of the noise source at the input) and that (7.46) holds in the special case of $T_S = T_P$.

3.3. The atmosphere as an "attenuator"

An illustration of the above concepts in noise analysis is the impact of the atmosphere on the sky noise, e.g., the cosmic microwave background ($T_{\mathrm{CMB}} \approx$ 2.725° K). The atmosphere, depending on the time of the day and the weather conditions, exhibits loss, which we describe by the loss factor L_{atm}. L_{atm} can be calculated if we know the averaged attenuation constant in the atmosphere α_{atm} and its thickness H, e.g., $L_{\mathrm{atm}} \approx \exp(2\alpha_{\mathrm{atm}}H)$. This atmospheric "attenuator" lies between the cosmic microwave background noise source and the antenna. Therefore, the actual external noise temperature perceived by the antenna is

$$T_{\mathrm{sky}} = L_{\mathrm{atm}}^{-1}T_{\mathrm{CMB}} + (1 - L_{\mathrm{atm}}^{-1})T_{\mathrm{atm},P}\,, \tag{7.48}$$

where $T_{\mathrm{atm},P}$ is the physical temperature of the atmosphere, as per (7.37). The 1st term in (7.48) is the *space noise* whereas the 2nd one is the *atmospheric noise*. For a pencil-beam antenna placed on Earth and pointed at the sky, $T_A = T_{\mathrm{sky}}$.

3.4. Antenna noise due to the antenna physical temperature

If the antenna has losses, the noise temperature at its terminals includes not only the antenna temperature T_A due to the environment surrounding the antenna (the *external* antenna temperature) but also the antenna equivalent noise temperature T_{AP} due to its physical temperature T_P. Here, we note that the antenna acts as an "attenuator" in the cascaded network consisting of the external noise, the antenna, the waveguide, and the receiver; see Figure on p. 11.

We first describe the antenna noise contribution at reference location #1, the antenna aperture, or, equivalently, its "input". Here, we view the antenna as a lossy two-port component. From (7.45), we obtain the antenna ***input*** equivalent noise temperature T_{AP} as

$$T_{AP} = \left(\frac{1}{e_A} - 1\right) T_P = \frac{R_l}{R_r} T_P, \text{ K} \quad (7.49)$$

where $e_A = L_A^{-1}$ is the radiation efficiency $(0 \le e_A \le 1)$, R_l is the antenna loss resistance, and R_r is its radiation resistance. As a reminder, $e_A = P_{\text{rad}} / P_{\text{in}} = R_r / (R_l + R_r)$. Eq. (7.49) describes the thermal noise contribution of the antenna due to its physical temperature T_P referred to its "input" (the antenna aperture). T_{AP} must be added to T_A in order to obtain the overall antenna noise temperature (external and internal) at location #1. However, additional terms exist in the whole receiver system due to the noise contributions of the lossy TL (or waveguide) and the receiver electronics.

3.5. Noise due to the physical temperature of the transmission line

We now consider the transmission line (TL) as a source of noise when it has conduction losses. In a manner analogous to the one applied to the antenna, the TL is considered as a two-port "attenuator". Thus, its noise contribution at the antenna terminals (the input to the TL or reference location #2) is

$$T_{L2} = \left(\frac{1}{e_L} - 1\right) T_{LP}, \text{ K.} \quad (7.50)$$

Here, $e_L = e^{-2\alpha l} = L_L^{-1}$ is the ***line efficiency*** $(0 \le e_L \le 1)$, T_{LP} is the physical temperature of the TL, α (Np/m) is the attenuation constant of the TL, and l is its length.

To transfer the TL noise contribution to the reference location #1, we use (7.44) which leads to

$$T_{L1} = L_A^{-1} T_{L2} = \frac{T_{L2}}{e_A} = \frac{1}{e_A}\left(\frac{1}{e_L} - 1\right) T_{LP}. \quad (7.51)$$

Together with T_{AP}, T_{L1} must be added to T_A in order to obtain the system operating noise temperature at location #1.

3.6. System noise referred to the antenna aperture (location #1)

The system temperature referred to the antenna aperture includes the contributions of the antenna (external noise temperature plus equivalent input antenna thermal noise temperature), the transmission line and the receiver as

$$T_{sys}^{A} = \underbrace{T_A}_{\substack{\text{antenna}\\ \text{external}}} + \underbrace{T_P\left(\frac{1}{e_A}-1\right)}_{T_{AP}\text{, antenna internal}} + \underbrace{\frac{1}{e_A}T_{LP}\left(\frac{1}{e_L}-1\right)}_{\text{TL internal}} + \underbrace{\frac{1}{e_A e_L}T_R}_{\text{receiver}}. \qquad (7.52)$$

Here, T_A is the external temperature that corresponds to the antenna temperature provided the antenna is loss-free, as discussed in Section 2. T_R is the receiver noise temperature (at its input, reference location #3). It is given by

$$T_R = T_1 + \frac{T_2}{G_1} + \frac{T_3}{G_1 G_2} + \cdots, \text{ K}. \qquad (7.53)$$

Here,

- T_1 is the noise temperature of the first amplifying stage;
- G_1 is the gain of the first amplifying stage ($G_1 = L_1^{-1}$, see (7.44));
- T_2 is the noise temperature of the second amplifying stage;
- G_2 is the gain of the second amplifying stage ($G_2 = L_2^{-1}$).

Notice that T_R is divided by the efficiencies e_L and e_A in order to refer it to the TL input (location #2) and on to the antenna aperture (location #1); see (7.44).

3.7. <u>System noise referred to the antenna terminals (TL input, location #2)</u>

The reference location is changed by considering the efficiency of the antenna. As per (7.44), we have

$$T_{sys}^{TL} = T_{sys}^{A} \cdot e_A \qquad (7.54)$$

since T_{sys}^{TL} is the system noise temperature at the antenna "output" and T_{sys}^{A} is that at its "input". Substituting (7.52) into (7.54) produces

$$T_{sys}^{TL} = \underbrace{T_A e_A}_{\substack{\text{antenna}\\ \text{external}}} + \underbrace{T_P(1-e_A)}_{\substack{\text{antenna}\\ \text{internal}}} + \underbrace{T_{LP}\left(\frac{1}{e_L}-1\right)}_{\text{TL internal}} + \underbrace{\frac{1}{e_L}T_R}_{\text{receiver}}. \qquad (7.55)$$

3.8. <u>System noise referred to the receiver input (location #3)</u>

The reference location is changed once again by considering the efficiency of the TL:

$$T_{sys}^{R} = T_{sys}^{TL} \cdot e_L . \tag{7.56}$$

Therefore,

$$T_{sys}^{R} = \underbrace{T_A e_A e_L}_{\substack{\text{antenna}\\ \text{external}}} + \underbrace{T_P (1 - e_A) e_L}_{\substack{\text{antenna}\\ \text{internal}}} + \underbrace{T_{LP} (1 - e_L)}_{\text{TL}} + \underbrace{T_R}_{\text{receiver}}, \text{ K.} \tag{7.57}$$

Example (from Kraus, p. 410, modified): A receiver has an antenna with an external noise temperature 50° K, a physical temperature of 300° K, and an efficiency of 99%. Its transmission line has a physical temperature of 300° K and an efficiency of 90%. The first three stages of the receiver all have 80° K noise temperature and 13 dB gain (13 dB is about 20 times the power). Find the system temperature at: (a) the antenna aperture, (b) the antenna terminals, and (c) the receiver input.

The receiver noise temperature is

$$T_R = 80 + \frac{80}{20} + \frac{80}{20^2} = 84.2 \ \text{°K}. \tag{7.58}$$

(a) Then, the system temperature at the antenna aperture is

$$\begin{aligned} T_{sys}^{A} &= T_A + T_P \left(\frac{1}{e_A} - 1 \right) + \frac{1}{e_A} T_{LP} \left(\frac{1}{e_L} - 1 \right) + \frac{1}{e_A e_L} T_R, \\ T_{sys}^{A} &= 50 + 300 \left(\frac{1}{0.99} - 1 \right) + \frac{300}{0.99} \left(\frac{1}{0.9} - 1 \right) + \frac{84.2}{0.99 \cdot 0.9} \approx 181.2009 \text{ K.} \end{aligned} \tag{7.59}$$

(b) The system temperature at the antenna terminals is

$$T_{sys}^{TL} = T_{sys}^{A} \cdot e_A \approx 181.2009 \cdot 0.99 \approx 180.3889 \ \text{°K}.$$

(c) The system temperature at the receiver input is

$$T_{sys}^{R} = T_{sys}^{TL} \cdot e_L = 180.3889 \cdot 0.9 \approx 162.35 \ \text{°K}.$$

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4. Minimum Detectable Temperature (Sensitivity) of the System

The minimum detectable temperature, or sensitivity, of a noise-power receiving system $\Delta T_{\min}$ is the minimum difference in the brightness temperatures of two objects that can be reliably measured by the receiver system. $\Delta T_{\min}$ is usually taken at the antenna aperture (reference location #1). $\Delta T_{\min}$ represents the system's uncertainty in measuring noise temperature. Thus, if the incremental antenna temperature ΔT_A obtained with a given bright object is less than $\Delta T_{\min}$, the object is deemed undetectable.

$\Delta T_{\min}$ is defined as the *RMS* deviation (or *RMS* uncertainty) in acquiring T_A:

$$\Delta T_{\min} = \Delta T_{\text{rms}}. \tag{7.60}$$

ΔT_{rms}, referred to as *RMS noise temperature*, is determined experimentally by pointing the antenna at a background portion of space, which does not contain distinct bright bodies, and recording many T_A values over a long period of time. Acquiring a large number of T_A samples *versus* time is critical since the formula for calculating ΔT_{rms} relies on a statistical model. This model assumes that the thermal-noise voltage at the antenna terminals is a zero-mean randomly fluctuating time-waveform $v(t)$ of Gaussian probability distribution.

Assume the output of the receiver is in the form of real-positive numbers proportional to the received noise power. Modern receivers are digital, and their output is in the form of integers. Each noise-power measurement is taken over a period of time τ, referred to as post-detection (or integration) time constant. The measurement is in effect the variance of the noise voltage $v(t)$ over the period τ and is thus proportional to $P_{sys}^R = k\Delta f\, T_{sys}^R$.

The *RMS* deviation D_{rms} of the numbers produced by the receiver represents (is proportional to) the *RMS* noise temperature at the receiver:

$$D_{\text{rms}} = \sqrt{\frac{1}{N}\sum_{n=1}^{N}(a_n - a_{\text{av}})^2} = \kappa \cdot \Delta T_{\text{rms}}^R \quad \text{where} \quad a_{\text{av}} = \frac{1}{N}\sum_{n=1}^{N} a_n. \tag{7.61}$$

Here, κ is a known constant converting the integers to noise-temperature values. ΔT_{rms} (at reference location #1) can be obtained from ΔT_{rms}^R by

$$\Delta T_{\text{rms}} = \frac{\Delta T_{\text{rms}}^R}{e_A e_L} = \Delta T_{\min}. \tag{7.62}$$

This is the *sensitivity* of the system in terms of noise temperature.

In order a source to be detected, it has to create an incremental antenna temperature ΔT_A which exceeds $\Delta T_{\min}$, $\Delta T_A > \Delta T_{\min}$. Thus, the *minimum detectable power* $P_{\min}$ is

$$P_{\min} = 0.5 A_e p_{\min} = k\Delta T_{\min}\Delta f, \tag{7.63}$$

where A_e is the effective antenna area, $p_{\min}$ is the power-flux density (magnitude of Poynting vector) due to the source at the location of the antenna, and the factor of 0.5 accounts for the randomness of the wave polarization. It follows that the minimum detectable power-flux density is

$$p_{\min} = \frac{2k\Delta T_{\min}\Delta f}{A_e}, \text{ W/m}^2. \tag{7.64}$$

The signal-to-noise ratio (SNR) for a noise-power signal of incremental antenna temperature ΔT_A is given by

$$SNR = \frac{\Delta T_A}{\Delta T_{\min}}. \tag{7.65}$$

This SNR is used in radioastronomy and remote sensing.

Theoretically, ΔT_{rms} can be estimated from the average system temperature T_{sys}^A, the bandwidth of the receiver Δf, and the time τ over which one measurement is taken:

$$\Delta T_{\min} = \Delta T_{\text{rms}} = \frac{T_{sys}^A}{\sqrt{\Delta f\, \tau}}. \tag{7.66}$$

This equation is well known as the *radiometry equation*. It is derived from statistical principles, but its derivation is not going to be discussed here. We only mention that (7.66) is often corrected as:

$$\Delta T_{\min} = \Delta T_{\text{rms}} = \frac{k' T_{sys}^A}{\sqrt{\Delta f\, \tau}}, \tag{7.67}$$

where k' is termed the *system constant*, and it is determined experimentally.

In the previous example, we found that the system temperature at the antenna aperture is $T_{sys}^A \approx 181.2009$ K. Assume that the receiver bandwidth is $\Delta f = 100$ Hz, that the system constant is $k' = 1$ and that the post-detection constant is $\tau = 1$ s. Find the minimum detectable noise power at the antenna aperture $P_{\min}$.

$$P_{\min} = k\Delta f\, \Delta T_{\min} = k\Delta f \frac{k' T_{sys}^A}{\sqrt{\Delta f \cdot \tau}} = k\sqrt{\frac{\Delta f}{\tau}} k' T_{sys}^A$$

$$\approx 1.38 \cdot 10^{-23} \cdot \sqrt{100} \cdot 181.2009 \approx \underline{\underline{2.5 \cdot 10^{-20} \text{ W}}}$$

5. System Signal-to-Noise Ratio (SNR) in Communication Links

The system noise power at the antenna terminals (location #2) is

$$P_N = kT_{sys}^{TL}\Delta f \text{ , W,} \tag{7.68}$$

where $T_{sys}^{TL} = e_A T_{sys}^{A}$. Using Friis' transmission equation, we can express the received power at the antenna terminals as

$$P_r = (1-|\Gamma_t|^2)(1-|\Gamma_r|^2)\text{PLF}\left(\frac{\lambda}{4\pi R}\right)^2 G_t(\theta_t,\varphi_t)G_r(\theta_r,\varphi_r)P_t. \tag{7.69}$$

Finally, the SNR becomes

$$SNR = \frac{P_r}{P_N} = \frac{(1-|\Gamma_t|^2)(1-|\Gamma_r|^2)\text{PLF}\left(\frac{\lambda}{4\pi R}\right)^2 G_t G_r P_t}{kT_{sys}^{TL}\Delta f}. \tag{7.70}$$

The above equation is fundamental in the design of telecommunication systems. More specifically, if the SNR necessary for the adequate operation of the receiver is known, Eq. (7.70) allows for determining the maximum range over which the communication link is stable.

APPENDIX: Basic Radiometric Receiver Theory

Any dissipative body emits spontaneous electromagnetic (EM) radiation of thermal origin (known as *blackbody radiation*). Planck's law of blackbody radiation states the *spectral radiance* (or *spectral brightness*) $\mathcal{B}(T, f)$ of a body at absolute temperature *T* (°K) above 0°K as a function of frequency *f* as:

$$\mathcal{B}(T, f) = \frac{2hf^3}{c^2\left[e^{hf/(kT)} - 1\right]}, \text{ W·m}^{-2}\text{·sr}^{-1}\text{·Hz}^{-1} \quad (1)$$

where c is the speed of light in vacuum, h is Planck's constant (h = 6.62607015×10^{-34}, J·s or J·Hz^{-1}), and k is Boltzmann's constant ($k \approx 1.380649\times10^{-23}$ J/K). The spectral brightness describes the amount of thermal power radiated per unit area, per unit solid angle, and per unit frequency for given frequency and temperature.

At microwave frequencies and moderate temperatures, the term $hf/(kT)$ is very small. Thus, the approximation $e^{hf/(kT)} \approx 1 + hf/(kT)$ (first 2 terms in the Taylor series expansion) can be made, leading to the approximation of (1) known as the Rayleigh-Jeans law:

$$\mathcal{B}(T, f) \approx \frac{2f^2kT}{c^2} = \frac{2kT}{\lambda^2}. \quad (2)$$

We notice that the thermal radiance decreases rapidly with decreasing frequency ($\mathcal{B} \sim f^2$). Importantly, the radiance is proportional to the body's temperature *T*. This allows for determining the temperature of the body by measuring the power it radiates. This is the underlying principle of thermography in microwave radiometry and radioastronomy.

The total thermal-noise power per *hertz* p_h of an isothermal body radiating isotropically can be estimated as

$$p_{h,\text{B}}(T, f) = \mathcal{B}(T, f) \cdot A_{e,\text{B}} \cdot 4\pi, \text{ W/Hz} \quad (3)$$

where $A_{e,\text{B}}$ is the effective radiating area of the bright body and 4π is the solid angle of 3-D space. Since the body radiates isotropically, it has a directivity $D_\text{B} = 1$; therefore, its effective area is $A_{e,\text{B}} = \lambda^2/(4\pi)$. Substituting in (3) yields

$$p_{h,\text{B}}(T, f) \approx \frac{2kT}{\lambda^2} \cdot \frac{\lambda^2}{4\pi} \cdot 4\pi = 2kT. \quad (4)$$

Thus, the bright body can be viewed as a source of microwave power with the spectral density given by (4). Note that the result in (4) no longer depends on frequency.

An antenna can receive at best half of the incident power and deliver it to the radiometer (provided it is loss-free and perfectly impedance matched). Assuming the antenna captures all the power radiated by the bright body, the received power is:

$$p_h(T) = 0.5\, p_{h,\mathrm{B}}(T) \approx kT, \ \mathrm{W/Hz}. \tag{5}$$

This is the Nyquist relation given in equation (7.1). The cases where the antenna captures only a portion of the black-body radiation are discussed in this Lecture.

Since the receiver has a limited bandwidth of Δf, the total noise power it can receive is

$$P_\mathrm{N} = kT\Delta f, \ \mathrm{W}. \tag{6}$$

This is equation (7.2).