LECTURE 13: LINEAR ARRAY THEORY - PART I

(*Linear arrays: the two-element array. N-element array with uniform amplitude and spacing. Broad-side array. End-fire array. Phased array.*)

1. Introduction

Usually, the radiation patterns of single-element antennas are relatively wide, i.e., they often have lower directivity than what is needed. In long-distance and satellite communications as well as in radar, antennas with high directivity are often required. Such antennas are possible to construct by enlarging the dimensions of the radiating aperture (size much larger than λ). This approach, however, may lead to the appearance of multiple side lobes, which are difficult to control. Besides, large antennas, e.g., reflector antennas, are often difficult to fabricate.

Another way to increase the electrical size of an antenna is to construct it as an assembly of radiating elements in a proper electrical and geometrical configuration – ***antenna array***. Often, the array elements are identical. This is not necessary, but it is practical and simpler to design. The individual elements may be of any type (wire dipoles, loops, apertures, printed antennas, etc.)

The total field of an array is a vector superposition of the fields radiated by the individual elements. To provide very directive pattern, it is necessary that the partial fields (generated by the individual elements) interfere constructively in the desired direction and interfere destructively elsewhere.

There are six factors that impact the overall antenna pattern:

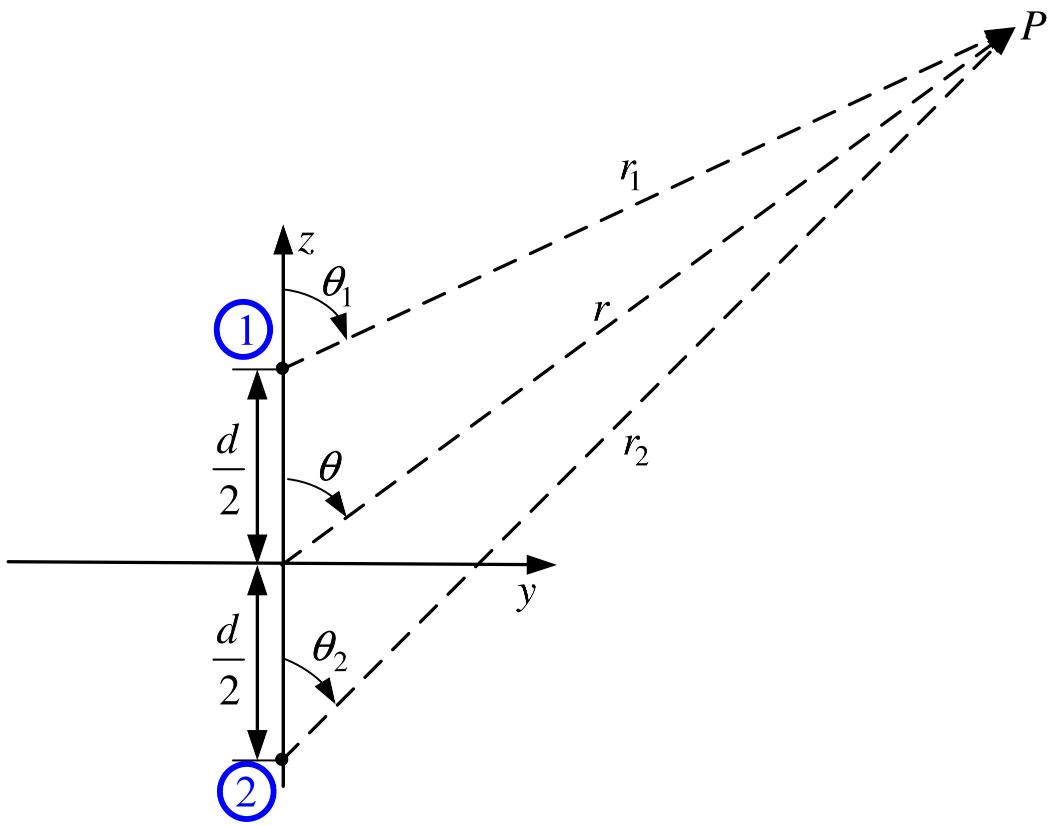
a) the type of the array (linear, circular, spherical, rectangular, etc.),
b) the overall size of the array,
c) the relative placement of the elements,
d) the excitation amplitude of each elements,
e) the excitation phase of each element,
f) the individual pattern of each element.

2. Two-element Array

Let us state the far-zone electric fields of the two array elements as

$$\mathbf{E}_1 = M_1 \bar{F}_1(\theta_1,\phi_1) \frac{e^{-j\left(kr_1 - \frac{\beta}{2}\right)}}{r_1} \hat{\mathbf{p}}_1, \tag{13.1}$$

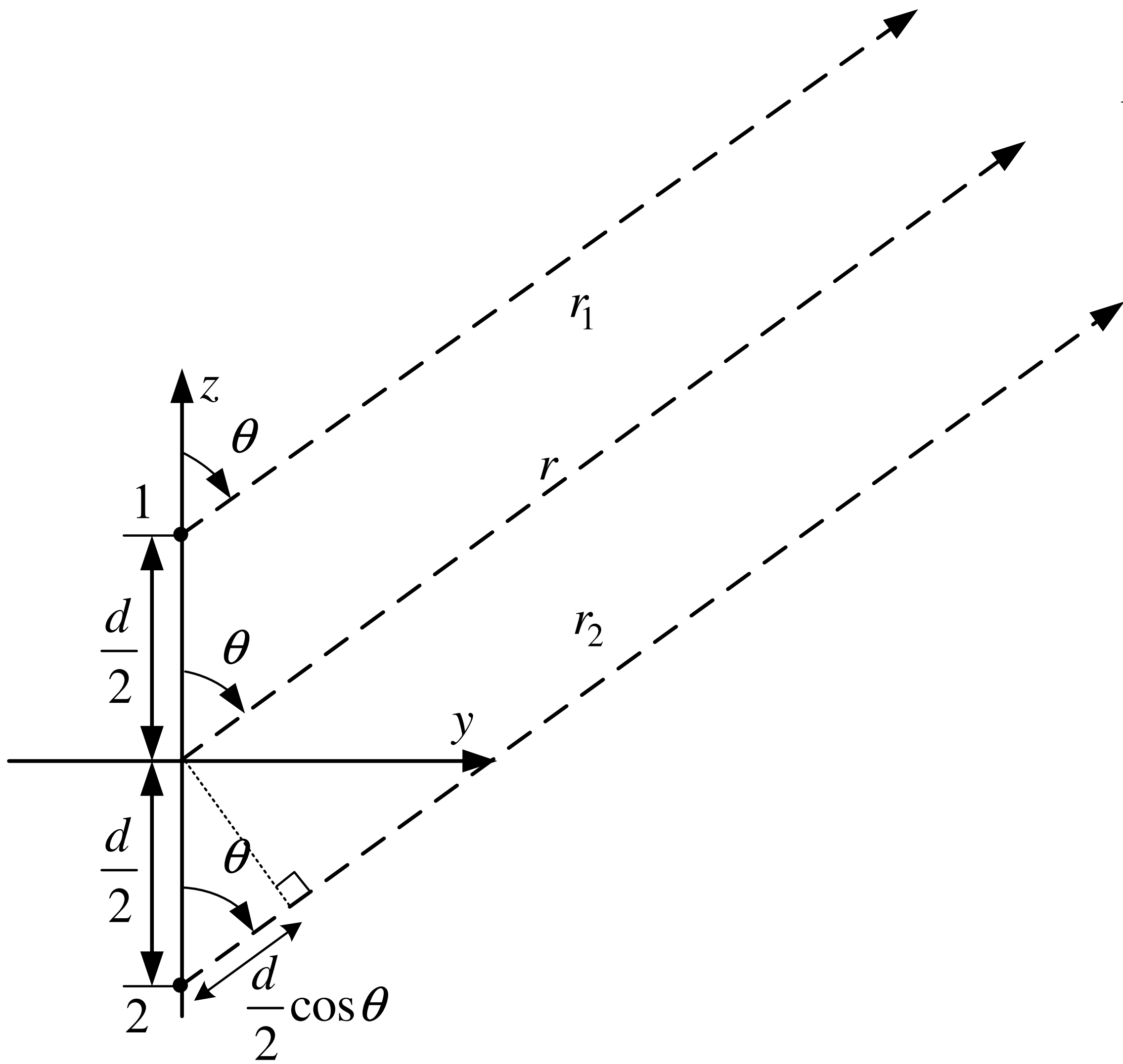
$$\mathbf{E}_2 = M_2 \bar{F}_2(\theta_2,\phi_2) \frac{e^{-j\left(kr_2 + \frac{\beta}{2}\right)}}{r_2} \hat{\mathbf{p}}_2. \tag{13.2}$$



Here,

$M_1,\ M_1$	field magnitudes (do not include the $1/r$ factor);
$\bar{F}_1,\ \bar{F}_2$	normalized field patterns;
$r_1,\ r_2$	distances to the observation point P;
β	phase difference between the feeds of the two array elements;
$\hat{\mathbf{p}}_1,\ \hat{\mathbf{p}}_2$	polarization vectors of the far-zone $\mathbf{E}$ fields.

The far-field approximation of the two-element array problem:



Let us assume that:

1) the array elements are identical, i.e.,

$$\bar{F}_1(\theta,\phi) = \bar{F}_2(\theta,\phi) = \bar{F}(\theta,\phi), \tag{13.3}$$

2) they are oriented in the same way in space (they have identical polarizations), i.e.,

$$\hat{\mathbf{p}}_1 = \hat{\mathbf{p}}_2 = \hat{\mathbf{p}}, \tag{13.4}$$

3) their excitation is of the same amplitude, i.e.,

$$M_1 = M_2 = M. \tag{13.5}$$

Then, the total field is

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2, \tag{13.6}$$

$$\mathbf{E}=\hat{\mathbf{p}}M\bar{F}(\theta,\phi)\frac{1}{r}\left[e^{-jk\left(r-\frac{d}{2}\cos\theta\right)+j\frac{\beta}{2}}+e^{-jk\left(r+\frac{d}{2}\cos\theta\right)-j\frac{\beta}{2}}\right],$$

$$\mathbf{E}=\hat{\mathbf{p}}\frac{M}{r}e^{-jkr}\bar{F}(\theta,\phi)\left[e^{j\left(\frac{kd}{2}\cos\theta+\frac{\beta}{2}\right)}+e^{-j\left(\frac{kd}{2}\cos\theta+\frac{\beta}{2}\right)}\right],$$

$$\boxed{\mathbf{E}=\underbrace{\hat{\mathbf{p}}M\frac{e^{-jkr}}{r}\bar{F}(\theta,\phi)}_{\text{element factor}}\times\underbrace{2\cos\left(\frac{kd\cos\theta+\beta}{2}\right)}_{AF}}. \tag{13.7}$$

The total field of the array is equal to the product of the field created by a single element if located at the origin (element factor) and the ***array factor*** (AF):

$$AF=2\cos\left(\frac{kd\cos\theta+\beta}{2}\right). \tag{13.8}$$

Using the normalized field pattern of a single element, $\bar{F}(\theta,\phi)$, and the normalized AF,

$$AF_{\text{n}}=\cos\left(\frac{kd\cos\theta+\beta}{2}\right), \tag{13.9}$$

the normalized field pattern of the array is expressed as their product:

$$\bar{f}(\theta,\phi)=\bar{F}(\theta,\phi)\times AF_{\text{n}}(\theta,\phi). \tag{13.10}$$

The concept expressed by (13.10) is the so-called ***pattern multiplication rule*** valid for arrays of identical elements. This rule holds for any array consisting of decoupled identical elements, where the excitation magnitudes, the phase shift between the elements and the displacement between them are not necessarily the same. The total pattern, therefore, can be controlled via the single–element pattern $\bar{F}(\theta,\phi)$ or via the AF. The AF, in general, depends on the:

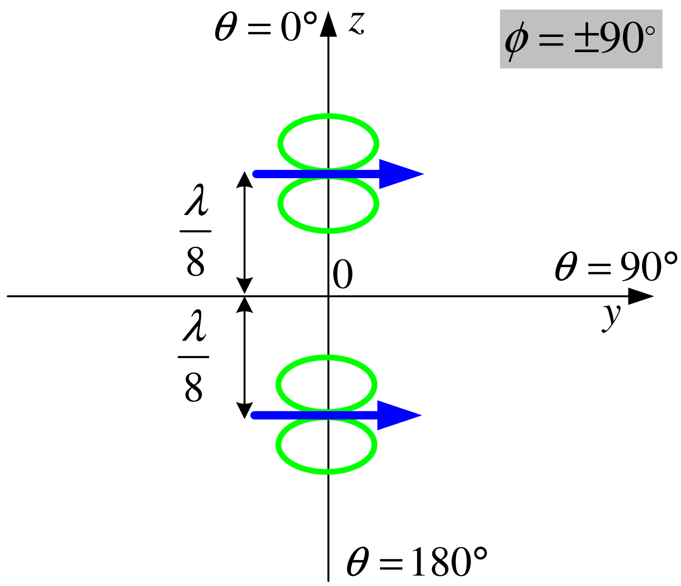
- number of elements,
- mutual placement,
- relative excitation magnitudes and phases.

Example 1: An array consists of two horizontal infinitesimal dipoles a distance $d = \lambda / 4$ apart. Find the nulls of the total field in the elevation zy plane (azimuth angle is fixed at $\phi = \pm 90°$), if the excitation magnitudes are the same and the phase difference is:

a) $\beta = 0$

b) $\beta = \pi / 2$

c) $\beta = -\pi / 2$



The element pattern is $\bar{F}(\theta,\phi) = \sqrt{1 - \sin^2\theta \sin^2\phi}$ (we obtained this element factor in Lecture 9 when discussing the horizontal dipole over a ground plane). This pattern does not depend on β and is thus the same in all three cases. At $\phi = \pm 90^\circ$, $\bar{F}(\theta,\phi) = |\cos\theta|$ and the null is at

$$\theta_1 = \pi / 2. \tag{13.11}$$

The AF depends on β, and it produces the following results in the 3 cases:

a) $\beta = 0$

$$AF_{\text{n}} = \cos\left(\frac{kd\cos\theta_n}{2}\right) = 0 \quad \Rightarrow \cos\left(\frac{\pi}{4}\cos\theta_n\right) = 0,$$

$$\Rightarrow \frac{\pi}{4}\cos\theta_n = (2n+1)\frac{\pi}{2} \quad \Rightarrow \cos\theta_n = 2(2n+1), \;\; n = 0, \pm 1, \pm 2, \ldots.$$

A solution with a real-valued angle does not exist. In this case, the total field pattern has only 1 null at $\theta_1 = 90°$, which is due to the element factor.

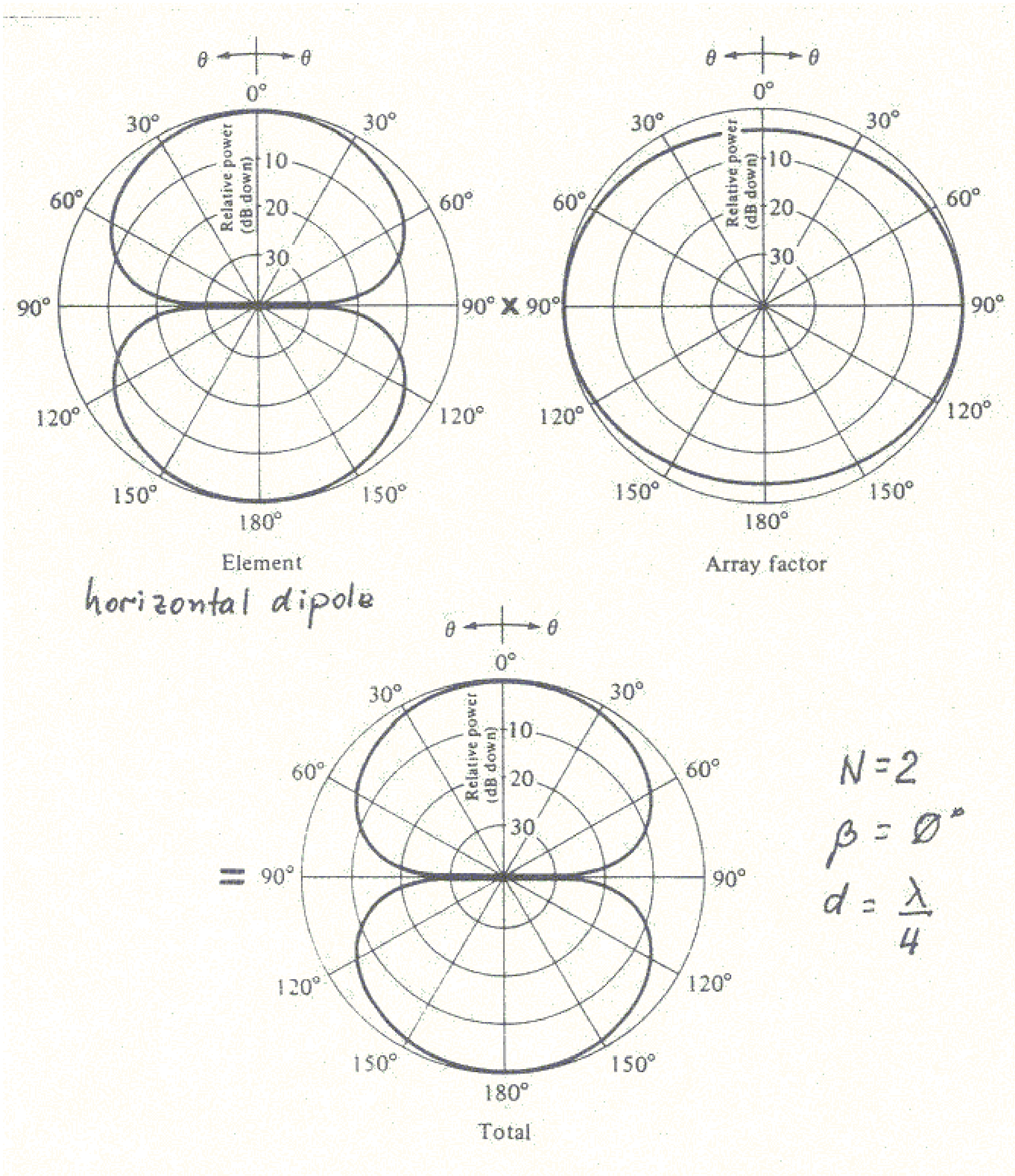


Fig. 6.3, p. 255, Balanis

b) $\beta = \pi / 2$

$$AF_{\mathrm{n}} = \cos\left(\frac{\pi}{4}\cos\theta_n + \frac{\pi}{4}\right) = 0 \quad \Rightarrow \frac{\pi}{4}\left(\cos\theta_n + 1\right) = (2n+1)\frac{\pi}{2},$$

$$\Rightarrow \cos\theta_n + 1 = 2(2n+1) \quad \Rightarrow \cos\theta_{(n=0)} = 1 \quad \Rightarrow \boxed{\theta_2 = 0}.$$

The solution for $n = 0$ is the only real-valued solution. Thus, the total field pattern has 2 nulls: at $\theta_1 = 90°$ and at $\theta_2 = 0°$:

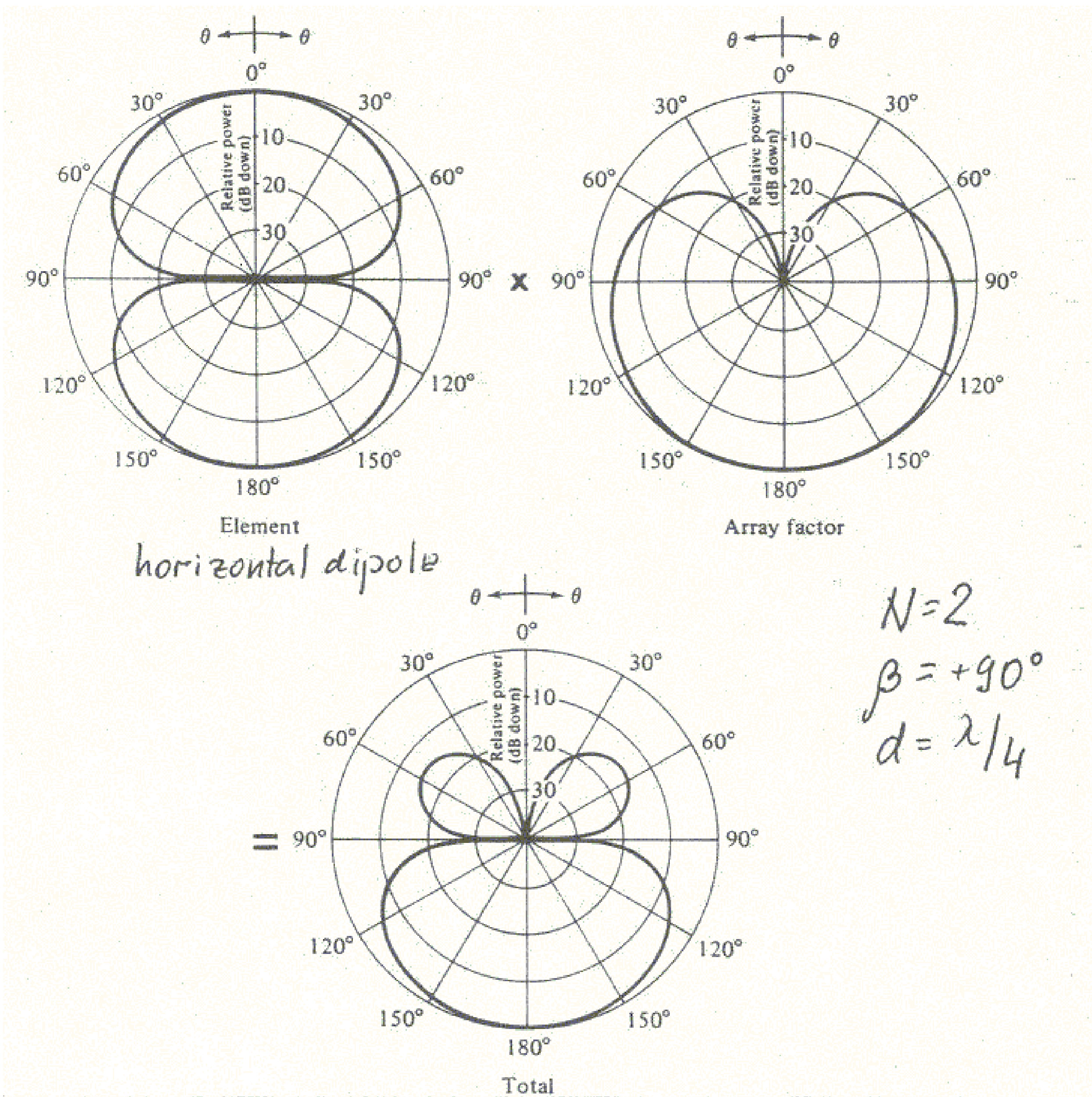


Fig. 6.4, p. 256, Balanis

c) $\beta = -\pi / 2$

$$AF_{\rm n} = \cos\left(\frac{\pi}{4}\cos\theta_n - \frac{\pi}{4}\right) = 0 \quad \Rightarrow \frac{\pi}{4}(\cos\theta_n - 1) = (2n+1)\frac{\pi}{2},$$

$$\Rightarrow \cos\theta_n - 1 = (2n+1)\cdot 2 \quad \Rightarrow \cos\theta_{(n=-1)} = -1 \quad \Rightarrow \boxed{\theta_2 = \pi}.$$

The total field pattern has 2 nulls: at $\theta_1 = 90°$ and at $\theta_2 = 180°$.

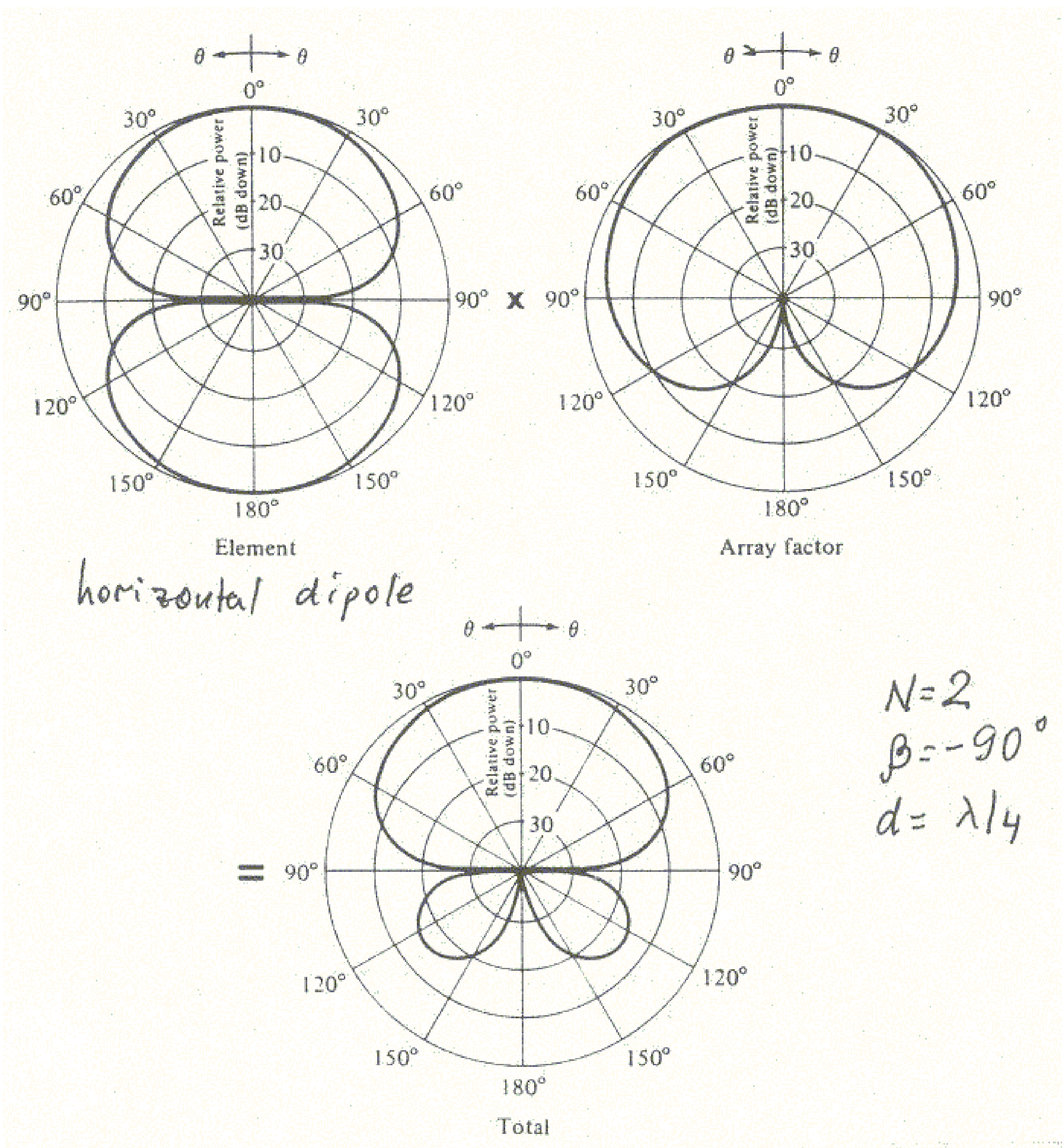


Fig. 6.4b, p. 257, Balanis

<u>Example 2:</u> Consider a 2-element array of identical infinitesimal dipoles oriented along the y-axis. Find the expression for the angles of observation where the nulls of the pattern occur in the plane $\phi = \pm 90°$ as a function of the distance d between the dipoles and the phase difference β.

The normalized total field pattern is

$$\bar{f} = |\cos\theta| \cos\left(\frac{kd\cos\theta + \beta}{2}\right). \tag{13.12}$$

In order to find the nulls, the equation

$$\bar{f} = |\cos\theta| \cos\left(\frac{kd\cos\theta + \beta}{2}\right) = 0 \tag{13.13}$$

is solved. The element factor $|\cos\theta|$ produces one null at

$$\theta_1 = \pi / 2. \tag{13.14}$$

The AF leads to the following solution:

$$\cos\left(\frac{kd\cos\theta + \beta}{2}\right) = 0 \;\Rightarrow\; \frac{kd\cos\theta + \beta}{2} = (2n+1)\frac{\pi}{2},\; n = 0, \pm 1, \pm 2 \ldots$$

$$\theta_n = \arccos\left\{\frac{\lambda}{2\pi d}\left[-\beta \pm (2n+1)\pi\right]\right\}. \tag{13.15}$$

When there is no phase difference between the two element feeds $(\beta = 0)$, the separation d must satisfy

$$d \ge \lambda / 2$$

in order at least one real-valued null to occur. Real-valued solutions to (13.15) occur when the argument within the braces is between −1 and +1.

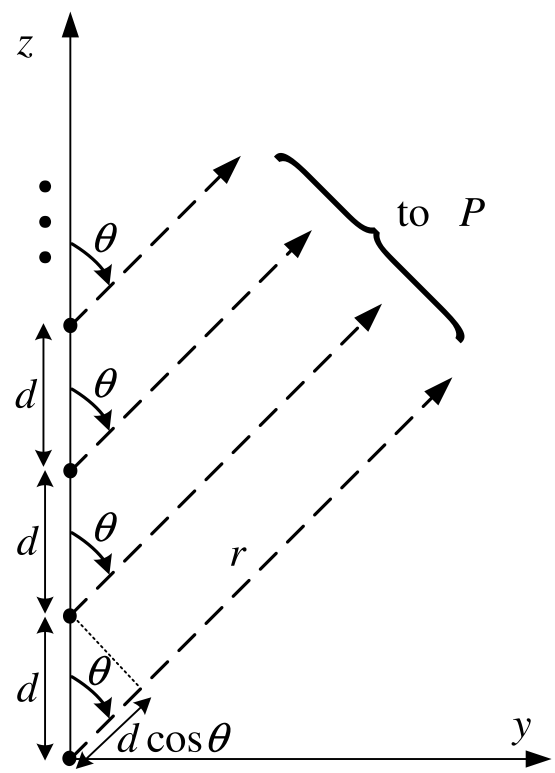
3. *N*-element Linear Array with Uniform Amplitude and Spacing

We assume that each succeeding element has a β *progressive phase* lead in the excitation relative to the preceding one. An array of identical elements with identical magnitudes and with a progressive phase is called a ***uniform array***. The *AF* of the uniform array can be obtained by considering the individual elements as point (isotropic) sources. Then, the total field pattern can be obtained by simply multiplying the *AF* by the element factor (the normalized field pattern of the individual element), provided the elements are not coupled.

The *AF* of an *N*-element linear array of isotropic sources is a superposition:

$$AF = 1 + e^{j(kd\cos\theta+\beta)} + e^{j2(kd\cos\theta+\beta)} + \ldots + e^{j(N-1)(kd\cos\theta+\beta)}. \quad (13.16)$$

The *AF* (at any angle of observation θ) depends on both the inter-element spacing d $(Nkd\cos\theta)$, which determines the far-zone phase delay, and the progressive phase shift β:



Equation (13.16) can be re-written as a sum:

$$AF = \sum_{n=1}^{N} e^{j(n-1)(kd\cos\theta+\beta)}, \tag{13.17}$$

$$AF = \sum_{n=1}^{N} e^{j(n-1)\psi}, \tag{13.18}$$

where $\psi = kd\cos\theta + \beta$. We refer to ψ as *elemental phase*.

From (13.18), we see that the *AF*s of uniform linear arrays can be controlled by the relative phase β between the elements. The *AF* in (13.18) can be expressed in a closed form, which is more convenient for pattern analysis. The closed-form expression is derived as:

$$AF\, e^{j\psi} = \sum_{n=1}^{N} e^{jn\psi}, \tag{13.19}$$

$$AF\, e^{j\psi} - AF = AF(e^{j\psi} - 1) = \sum_{n=1}^{N} e^{jn\psi} - \sum_{n=1}^{N} e^{j(n-1)\psi} = e^{jN\psi} - 1,$$

$$AF = \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} = \frac{e^{j\frac{N}{2}\psi}\left(e^{j\frac{N}{2}\psi} - e^{-j\frac{N}{2}\psi}\right)}{e^{j\frac{\psi}{2}}\left(e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}\right)},$$

$$AF = e^{j\left(\frac{N-1}{2}\right)\psi} \frac{\sin(N\psi/2)}{\sin(\psi/2)}. \tag{13.20}$$

Here, the phase factor $\exp\left[j(N-1)\psi/2\right]$ reflects a phase advancement associated with the last (*N*th) array element relative to the center of the linear array. It represents the phase shift of the array's centre relative to the origin, and it would be equal to one if the origin were to coincide with the array's centre. This factor is not important unless the array is further combined with another antenna. As we aim at obtaining the normalized *AF*, we neglect this phase factor, leading to

$$AF = \frac{\sin(N\psi/2)}{\sin(\psi/2)}. \tag{13.21}$$

For small values of $\psi = kd\cos\theta + \beta$, (13.21) is approximated as

$$AF \approx \frac{\sin(N\psi/2)}{\psi/2}. \tag{13.22}$$

To normalize (13.21), we need the maximum of the AF. We re-write (13.21) as

$$AF = N \cdot \frac{\sin(N\psi/2)}{N\sin(\psi/2)}. \tag{13.23}$$

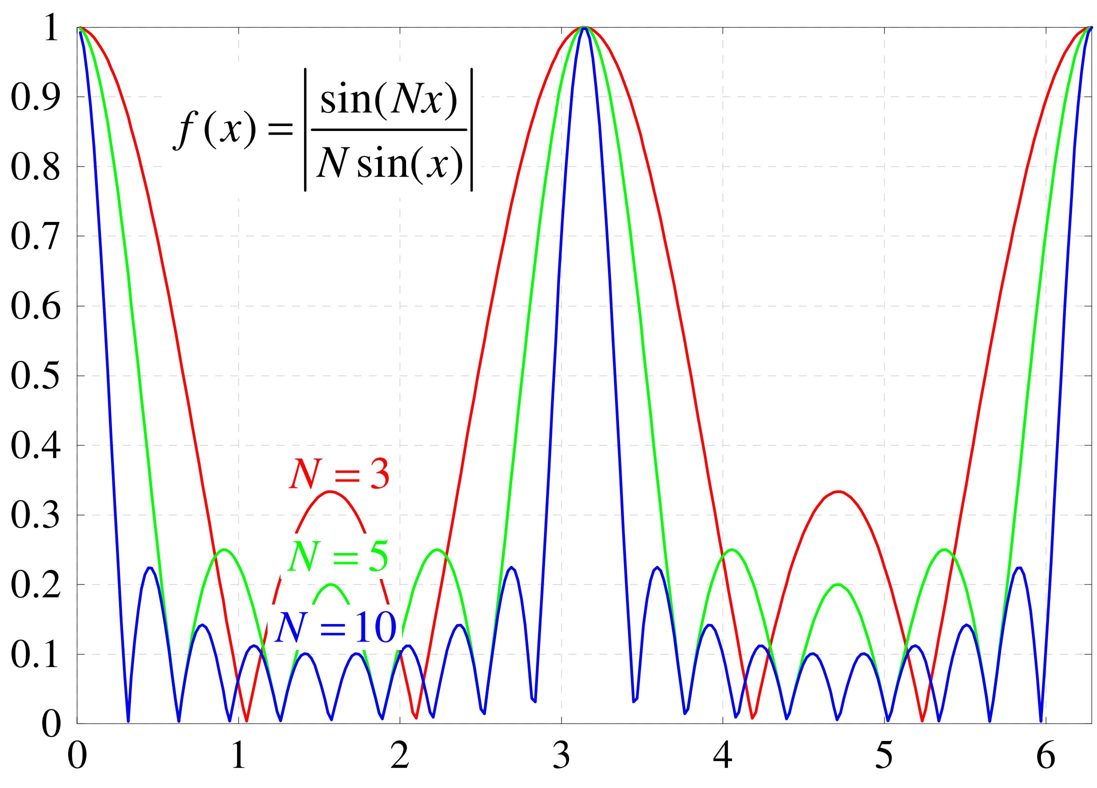
The function

$$f(x) = \frac{\sin(Nx)}{N\sin(x)}$$

has its maxima at $x = 0, \pi, \ldots,$ all having the value $f_{\max} = 1$. Therefore, $AF_{\max} = N$. The normalized AF is thus obtained as

$$\boxed{AF_{\mathrm{n}} = \frac{\sin(N\psi/2)}{N\sin(\psi/2)}}. \tag{13.24}$$

The function $|f(x)|$ representing AF_{n} is plotted below. Note that, in this plot, $x = 0.5\psi = 0.5(kd\cos\theta + \beta)$.



For small ψ, the normalized for of (13.22) is

$$\boxed{AF_{\mathrm{n}} \approx \frac{1}{N}\left[\frac{\sin(N\psi/2)}{\psi/2}\right]}. \tag{13.25}$$

Nulls of the *AF*

To find the nulls of the *AF*, equation (13.24) is set equal to zero:

$$\sin\left(\frac{N}{2}\psi\right)=0 \quad \Rightarrow \frac{N}{2}\psi = \pm n\pi \quad \Rightarrow \frac{N}{2}(kd\cos\theta_n + \beta) = \pm n\pi, \tag{13.26}$$

$$\theta_n = \arccos\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2n}{N}\pi\right)\right], \; n = 1,2,3\ldots (n \neq 0, N, 2N, 3N\ldots). \tag{13.27}$$

When $n = 0, N, 2N, 3N\ldots$, the *AF* attains its maximum values, not nulls (see the case considered below). The values of n determine the order of the nulls. For a null to exist, *the argument of the arccosine must be between* -1 *and* $+1$.

Major maxima of the *AF*

They are studied to determine the maximum directivity, the *HPBW*s, and the direction of maximum radiation. The maxima of (13.24) occur when (see the plot in page 12)

$$\frac{\psi}{2} = \frac{1}{2}(kd\cos\theta_m + \beta) = \pm m\pi, \tag{13.28}$$

$$\theta_m = \arccos\left[\frac{\lambda}{2\pi d}(-\beta \pm 2m\pi)\right], \quad m = 0,1,2\ldots. \tag{13.29}$$

At $\theta = \theta_m$, $AF_{\mathrm{n}} = 1$, i.e., these are all major lobes. The index m shows the maximum's order. It is usually desirable to have a single major lobe (main beam), i.e., m = 0 only. This can be achieved by choosing d/λ sufficiently small ($d/\lambda < 1 + |\beta|/(2\pi)$). Then the argument of the *arccosine* function in (13.29) becomes greater than unity for $m = 1, 2\ldots$ and equation (13.29) has a single real-valued solution giving the direction of the major lobe:

$$\theta_0 = \arccos\left(-\frac{\beta\lambda}{2\pi d}\right). \tag{13.30}$$

The *HPBW* of a major lobe

The *HPBW* of a major lobe is calculated by setting the value of AF_n equal to $1/\sqrt{2}$. For the approximate AF_n in (13.25),

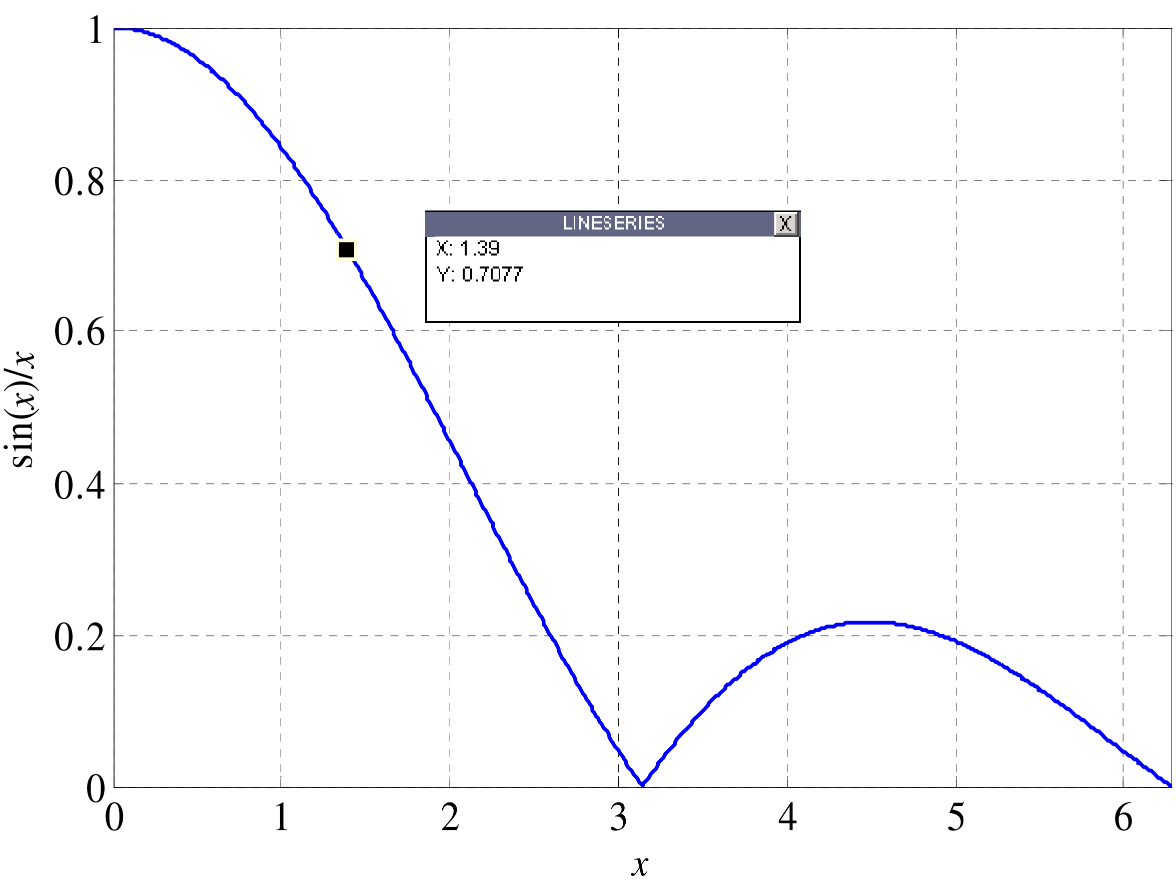
$$\frac{N}{2}\psi = \frac{N}{2}\left(kd\cos\theta_h + \beta\right) \approx \pm 1.391.$$

See the plot of $(\sin x)/x$ below.

$$\Rightarrow \theta_h \approx \arccos\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]. \tag{13.31}$$

For a symmetrical pattern with a single major beam at θ_0 (the angle at which maximum radiation occurs), the *HPBW* is calculated as

$$HPBW = 2\,|\,\theta_0 - \theta_h\,|. \tag{13.32}$$



Maxima of minor lobes (secondary maxima)

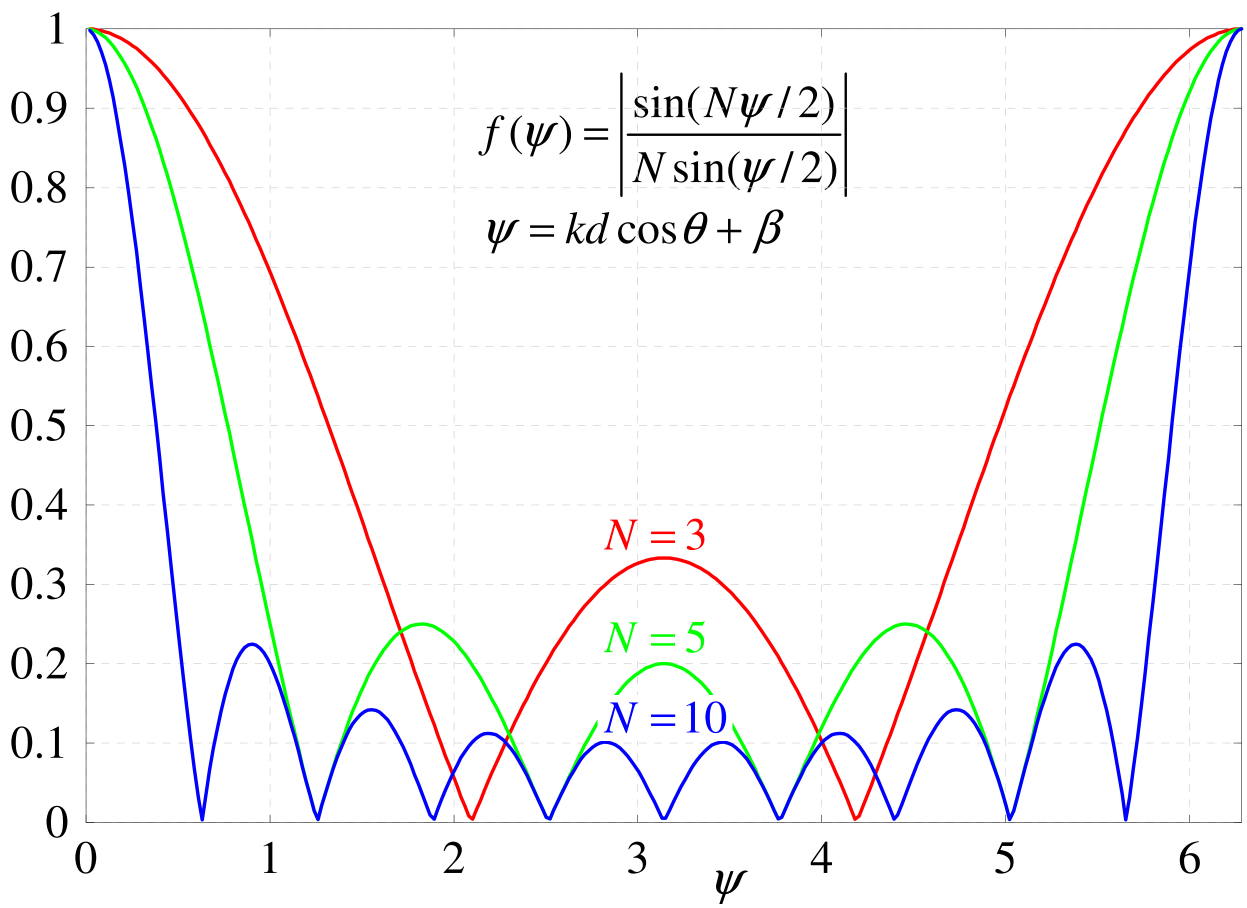
They are the maxima where $AF_n < 1$. These are illustrated in the plot below, which shows the array factors as a function of $\psi = kd\cos\theta + \beta$ for a uniform equally spaced linear array with $N = 3, 5, 10$.

The secondary maxima occur where the numerator attains a maximum and the AF is beyond its 1st null:

$$\sin\left(\frac{N}{2}\psi\right) = \pm 1 \quad \Rightarrow \frac{N}{2}\left(kd\cos\theta + \beta\right) = \pm(2s+1)\frac{\pi}{2}, \tag{13.33}$$

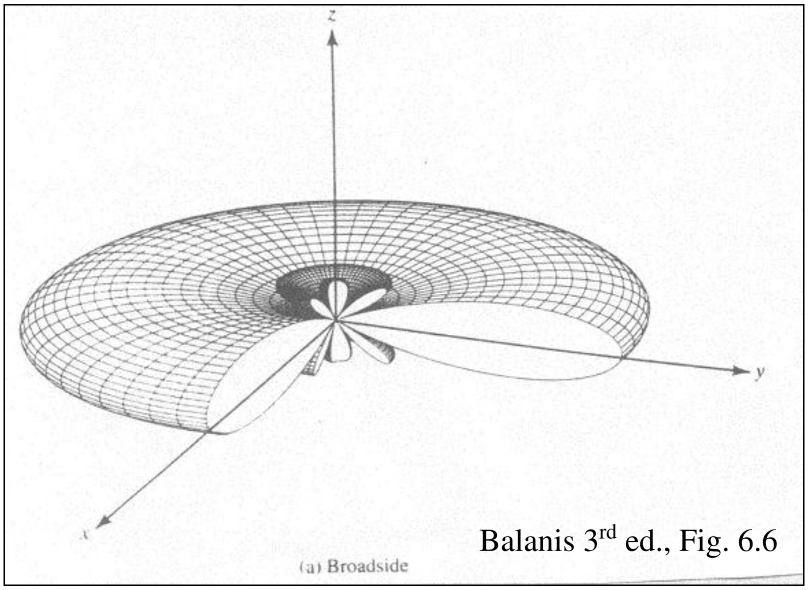
$$\Rightarrow \theta_s = \arccos\left\{\frac{\lambda}{2\pi d}\left(-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right)\right\} \text{ or} \tag{13.34}$$

$$\theta_s = \frac{\pi}{2} - \arccos\left\{\frac{\lambda}{2\pi d}\left(-\beta \pm \left(\frac{2s+1}{N}\right)\pi\right)\right\}. \tag{13.35}$$



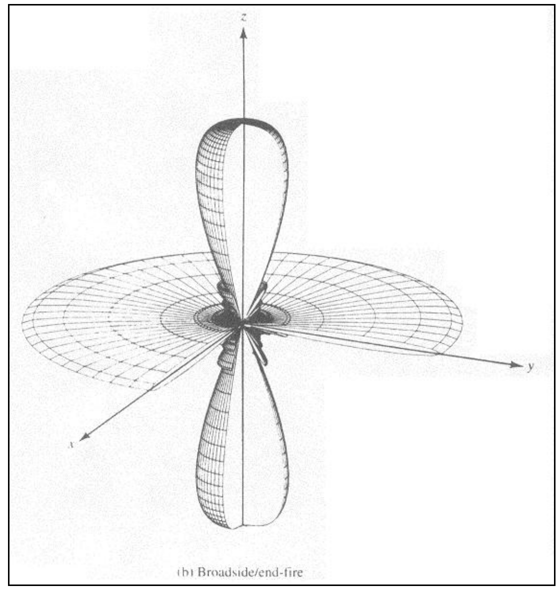
4. Broadside Array

A broadside array is an array, which has maximum radiation at $\theta = 90°$ (in the plane orthogonal to the axis of the array); see figure (a) below. For best total array pattern, both the element factor and the AF should have their maxima at $\theta = 90°$.



Balanis 3rd ed., Fig. 6.6

(a) $\beta = 0$, $d = \lambda / 4$



(b) $\beta = 0$, $d = \lambda$

From (13.28), the AF major maxima occur when $\psi = \pm m 2\pi$ $(m = 0,1,2\ldots)$:

$$\psi = kd\cos\theta_m + \beta = \pm m 2\pi . \qquad (13.36)$$

For the 0th order maximum ($m = 0$) to be at $\theta_0 = 90°$,

$$\boxed{\beta = 0}. \qquad (13.37)$$

The uniform linear array has its maximum radiation at θ = 0°, if all array elements are fed in phase.

To ensure that there are no major maxima in other directions (grating lobes), the separation between the elements d must be smaller than the wavelength:

$$d < \lambda. \qquad (13.38)$$

In fact, a value of $d \approx \lambda / 2$ ensures that the side-lobe level of the broadside array never exceeds the level of the minor lobes. With $\beta = 0$, we have $\psi(\theta) = kd\cos\theta$, where $kd = 2\pi d / \lambda$. If $d \approx \lambda / 2$, then $\psi(\theta) \approx \pi\cos\theta$. Since $-1 \le \cos\theta \le 1$, $|\psi(\theta)| \le \pi$. The plot in p. 15 shows that, in this case, the AF has only one major lobe and some minor lobes (depending on N).

To illustrate the appearance of additional major maxima with $AF_n = 1$, consider the case of $d = \xi\lambda$, where $\xi \ge 1$. Then the elemental phase is

$$\psi_{(\beta=0)} = kd\cos\theta = \frac{2\pi}{\lambda}\xi\lambda\cos\theta = 2\pi\xi\cos\theta. \qquad (13.39)$$

The condition for a major maximum ($AF_n = 1$) from (13.36) requires that

$$\psi_m = 2\pi\xi\cos\theta = m2\pi, \quad m = 0, \pm1, \pm2\ldots \qquad (13.40)$$

Aside from $\theta_0 = \pi/2$, there are other major maxima, describing *grating lobes*:

$$\theta_g = \arccos(m/\xi), \quad m = \pm1, \pm2\ldots. \qquad (13.41)$$

As long as $m \le \xi$ (remember that $\xi \ge 1$), real-valued solutions for θ_g exist, and grating lobes are present.

If, for example, $d = \lambda$ $(\xi = 1)$, equation (13.41) results in two additional major lobes at

$$\theta_g = \arccos(\pm1) \Rightarrow \theta_{g_{1,2}} = 0°, 180°.$$

The resulting *AF* is illustrated in figure (b) above.

If $d = 2\lambda$ $(\xi = 2)$, equation (13.41) results in four additional major lobes at

$$\theta_g = \arccos(\pm0.5, \pm1) \Rightarrow \theta_{g_{1,2,3,4}} = 0°, 60°, 120°, 180°.$$

If $d = 1.25\lambda$ $(\xi = 1.25)$, then $\theta_g = \arccos(\pm0.8) \Rightarrow \theta_{g_{1,2}} \approx 37°, 143°$.

5. Ordinary End-fire Array

An end-fire array (EFA) has maximum radiation along the axis of the array $(\theta = 0°, 180°)$. Usually, the array is required to radiate only in one direction – either $\theta = 0°$ or $\theta = 180°$. For *AF* main lobe (*m* = 0) at $\theta_0 = 0°$,

$$\psi = kd\cos\underbrace{\theta_0}_{0} + \beta = kd + \beta = m2\pi = 0, \qquad (13.42)$$

$$\Rightarrow \boxed{\beta = -kd, \text{ for } \theta_0 = 0°}. \qquad (13.43)$$

For *AF* main lobe (*m* = 0) at $\theta_0 = 180°$,

$$\psi = kd\cos\underbrace{\theta_0}_{180°} + \beta = -kd + \beta = 0,$$

$$\Rightarrow \boxed{\beta = kd, \;\; \text{for} \;\; \theta_0 = 180°}. \tag{13.44}$$

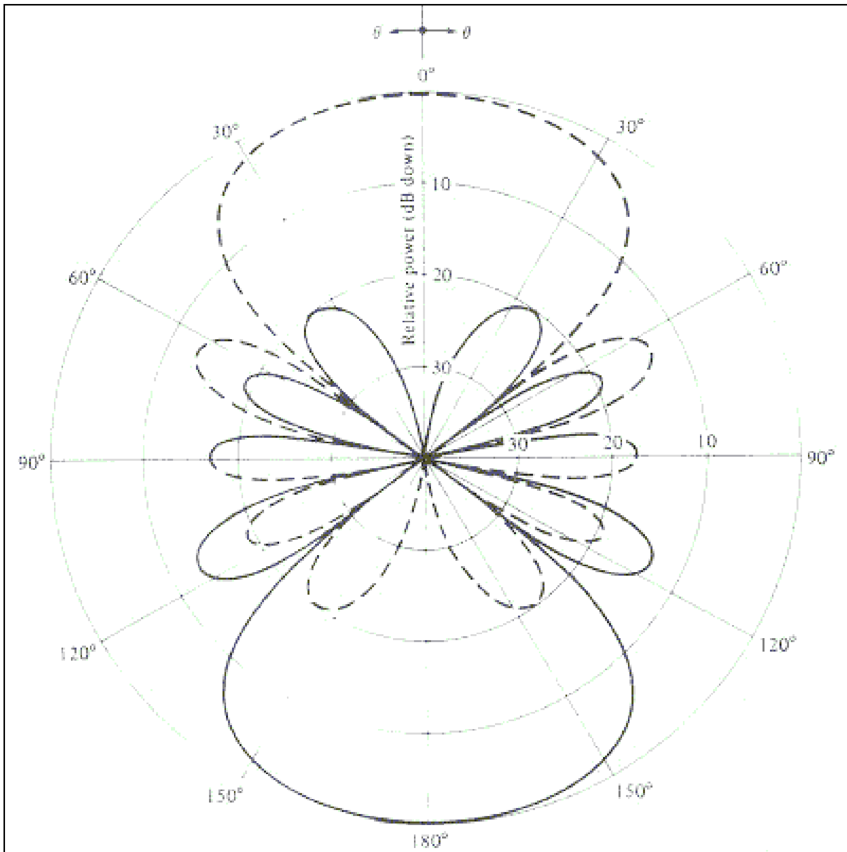
If the element separation is multiple of a wavelength, $d = n\lambda$, then in addition to the end-fire maxima there also exists a major maximum (grating lobe) in the broadside direction ($\theta_m = 90°$). As with the broadside array, to avoid grating lobes, the maximum spacing between the element should be less than λ:

$$d < \lambda.$$

In fact, a value of $d \approx \lambda / 4$ ensures that the side-lobe level of the ordinary EFA will never exceed the level of the minor lobes. For example, for the EFA, we have $\psi(\theta) = kd(\cos\theta - 1)$ if $\theta_0 = 0°$. If $d \approx \lambda / 4$, then $\psi(\theta) \approx 0.5\pi(\cos\theta - 1)$. Since $-1 \le \cos\theta \le 1$, $|\psi(\theta)| \le \pi$, and the AF has only one major lobe and some minor lobes.

Homework: Show that an end-fire array with $d = \lambda / 2$ has 2 maxima when $\beta = -kd$: not only at $\theta = 0°$ but also at $\theta = 180°$.

AF pattern of an *EFA*: $N = 10$, $d = \lambda / 4$



——— $\beta = kd$ (θ_0=180°)

– – – – $\beta = -kd$ (θ_0=0°)

Fig. 6-11, p. 270, Balanis

6. Phased (Scanning) Arrays

It was already shown that the 0th order maximum (m=0) of AF_n occurs when

$$\psi = kd\cos\theta_0 + \beta = 0. \qquad (13.45)$$

This gives the relation between the direction of the main beam θ_0 and the phase difference β. Therefore, the direction of the main beam can be controlled by β. This is the basic principle of ***electronic scanning*** for phased arrays.

When the scanning is required to be continuous, the feeding system must be capable of continuously varying the progressive phase β between the elements. This is accomplished by ferrite or diode (varactor) phase shifters.

Example: Derive the values of the progressive phase shift β as dependent on the direction of the main beam θ_0 for a uniform linear array with $d = \lambda / 4$.

From equation (13.45):

$$\beta = -kd\cos\theta_0 = -\frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\theta_0 = -\frac{\pi}{2}\cos\theta_0$$

θ_0	β
0°	−90°
60°	−45°
120°	+45°
180°	+90°

The approximate *HPBW* of a scanning array is obtained using (13.31) with $\beta = -kd\cos\theta_0$:

$$\theta_{h1,2} \approx \arccos\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right]. \qquad (13.46)$$

The total beamwidth is

$$HPBW = \theta_{h1} - \theta_{h2}, \tag{13.47}$$

$$HPBW \approx \arccos\left[\frac{\lambda}{2\pi d}\left(kd\cos\theta_0 - \frac{2.782}{N}\right)\right] - \arccos\left[\frac{\lambda}{2\pi d}\left(kd\cos\theta_0 + \frac{2.782}{N}\right)\right] \tag{13.48}$$

Since $k = 2\pi/\lambda$,

$$\boxed{HPBW \approx \arccos\left(\cos\theta_0 - \frac{2.782}{Nkd}\right) - \arccos\left(\cos\theta_0 + \frac{2.782}{Nkd}\right)}. \tag{13.49}$$

We can also use the substitution $N = (L+d)/d$ to obtain

$$\boxed{\begin{aligned} HPBW \approx \arccos\left[\cos\theta_0 - 0.443\left(\frac{\lambda}{L+d}\right)\right] - \\ \arccos\left[\cos\theta_0 + 0.443\left(\frac{\lambda}{L+d}\right)\right]. \end{aligned}} \tag{13.50}$$

Here, L is the length of the array.

Be aware that the equations in (13.49) and (13.50) can be used to calculate the *HPBW* of an array as long as it is not an end-fire array. End-fire arrays have circularly symmetric beams around the end-fire direction, in which case

$$HPBW \approx 2\arccos\left(1 - \frac{2.782}{Nkd}\right). \tag{13.51}$$