FREQUENCY RESPONSE

* We have already observed that capacitors and inductors have different impedances for sinusoids of different frequencies.

* We now provide a framework for analyzing this effect for more general signals

* A linear (time-invariant) circuit changes the amplitude + phase of a sinusoid

* Moreover the amount of change varies with frequency

* Controlling these variations is the key to building filters for stereo equipment, radio reception, cell phones, biomedical equipment, etc.

* Note "Frequency response" refers to steady-state behaviour

Basic Definitions

If $v_{in}(t) = A\cos(\omega t)$

$v_{out}(t) = B\cos(\omega t + \phi)$

then.

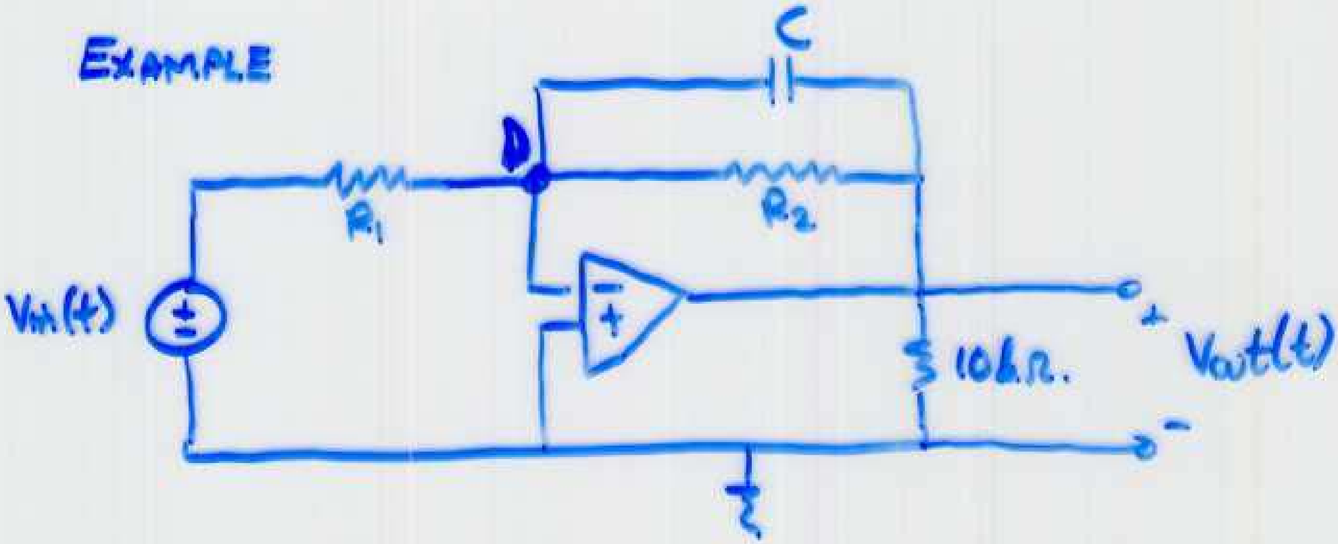
$$\text{gain} = B/A.$$

$$\text{phase shift} = \phi - 0 = \phi$$

* The gain and phase can be combined to form a single complex number $\underline{H} = \frac{B}{A} e^{j\phi}$
* This is actually the ratio of the phasors for vin(t) and vout(t), ie $\underline{H} = \frac{\underline{V}_{out}}{\underline{V}_{in}}$
* The value of $\underline{H}$ is different for different frequencies. Make this explicit by $\underline{H}(\omega)$
* $$\text{Gain} = |\underline{H}(\omega)|$$

$$\text{phase shift} = \measuredangle\, \underline{H}(\omega)$$

EXAMPLE



Let $v_{in}(t) = A \quad \omega t$

then its phasor is $\underline{V}_{in} = Ae^{j0} = A$.

Using KCL at node D, and phasor analysis

(sum currents going in, ideal op amp $\Rightarrow V_D = 0$)

$$\Rightarrow \quad \underline{H}(\omega) = \frac{\underline{V}_{out}}{\underline{V}_{in}} = \frac{-R_2/R_1}{1 + j\omega C R_2}$$

$$\text{gain} = \frac{R_2/R_1}{\sqrt{1 + \omega^2 C^2 R_2^2}}$$

$$\text{phase shift} = \pi - \tan^{-1}(\omega C R_2)$$

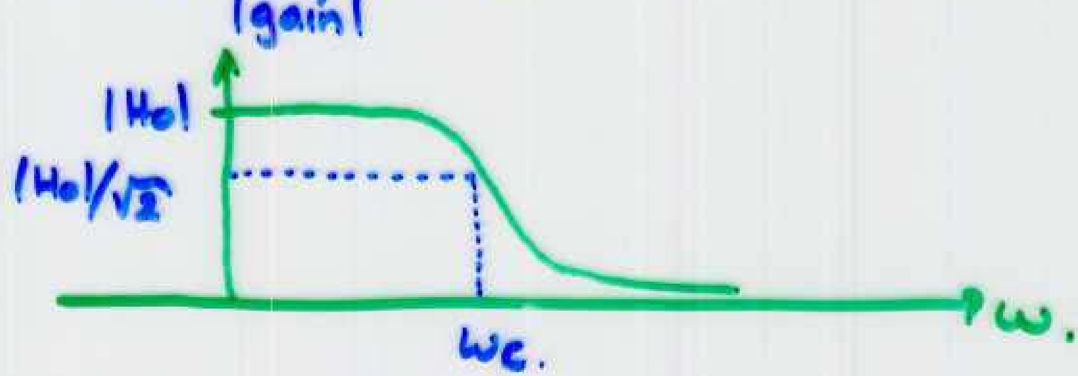
* This analysis is useful because it applies to all frequencies, not just one.

What we have just seen is an example of a first order low-pass filter

$$H(\omega) = \frac{H_0}{1 + j\,\omega/\omega_0}$$

Here $H_0 = -R_2/R_1$, $\omega_0 = \frac{1}{R_2 C}$

Plot gain. (sketch).



At low frequencies, gain is large

At high frequencies, gain is small

ω_c is one way to measure the "bandwidth" of the filter; it tells us which frequencies are small + which ones are large

What is the significance of ω_c?

Consider case where $H_0 = 1$

If $v_{in}(t) = A \sin \omega_c t$

then $v_{out}(t) = \frac{A}{\sqrt{2}} \sin(\omega_c t + \theta)$

The average power of $v_{in}(t)$ is energy per unit time

$$\frac{1}{T} \int_0^T v_{in}^2(t)\, dt. \quad , \text{where } T = \frac{2\pi}{\omega_c}$$

$$= \frac{A^2}{T} \int_0^T \sin^2(\omega_c t)\, dt$$

$$= \frac{A^2}{2T} \int_0^T 1 - \cos(2\omega_c t)\, dt.$$

$$= \frac{A^2}{2} - \underbrace{\frac{A^2}{2T} \int_0^T \cos(2\omega_c t)\, dt}_{\text{Integral of 2 periods of a cosine} = 0}$$

$$\Rightarrow \quad P_{in} = \frac{A^2}{2}$$

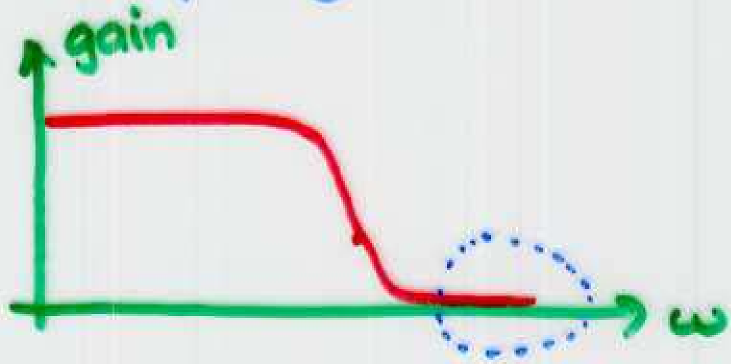
The average power of the output, $P_{out} = \frac{A^2}{4} = \frac{P_{in}}{2}.$

$\Rightarrow \omega_c$ is the point at which half the power of the sinusoid is attenuated

Bode Plots

* These are simply special plots of gain + phase versus frequency
* The plots are constructed in such a way that they
    * illustrate how the circuit operates
    * can be approximated quite quickly for "back of the envelope" type computation

* One way to see what is happening is to simply plot gain vs frequency



* However this is of limited use, because the dashed area is important, but hard to see.
* The dashed area may also be important for a broad range of frequencies.

* How can we plot all of these things on one graph?
Take logarithms

* Define logarithmic gain

$$\text{log. gain} = 20 \log_{10}\left(|H(\omega)| \right)$$

* Unit is "decibel", abbrev. dB.

log. gain often called "gain in dB"

* where does the name come from?

* Power ratios.

Given two powers, P_1, P_2

both P_1/P_2 and $\log_{10}(P_1/P_2)$ measure relative size

The units of $\log_{10}(P_1/P_2)$ are "bels", after

Alexander Graham Bell.

* The log gain is in decibels because.

$$10 \log_{10}\left(\frac{P_{out}}{P_{in}}\right) = 10 \log_{10}\left(|H(\omega)|^2 \right)$$

$$=$$

A **Bode Plot** consists of two graphs

a) gain in dB vs $\log_{10}(\omega)$

b) phase shift in degrees vs $\log_{10}(\omega)$

* Choice of logarithmic measures allows us to capture all relevent information on one graph.

* it also allows us to approximate the plots "by hand"

Drawing Bode Plots by Hand

Let us try this for the simple first-order low-pass filter

$$\underline{H}(\omega) = \frac{1}{1 + j(\omega/\omega_0)}$$

The gain is

$$\text{gain} = |\underline{H}(\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

The log. gain is

$$\text{log. gain} = -20 \log_{10}\left(\sqrt{1 + (\omega/\omega_0)^2}\right)$$

To get an idea of what this looks like, observe that for $\omega \ll \omega_0$,

$$1 + (\omega/\omega_0)^2 \approx 1$$

$$\Rightarrow \quad \text{gain in dB} \approx 0\,\text{dB}$$

This corresponds to a horizontal line

For large ω, i.e. $\omega \gg \omega_0$.

$$1 + (\omega/\omega_0)^2 \approx (\omega/\omega_0)^2.$$

$$\Rightarrow \text{gain in dB} \approx -20\log_{10}\left(\frac{\omega}{\omega_0}\right)$$

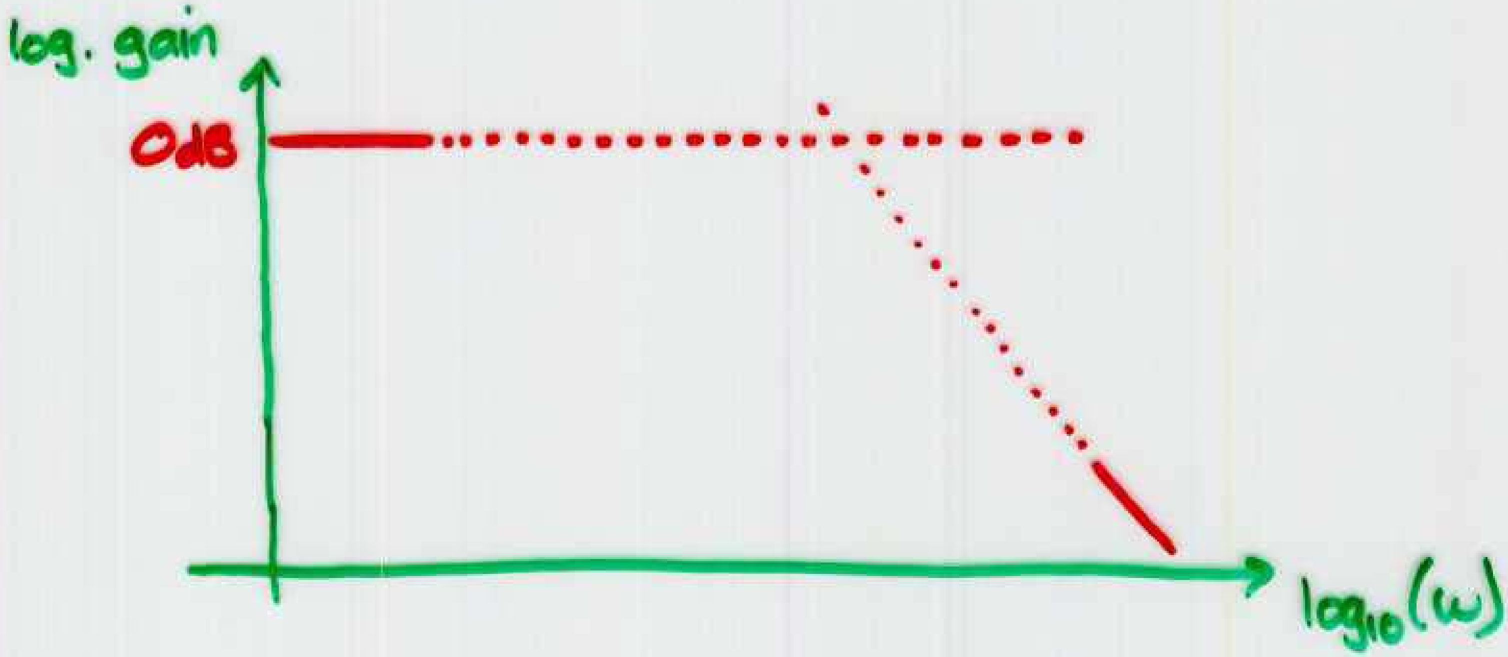
$$= +20\log_{10}\omega_0 - 20\log_{10}\omega.$$

This is a straight line with slope -20 as a function of $\log_{10}\omega$.

This slope is often said to be -20dB per decade,

where 'decade' is a factor of 10 in frequency

So what do we know?



where do these asymptotic approximations intersect?

$$0 = 20\log_{10}(\omega_0) - 20\log_{10}(\omega)$$

$$\Rightarrow \omega = \omega_0.$$

What is the value of the gain at ω_0?

$$\text{gain} = \frac{1}{\sqrt{1+\left(\frac{\omega_0}{\omega_0}\right)^2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \text{log. gain} = 20\log_{10}\left(\frac{1}{\sqrt{2}}\right) \approx -3\text{dB}$$

Now what about the phase?

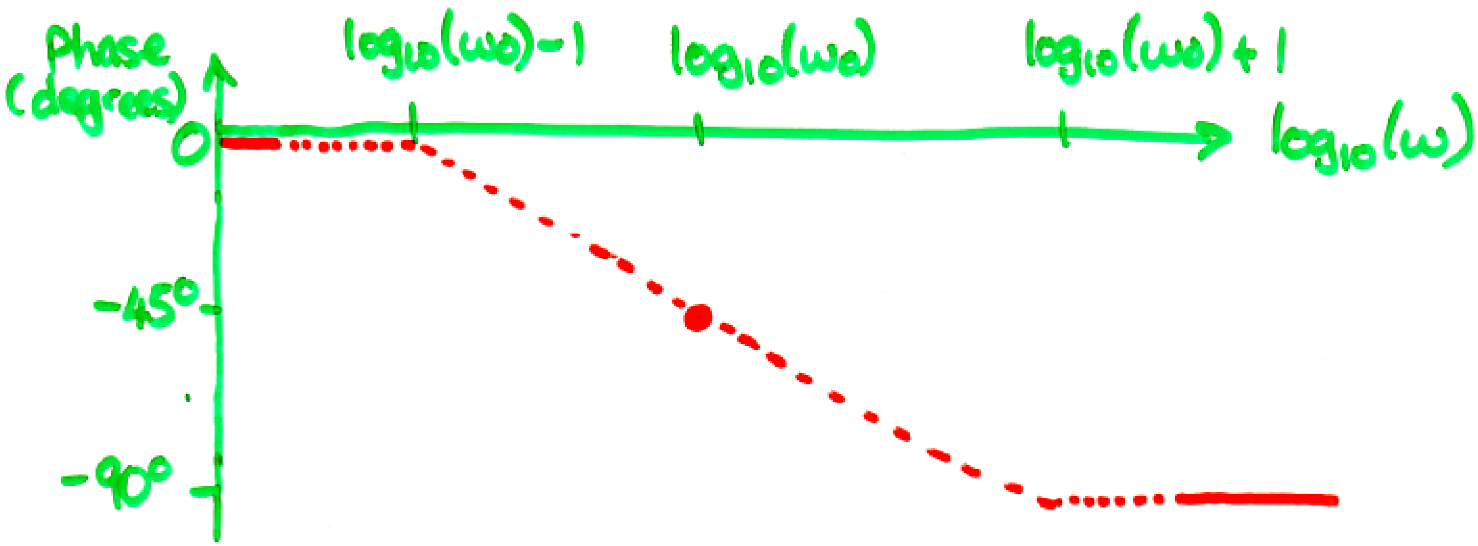
$$\underline{H}(\omega) = \frac{1}{1 + j\,\omega/\omega_0}$$

$$\Rightarrow \angle\underline{H}(\omega) = -\mathrm{atan}(\omega/\omega_0)$$

for $\omega \ll \omega_0$, $\angle\underline{H}(\omega) \approx -\mathrm{atan}(0) = 0^\circ$

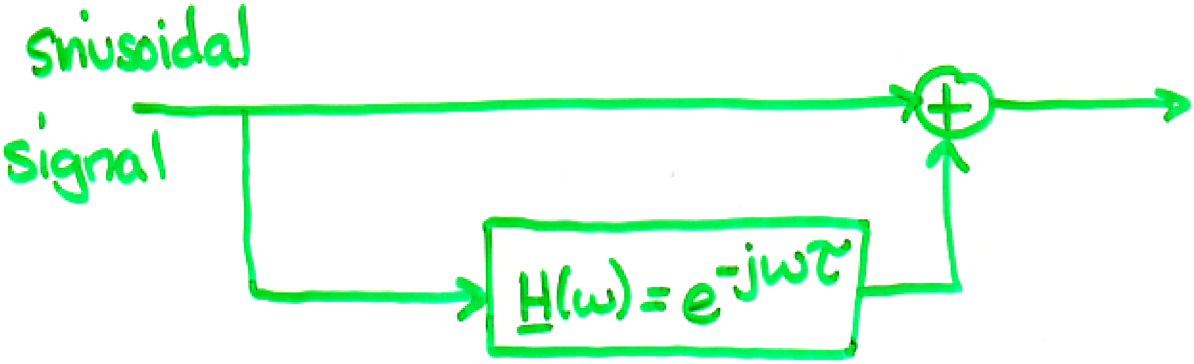
for $\omega \gg \omega_0$, $\angle\underline{H}(\omega) \approx -\mathrm{atan}(\infty) = -90^\circ$

for $\omega = \omega_0$ $\angle\underline{H}(\omega) = -\mathrm{atan}(1) = -45^\circ$



Accurate to within 6° !

Why is phase important?



Note that $|\underline{H}(\omega)| = 1$

At what frequencies is the output zero?