EE3CL4: Introduction to Linear Control Systems
Section 8: Frequency Domain Techniques

Tim Davidson

McMaster University

Winter 2020
Transfer functions

Frequency Response

Plotting the freq. resp.

Mapping Contours

Nyquist's criterion

Nyquist's Stability Criterion as a Design Tool

Relative Stability

Gain margin and Phase margin

Relationship to transient response
Transfer Functions: A Quick Review

- Consider a transfer function
  \[ G(s) = K \prod_i \frac{s + z_i}{\prod_j \left(s + p_j\right)} \]

- Zeros: \(-z_i\); Poles: \(-p_j\)

- Note that \(s + z_i = s - (-z_i)\),

- This is the vector from \(-z_i\) to \(s\)

- Magnitude:
  \[ |G(s)| = |K| \prod_i \left| \frac{s + z_i}{\prod_j \left(s + p_j\right)} \right| = |K| \frac{\text{prod. dist’s from zeros to } s}{\text{prod. dist’s from poles to } s} \]

- Phase:
  \[ \angle G(s) = \angle K + \text{sum angles from zeros to } s - \text{sum angles from poles to } s \]
Frequency Response

- For a stable, linear, time-invariant (LTI) system, the steady-state response to a sinusoidal input is a sinusoid of the same frequency but possibly different magnitude and different phase.

- Sinusoids are the eigenfunctions of convolution.

- If input is $A \cos(\omega_0 t + \theta)$ and steady-state output is $B \cos(\omega_0 t + \phi)$, then the complex number $B/Ae^{i(\phi-\theta)}$ is called the frequency response of the system at frequency $\omega_0$. 
Frequency Response, II

- If a stable LTI system has a transfer function $G(s)$, then the frequency response at $\omega_0$ is $G(s)|_{s=j\omega_0}$

- What if the system is unstable?
Plotting the frequency response

- For each $\omega$, $G(j\omega)$ is a complex number.

- How should we plot it?

  - $G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)}$
    Plot $|G(j\omega)|$ versus $\omega$, and $\angle G(j\omega)$ versus $\omega$

  - Plot $20 \log_{10}(|G(j\omega)|)$ versus $\log_{10}(\omega)$, and $\angle G(j\omega)$ versus $\log_{10}(\omega)$

  - $G(j\omega) = \text{Re}(G(j\omega)) + j \text{Im}(G(j\omega))$
    Plot the curve $(\text{Re}(G(j\omega)), \text{Im}(G(j\omega)))$ on an “$x$–$y$” plot
    Equiv. to curve $|G(j\omega)| e^{j\angle G(j\omega)}$ as $\omega$ changes (polar plot)
Polar plot, example 1

Let's consider the example of an RC circuit

\[ G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{1+sRC} \]

\[ G(j\omega) = \frac{1}{1+j\omega/\omega_1}, \text{ where } \omega_1 = 1/(RC). \]

\[ G(j\omega) = \frac{1}{1+(\omega/\omega_1)^2} - j\frac{\omega/\omega_1}{1+(\omega/\omega_1)^2} \]

\[ G(j\omega) = \frac{1}{\sqrt{1+(\omega/\omega_1)^2}} e^{-j\tan(\omega/\omega_1)} \]
Polar plot, example 1

- \( G(j\omega) = \frac{1}{1+(\omega/\omega_1)^2} - j\frac{\omega/\omega_1}{1+(\omega/\omega_1)^2} \)

- \( G(j\omega) = \frac{1}{\sqrt{1+(\omega/\omega_1)^2}} e^{-j\text{atan}(\omega/\omega_1)} \)
Consider $G(s) = \frac{K}{s(s\tau+1)}$.

- Poles at origin and $s = -1/\tau$.
- To use geometric insight to plot polar plot, rewrite as $G(s) = \frac{K/\tau}{s(s+1/\tau)}$.

Then $|G(j\omega)| = \frac{K/\tau}{|j\omega| |j\omega + 1/\tau|}$ and $\angle G(j\omega) = -\angle(j\omega) - \angle(j\omega + 1/\tau)$.
Polar plot, ex. 2, \( G(s) = \frac{K/\tau}{s(s+1/\tau)} \)

- When \( \omega \to 0^+ \), \( |G(j\omega)| \to \infty \), \( \angle G(j\omega) \to -90^\circ \) from below

  - Tricky

- To get a better feel, write \( G(j\omega) = \frac{-K\omega^2\tau}{\omega^2+\omega^4\tau^2} - j\frac{\omega K}{\omega^2+\omega^4\tau^2} \)

  - Hence, as \( \omega \to 0^+ \), \( G(j\omega) \to -K\tau - j\infty \)

- As \( \omega \) increases, distances from poles to \( j\omega \) increase.

  - Hence \( |G(j\omega)| \) decreases

- As \( \omega \) increases, angle from pole at \(-1/\tau\) increases.

  - Hence \( \angle G(j\omega) \) becomes more negative
Polar plot, ex. 2, \( G(s) = \frac{K/\tau}{s(s+1/\tau)} \)

- When \( \omega = 1/\tau \), \( G(j\omega) = (K/\tau)/((1/\tau)(\sqrt{2}/\tau)) e^{-j(90^\circ + 45^\circ)} \)
i.e., \( G(j\omega)|_{\omega=1/\tau} = (K\tau/\sqrt{2})e^{-j135^\circ} \)

- As \( \omega \) approaches \(+\infty\), both distances from poles get large. Hence \( |G(j\omega)| \rightarrow 0 \)

- As \( \omega \) approaches \(+\infty\), angle from \(-1/\tau\) approaches \(-90^\circ\) from below. Hence \( \angle G(j\omega) \) approaches \(-180^\circ\) from below.
Polar plot, ex. 2, $G(s) = \frac{K/\tau}{s(s+1/\tau)}$

Summary

- As $\omega \to 0^+$, $G(j\omega) \to -K\tau - j\infty$
- As $\omega$ increases,
  $|G(j\omega)|$ decreases, $\angle G(j\omega)$ becomes more negative
- When $\omega = 1/\tau$, $G(j\omega) = (K/\sqrt{2})e^{-j135^\circ}$
- As $\omega$ approaches $+\infty$,
  $G(j\omega)$ approaches zero from angle $-180^\circ$
Polar plot, ex. 2, \( G(s) = \frac{K/\tau}{s(s+1/\tau)} \)
Bode Diagrams

- Bode magnitude plot
  \[ 20 \log_{10} |G(j\omega)| \text{ against } \log_{10} \omega \]

- Bode phase plot
  \[ \angle G(j\omega) \text{ against } \log_{10} \omega \]

- In 2CJ4 we developed rules to help sketch these plots
- In this course we will use these sketches to design controllers
Sketching Bode Diagrams

• Consider generic transfer function of LTI system

\[
G(s) = \frac{K \prod_i (s + z_i) \prod_k (s^2 + 2\zeta_k \omega_{n,k} s + \omega_{n,k}^2)}{s^N \prod_j (s + p_j) \prod_r (s^2 + 2\zeta_{d,k} \omega_{n,d,r} s + \omega_{n,d,r}^2)}
\]

where \( z_i \) and \( p_j \) are real.

• Unfortunately, not in the form that we are used to for Bode diagrams

• Divide numerator by \( \prod_i z_i \prod_k \omega_{n,k}^2 \)

• Similarly for denominator

• Then if \( \tilde{K} = K \prod_i z_i \prod_k \omega_{n,k}^2 / (\prod_j p_j \prod_r \omega_{n,d,r}^2) \),

\[
G(s) = \frac{\tilde{K} \prod_i (1 + s/z_i) \prod_k (1 + 2\zeta_k (s/\omega_{n,k}) + (s/\omega_{n,k})^2)}{s^N \prod_j (1 + s/p_j) \prod_r (1 + 2\zeta_{d,k} (s/\omega_{n,d,r}) + (s/\omega_{n,d,r})^2)}
\]
Sketching Bode Diagrams, II

• Now, frequency response can be written as:

\[ G(j\omega) = \frac{\tilde{K} \prod_i (1 + j\omega/z_i)}{(j\omega)^N \prod_j (1 + j\omega/p_j)} \times \frac{\prod_k (1 + 2\zeta_k(j\omega/\omega_{n,k}) + (j\omega/\omega_{n,k})^2)}{\prod_r (1 + 2\zeta_{d,k}(j\omega/\omega_{nd,r}) + (j\omega/\omega_{nd,r})^2)} \]

• Four key components:
  • Gain, \( \tilde{K} \)
  • Poles (or zeros) at origin
  • Poles and zeros on real axis
  • Poles and zeros in complex conjugate pairs

• Each contributes to the Bode Diagram
Bode Magnitude diagram

\[ G(j\omega) = \frac{\tilde{K} \prod_i (1 + j\omega/z_i)}{(j\omega)^N \prod_j (1 + j\omega/p_j)} \times \frac{\prod_k (1 + 2\zeta_k (j\omega/\omega_{n,k}) + (j\omega/\omega_{n,k})^2)}{\prod_r (1 + 2\zeta_d,k (j\omega/\omega_{nd,r}) + (j\omega/\omega_{nd,r})^2)} \]

- Bode Magnitude diagram:
  \[ 20 \log_{10} |G(j\omega)| \text{ against } \log_{10} \omega \]

- \[ 20 \log_{10} |G(j\omega)| \text{ is} \]
  
  Sum of \[ 20 \log_{10} \text{ of components of numerator} \]
  
  – sum of \[ 20 \log_{10} \text{ of components of denominator} \]
Components for magnitude

\[ G(j\omega) = \frac{\tilde{K} \prod_i (1 + j\omega/z_i)}{(j\omega)^N \prod_j (1 + j\omega/p_j)} \times \frac{\prod_k (1 + 2\zeta_k (j\omega/\omega_{n,k}) + (j\omega/\omega_{n,k})^2)}{\prod_r (1 + 2\zeta_{d,k} (j\omega/\omega_{nd,r}) + (j\omega/\omega_{nd,r})^2)} \]

- Poles at origin: slope starts at \(-20N \text{ dB/dec}\)
- Gain \(|\tilde{K}|\) incorporated in position of that sloping line
- First order component in numerator:
  increase slope by 20 dB/dec at \(\omega = z_i\)
- First order component in denominator:
  decrease slope by 20 dB/dec at \(\omega = p_j\)
- Second order components:
  increase or decrease slope by 40 dB/dec at \(\omega = \omega_n\)
Bode Phase Diagram

\[ G(j\omega) = \frac{\tilde{K} \prod_i (1 + j\omega/z_i)}{(j\omega)^N \prod_j (1 + j\omega/p_j)} \times \frac{\prod_k (1 + 2\zeta_k (j\omega/\omega_{n,k}) + (j\omega/\omega_{n,k})^2)}{\prod_r (1 + 2\zeta_{d,k} (j\omega/\omega_{nd,r}) + (j\omega/\omega_{nd,r})^2)} \]

- Bode Phase Diagram
  \[ \angle G(j\omega) \text{ against } \log_{10} \omega \]

- \( \angle G(j\omega) \) is
  
  Sum of phases of components of numerator
  – sum of phases of components of denominator
Components

\[ G(j\omega) = \frac{\tilde{K} \prod_i (1 + j\omega/z_i)}{(j\omega)^N \prod_j (1 + j\omega/p_j)} \]
\[ \times \frac{\prod_k (1 + 2\zeta_k(j\omega/\omega_{n,k}) + (j\omega/\omega_{n,k})^2)}{\prod_r (1 + 2\zeta_d,k(j\omega/\omega_{nd,r}) + (j\omega/\omega_{nd,r})^2)} \]

- Phase of \( \tilde{K} \)
- Poles at origin: \(-N90^\circ\)
- First order component in numerator: linear phase change of \(+90^\circ\) over \( \omega \in [z_i/10, 10z_i] \)
- First order component in denominator: linear phase change of \(-90^\circ\) over \( \omega \in [p_j/10, 10p_j] \)
- Second order components: phase change of \( \pm180^\circ\) around \( \omega = \omega_n \)
Graphically

Table 8.3 Asymptotic Curves for Basic Terms of a Transfer Function

| Term                  | Magnitude $20 \log |G| \ [\text{dB}]$ | Phase $\phi(\omega)$ |
|-----------------------|--------------------|---------------------|
| 1. Gain, $G(j\omega) = K$ |                     |                     |
|                      | $20 \log K \ [\text{dB}]$ | $\phi(\omega)$     |
|                      | 0                   | 0                   |
|                      | $-20$               | $-90^\circ$         |
|                      | $-40$               | $-90^\circ$         |
|                      | $20$                | $45^\circ$          |
|                      | $40$                | $0^\circ$           |

2. Zero, $G(j\omega) = 1 + j\omega/\omega_1$

|                      |                     |                     |
|                      | $0.1 \omega_1$      | $0.1 \omega_1$      |
|                      | $\omega_1$          | $\omega_1$          |
|                      | $10 \omega_1$       | $10 \omega_1$       |

Graphically
3. Pole,
\[ G(j\omega) = \frac{1}{1 + j\omega/\omega_1} \]

4. Pole at the origin,
\[ G(j\omega) = \frac{1}{j\omega} \]

5. Two complex poles, 
\[ 0.1 < \xi < 1, \quad G(j\omega) = \frac{1 + j2\xi u - u^2}{\omega/\omega_n} \]
\[ u = \omega/\omega_n \]
Accuracy of Bode Sketches

Isolated first order pole (analogous for zero)
Accuracy of Bode Sketches

Isolated complex conjugate pair of poles

\[ u = \frac{\omega}{\omega_n} = \text{Frequency ratio} \]

\[ 20 \log |G| \]

\[ \zeta = \{0.05, 0.10, 0.15, 0.20, 0.25\} \]

\[ \omega = \{0.3, 0.4, 0.5, 0.6, 0.8, 1.0\} \]
Accuracy of Bode Sketches

Isolated complex conjugate pair of poles

\[ \mu = \frac{\omega}{\omega_n} = \text{Frequency ratio} \]
Example

\[ G(j\omega) = \frac{5(1 + j\omega/10)}{j\omega(1 + j\omega/2)(1 + 0.6(j\omega/50) + (j\omega/50)^2)} \]
Example

\[ G(j\omega) = \frac{5(1 + j\omega/10)}{j\omega(1 + j\omega/2)(1 + 0.6(j\omega/50) + (j\omega/50)^2)} \]
Introduction

- We have seen techniques that determine stability of a system:
  - Routh-Hurwitz
  - root locus

- However, both of them require a model for the plant

- Today: frequency response techniques
  - Although they work best with a model
  - For an open-loop stable plant, they also work with measurements

- Key result: Nyquist’s stability criterion

- Design implications: Bode techniques based on gain margin and phase margin
To determine the stability of the system we need to examine the characteristic equation:

\[ F(s) = 1 + L(s) = 0 \]

where \( L(s) = G_c(s)G(s)H(s) \).

The key result involves mapping a closed contour of values of \( s \) to a closed contour of values of \( F(s) \).

We will investigate the idea of mappings first.
Simple example

- Set $F(s) = 2s + 1$
- Map the square in the "s-plane" to the contour in the "$F(s)$-plane"
Area enclosed

- How might we define area enclosed by a closed contour?
- We will be perfectly rigorous, but will go against mathematical convention
- Define area enclosed to be that to the right when the contour is traversed clockwise
- What you see when moving clockwise with eyes right
- Sometimes we say that this area is the area “inside” the clockwise contour
- Notions of “enclosed” or “inside” will be applied to contours in the s-plane
In the $F(s)$-plane, we will be interested in the notion of encirclement of the origin.

A contour is said to encircle the origin in the clockwise direction, if the contour completes a $360^\circ$ revolution around the origin in the clockwise direction.

A contour is said to encircle the origin in the anti-clockwise direction, if the contour completes a $360^\circ$ revolution around the origin in the anti-clockwise direction.

We will say that an anti-clockwise encirclement is a “negative” clockwise encirclement.
Example with rational $F(s)$

- A mapping for $F(s) = \frac{s}{s+2}$
- Note that $s$-plane contour encloses the zero of $F(s)$
- How many times does the $F(s)$-plane contour encircle the origin in the clockwise direction?
Cauchy’s Theorem

- Nyquist’s Criterion is based on Cauchy’s Theorem:
  - Consider a rational function $F(s)$
  - If the clockwise traversal of a contour $\Gamma_s$ in the $s$-plane encloses $Z$ zeros and $P$ poles of $F(s)$ and does not go through any poles or zeros
  - then the corresponding contour in the $F(s)$-plane, $\Gamma_F$ encircles the origin $N = Z - P$ times in the clockwise direction

- A sketch of the proof later.
- First, some examples
Example 1

- A mapping for $F(s) = \frac{s}{s + 1/2}$
- s-plane contour encloses a zero and a pole
- Theorem suggests no clockwise encirclements of origin of $F(s)$-plane
- This is what we have!
• s-plane contour encloses 3 zeros and a pole
• Theorem suggests 2 clockwise encirclements of the origin of the $F(s)$-plane
Example 3

- $s$-plane contour encloses one pole
- Theorem suggests -1 clockwise encirclements of the origin of the $F(s)$-plane
- That is, one anti-clockwise encirclement
Informal Justification of Cauchy’s Theorem

- Consider the case of $F(s) = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$
- $\angle F(s_1) = \phi_{z_1} + \phi_{z_2} - \phi_{p_1} - \phi_{p_2}$
- As the contour is traversed the nett contribution from $\phi_{z_1}$ is 360 degrees
- As contour is traversed, the nett contribution from other angles is 0 degrees
- Hence, as contour is traversed, $\angle F(s)$ changes by 360 degrees. One encirclement!
Informal Justification

- Extending this to any number of poles and zeros inside the contour
- For a closed contour, the change in $\angle F(s)$ is $360Z - 360P$
- Hence $F(s)$ encircles origin $Z - P$ times
Cauchy’s Theorem (Review)

- Consider a rational function $F(s)$
- If the clockwise traversal of a contour $\Gamma_s$ in the s-plane encloses $Z$ zeros and $P$ poles of $F(s)$ and does not go through any poles or zeros
- then the corresponding contour in the $F(s)$-plane, $\Gamma_F$ encircles the origin $N = Z - P$ times in the clockwise direction
Nyquist’s goal

- Nyquist was concerned about testing for stability
- How might one use Cauchy Theorem to examine this?
- Perhaps choose $F(s) = 1 + L(s)$, as this determines stability
- Which contour should we use?
Actually, we have to be careful regarding poles and zeros on the $j\omega$-axis, including the origin.

Standard approach is to indent contour so that it goes to the right of any such poles or zeros.
Modified Nyquist contour

Here’s an example for a model like that of the motor in the lab.

![Modified Nyquist contour diagram](attachment:image.png)
Coarse Applic. of Cauchy

- Recall that the zeros of $F(s) = 1 + L(s)$ are the poles of the closed loop.
- Let $P$ denote the number of right half plane poles of $F(s)$.
- The number of right half plane zeros of $F(s)$ is $N + P$, where $N$ is the number of clockwise encirclements of the origin made by the image of Nyquist’s contour in the $F(s)$ plane.
- A little difficult to parse.
- Perhaps we can apply Cauchy’s Theorem in a more sophisticated way.
Towards Nyquist’s Criterion

- $F(s) = 1 + L(s)$, where $L(s)$ is the open loop transfer function
- Encirclement of the origin in $F(s)$-plane is the same as encirclement of $-1$ in the $L(s)$-plane
- This is more convenient, because $L(s)$ is often factorized, and hence we can easily determine $P$
- Now that we are dealing with $L(s)$, $P$ is the number of right-half plane poles of the open loop transfer function
- If we handle the remainder of the components of Cauchy’s theorem carefully we obtain:
Nyquist’s Criterion: Simplified statement

- Consider a unity feedback system with an open loop transfer function $L(s) = G_c(s)G(s)H(s)$, with no z’s or p’s on $j\omega$-axis.
- Let $P_L$ denote the number of poles of $L(s)$ in RHP.
- Consider the Nyquist Contour in the $s$-plane.
- Let $\Gamma_L$ denote image of Nyquist Contour under $L(s)$.
- Let $N_L$ denote the number of clockwise encirclements that $\Gamma_L$ makes of the point $(-1, 0)$.

**Nyquist’s Stability Criterion:**

$$\text{Number of closed-loop poles in RHP} = N_L + P_L$$

- Note that for a stable open loop, the closed-loop is stable if the image of the Nyquist contour does not encircle $(-1, 0)$. 
Ex: \( L(s) = \frac{1000}{(s+1)(s+10)} \) (stable)

- For \( 0 \leq \omega < \infty \):
  - No zeros, two poles.
  - \(|L(0)| = 1000/(1 \times 10) = 100; \angle L(0) = -0 - 0 = 0\)
  - Distances from poles to \( j\omega \) is increasing; hence \(|L(j\omega)|\) is decreasing
  - Angles from poles to \( j\omega \) are increasing; hence \( \angle L(j\omega) \) is decreasing
  - As \( \omega \to \infty \), \(|L(j\omega)| \to 0, \angle L(j\omega) \to -180^\circ \)
- Recall that \( L(-j\omega) = L(j\omega)^* \)
- Remember to examine the \( r \to \infty \) part of the curve
Ex: \( L(s) = \frac{1000}{(s+1)(s+10)} \) (stable)

Note: No encirclements of \((-1, 0)\) \(\implies\) closed loop is stable
Nyquist’s Criterion: Refined statement

Consider a unity feedback system with an open loop transfer function \( L(s) = G_c(s)G(s)H(s) \),

Let \( P_L \) denote the number of poles of \( L(s) \) in open RHP.

Consider the modified Nyquist Contour in the \( s \)-plane looping to the right of any poles or zeros on the \( j\omega \)-axis.

Let \( \Gamma_L \) denote image of mod. Nyquist Contour under \( L(s) \).

Let \( N_L \) denote the number of clockwise encirclements that \( \Gamma_L \) makes of the point \((-1, 0)\).

**Nyquist’s Stability Criterion:**

\[
\text{Number of closed-loop poles in open RHP} = N_L + P_L
\]

Now we can handle open-loop poles and zeros on \( j\omega \)-axis.
Example: Pole of $L(s)$ at origin

- Consider
  \[ L(s) = \frac{K}{s(\tau s + 1)} \]

- Like in servomotor
- Problem with the original Nyquist contour
- It goes through a pole!
- Cauchy’s Theorem does not apply
- Must modify Nyquist Contour to go around pole
- Then Nyquist Criterion can be applied
Example: Pole of $L(s)$ at origin

Now three key aspects of the curve

- Around the origin
- Positive frequency axis; remember negative freq. axis yields conjugate
- At $\infty$
• \( L(s) = \frac{K}{s(\tau s+1)} \)

• Around the origin, \( s = \epsilon e^{j\phi} \), where \( \phi \) goes from \(-90^\circ\) to \(90^\circ\)

• In the \( L(s) \) plane: \( \lim_{\epsilon \to 0} L(\epsilon e^{j\phi}) \)

• This is: \( \lim_{\epsilon \to 0} \frac{K}{\epsilon e^{j\phi}} = \lim_{\epsilon \to 0} \frac{K}{\epsilon} e^{-j\phi} \)
Up positive $j\omega$-axis

- For $0 < \omega < \infty$, $L(j\omega) = \frac{K}{\omega \sqrt{1+\omega^2\tau^2}} e^{-j(90^\circ + \tan(\omega\tau))}$
- For small $\omega$, $L(j\omega)$ is large with phase $-90^\circ$
  Actually, as we worked out in a previous lecture, as $\omega \to 0^+$, $L(j\omega) \to -K\tau - j\infty$
- For large $\omega$, $L(j\omega)$ is small with phase $-180^\circ$
- For $\omega = 1/\tau$, $L(j\omega) = K\tau/\sqrt{2} e^{-j135^\circ}$
For $s = re^{j\theta}$ for large $r$

- For $s = re^{j\theta}$ with large $r$, and $\theta$ from $+90^\circ$ to $-90^\circ$,
- $\lim_{r \to \infty} L(re^{j\theta}) = \frac{K}{\tau r^2} e^{-j2\theta}$

- How many encirclements of $-1$ in $L(s)$ plane? None
- Implies that closed loop is stable for all positive $K$
- Consistent with what we know from root locus (Lab. 2)
Example with open loop RHP pole, proportional control

- Consider $G(s) = \frac{1}{s(s-1)}$
- Essentially the same as plant model for VTOL aircraft example in root locus section
- Consider prop. control, $G_c(s) = K_1$, and $H(s) = 1$.
- Hence, $L(s) = \frac{K_1}{s(s-1)}$
- Observe that $L(s)$ has a pole in RHP; hence $P_L = 1$
Ex. with open loop RHP pole

\[ L(s) = \frac{K_1}{s(s-1)}. \]

For \( s = j\omega \) and \( 0 < \omega < \infty \),

\[
L(j\omega) = \frac{-K_1}{1 + \omega^2} + j \frac{K_1}{\omega(1 + \omega^2)} = \frac{K_1}{\omega \sqrt{1 + \omega^2}} \angle(90^\circ + \arctan(\omega))
\]

- For \( \omega \to 0^+ \), \( L(j\omega) \to -K_1 + j\infty \)
- As \( \omega \) increases, real and imag. parts decrease, imag. part decreases faster
- Equiv. magnitude decreases, phase increases
- For \( \omega \to \infty \), \( L(j\omega) \) is small with angle \(+180^\circ\)
- Conjugate for \(-\infty < \omega < 0\)
- What about when \( s = \epsilon e^{j\theta} \) for \(-90^\circ \leq \theta \leq 90^\circ\)?

\[
L(s) = \frac{K_1}{\epsilon} \angle(-180^\circ - \theta)
\]
Example with open loop RHP pole

- Recall $P_L = 1$
- Number clockwise encirclements of $(-1, 0)$ is 1
- Hence there are two closed loop poles in the RHP for all positive values of $K_1$
- Consistent with root locus analysis
Root locus of $L(s) = \frac{1}{s(s-1)}$
Example with open loop RHP pole, PD control

\[ G(s) = \frac{1}{s(s-1)} \] and \( H(s) = 1. \) \( L(s) = G_C(s)G(s). \)

- In the VTOL aircraft example, showed that closed-loop can be stabilized by lead compensation, \( G_C(s) = \frac{K_c(s+z)}{s+p} \)

- It can also be stabilized by PD comp., \( G_C(s) = K_1(1 + K_2s). \)

(Under the presumption that this can be realized. It can be realized when we have “velocity” feedback.)

- Using the root locus, we can show that when \( K_2 > 0 \) there is a \( K_1 > 0 \) that stabilizes the closed loop (see next page)

- Can we see that in the Nyquist diagram?

- Plot the Nyquist diagram of \( L(s) = G_C(s)G(s), \) where \( G(s) = \frac{K_1}{s(s-1)} \) and \( G_c(s) = 1 + K_2s \)
Root locus analysis

Root locus of \((1 + K_2 s) \frac{1}{s(s-1)}\) for a given \(K_2 > 0\)

- Poles, zero and active sections of real axis

- Complete root locus

Conclusion: For any given \(K_2 > 0\) there is a \(\bar{K}_1 > 0\) such that closed loop is stable for all \(K_1 > \bar{K}_1\). We can find \(\bar{K}_1\) using Routh-Hurwitz
Nyquist diagram of 

\[
\frac{(1 + K_2s)K_1}{s(s-1)}
\]

- Recall that \( P_L = 1 \)
- If \( K_1K_2 > 1 \), there is one anti-clockwise encirc. of \(-1\)
- In that case, number closed-loop poles in RHP is \(-1 + 1 = 0\) and the closed loop is stable
- Consistent with root locus analysis; but gives \( \bar{K}_1 = 1/K_2 \) directly
One more example

\[ L(s) = \frac{K(s - 2)}{(s + 1)^2} \]

Open loop is stable, but has non-minimum phase (RHP) zero

\[ L(j\omega) = \frac{K\sqrt{\omega^2 + 4}}{\omega^2 + 1} \angle(180^\circ - \tan(\omega/2) - 2\tan(\omega)) \]

- For small positive \( \omega \), \( L(j\omega) \approx 2K\angle180^\circ \)
- For large positive \( \omega \), \( L(j\omega) \approx \frac{K}{\omega} \angle -90^\circ \)
- In between, phase decreases monotonically, \( 180^\circ \rightarrow -90^\circ \). magnitude decreases monotonically (Bode mag dia.)
- \( L(j\omega) = \frac{2K(2\omega^2-1+j\omega(5-\omega^2))}{(1+\omega^2)^2} \); When \( \omega = \sqrt{5} \), \( L(j\omega) = K/2 \)
- When \( s = re^{j\theta} \) with \( r \rightarrow \infty \) and \( \theta : 90^\circ \rightarrow -90^\circ \), \( L(s) \rightarrow (K/r)e^{-j\theta} \)
Nyquist's Stability Criterion as a Design Tool

- Number of open loop RHP poles: 0
- Number of clockwise encirclements of $-1$: if $K < 1/2$: 0; if $K > 1/2$: 1
- Hence closed loop is stable for $K < 1/2$; unstable for $K > 1/2$
- This is what we would expect from root locus
Root locus of $L(s) = \frac{s-2}{(s+1)^2}$
Nyquist’s Criterion (Review)

- Consider a unity feedback system with an open loop transfer function $L(s) = G_c(s)G(s)H(s)$,
- Let $P_L$ denote the number of poles of $L(s)$ in open RHP
- Consider the modified Nyquist Contour in the $s$-plane (looping to the right of any poles or zeros on the $j\omega$-axis)
- Let $\Gamma_L$ denote image of mod. Nyquist Contour under $L(s)$
- Let $N_L$ denote the number of clockwise encirclements that $\Gamma_L$ makes of the point $(-1, 0)$
- **Nyquist’s Stability Criterion:**
  
  Number of closed-loop poles in open RHP = $N_L + P_L$
Consider

\[ L(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)} \]

Nyquist Diagram:
Since $L(s)$ is minimum phase (no RHP zeros), we can zoom in on the Nyquist plot. For a given $K$,

- how much extra gain would result in instability? We will call this the gain margin.
- how much extra phase lag would result in instability? We will call this the phase margin.
Formal definitions

- **Gain margin:** \( \frac{1}{|L(j\omega_x)|} \), where \( \omega_x \) is the frequency at which \( \angle L(j\omega) \) reaches \(-180^\circ\) amplifying the open-loop transfer function by this amount would result in a marginally stable closed loop.

- **Phase margin:** \( 180^\circ + \angle L(j\omega_c) \), where \( \omega_c \) is the frequency at which \( |L(j\omega)| \) equals 1 adding this much phase lag would result in a marginally stable closed loop.

- These margins can be read from the Bode diagram.
Transfer functions
Frequency Response
Plotting the freq. resp.
Mapping Contours
Nyquist's criterion
Ex: servo, P control
Ex: unst., P control
Ex: unst., PD contr.
Ex: RHP Z, P contr.
Nyquist's Stability Criterion as a Design Tool
Relative Stability Gain margin and Phase margin
Relationship to transient response

\[ L(j\omega) = \frac{1}{j\omega(1 + j\omega)(1 + j\omega/5)} \]

- Gain margin \(\approx 15 \text{ dB}\)
- Phase margin \(\approx 43^\circ\)
Phase margin and damping

- Consider a second-order open loop of the form
  \[ L(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}, \text{ with } \zeta < 1 \]
- Closed-loop poles \( s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \)
- Let \( \omega_c \) be the frequency at which \( |L(j\omega)| = 1 \)
- Square and rearrange: \( \omega_c^4 + 4\zeta^2\omega_n^2\omega_c^2 - \omega_n^4 = 0 \)
  Equivalently, \( \frac{\omega_c^2}{\omega_n^2} = \sqrt{4\zeta^4 + 1 - 2\zeta^2} \)
- By definition, \( \phi_{pm} = 180^\circ + \angle L(j\omega_c) \)
- Hence
  \[ \phi_{pm} = \arctan\left(\frac{2}{\sqrt{(4 + 1/\zeta^4)^{1/2} - 2}}\right) \]
- Phase margin is an explicit function of damping ratio!
- Approximation: for \( \zeta < 0.7 \), \( \zeta \approx 0.01\phi_{pm} \), where \( \phi_{pm} \) is measured in degrees
\[ L(j\omega) = \frac{1}{j\omega(1 + j\omega)(1 + j\omega/5)} \]

- Phase margin \( \approx 43^\circ \)
- Damping ratio \( \approx 0.43 \)