# Back to the properties of Fourier Representations

## Differentiation in time (CT, non-periodic signals)

$x(t) \leftrightarrow X(j\omega)$ , what is the FT of $\frac{dx(t)}{dt}$ ?

$$x(t) \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega.$$

Differentiate both sides wrt 't' and use linearity

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, j\omega\, e^{j\omega t}\, d\omega$$

Hence

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$$

As you might expect, differentiation

1. accentuates the high frequency components
2. eliminates the constant component ($\omega = 0$)

Similarly, for CT periodic signals,

$$x(t) = \sum_k X[k] e^{jk\omega_0 t}$$

$$\Rightarrow \quad \frac{dx(t)}{dt} = \sum_k X[k]\, jk\omega_0 e^{jk\omega_0 t}$$

$\Rightarrow$ For periodic signals $x(t)$,

$$\frac{dx(t)}{dt} \xleftrightarrow{FS;\,\omega_0} jk\omega_0 X[k]$$

Integration in time (CT, non-periodic signals)

Let $\quad y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

Then $\quad \frac{dy(t)}{dt} = x(t)$

This suggests that

$$j\omega Y(j\omega) = X(j\omega)$$

$$\Rightarrow \quad Y(j\omega) = \frac{1}{j\omega} X(j\omega)$$

However this formula is indeterminate at $\omega = 0$
Infact it only applies if $X(j0) = 0$.

- To allow us to deal with signals with non-zero DC component, we must allow impulses in the FT.

- Proof is quite involved (maybe we'll try it later), but it turns out that the appropriate form is

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\,\delta(\omega)$$

- Since integration involves "averaging", it de-emphasises the high frequency components.

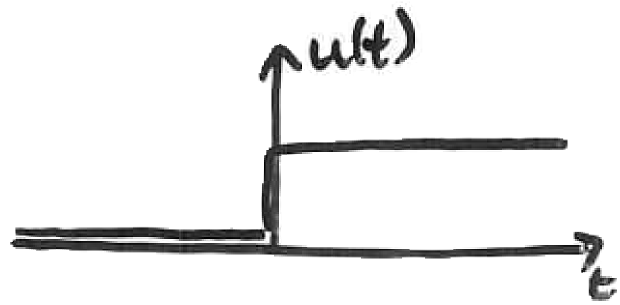
$$\int \left| x_{DC}(t) + x_{AC}(t) \right| dt < \infty \qquad |a+b| \le |a| + |b|$$

$$\int | \quad | \, dt \le \int |x_{DC}(t)|\,dt + \int |x_{AC}(t)|\,dt$$

## Application

- What is the FT of $u(t)$?
- Well $\int |u(t)|dt \to \infty$ so we'll have to have impulses in the FT if there's any chance that it'll exist

- 


Recall that $u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$

- Since $\delta(t) \overset{FT}{\longleftrightarrow}$ , we have

$$u(t) \overset{FT}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$$

Again, as you might expect, $u(t)$ has strong components at low frequency and weak components at higher frequencies

## Convolution Property

$$x(t) \longrightarrow \boxed{\begin{array}{l}\text{LTI}\\ \text{SYSTEM}\end{array}} \longrightarrow y(t)$$

$$x(t) \xleftrightarrow{FT} X(j\omega) \quad \text{and} \quad h(t \xleftrightarrow{FT} H(j\omega)$$

so that $H(j\omega)$ is the frequency response of the system, what is $Y(j\omega)$?

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

but $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

$$x(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega(t-\tau)}\, d\omega$$

$$\Rightarrow \quad y(t) = \int_{-\infty}^{\infty} h(\tau)\, \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t-\tau)}\, d\omega\, d\tau$$

$$\frac{1}{2\pi}\int X(j\omega) \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau \; e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int X(j\omega) H(j\omega)\, e^{j\omega t}\, d\omega.$$

$$= \frac{1}{2\pi}\int Y(j\omega) e^{j\omega t}\, d\omega$$

$$\Rightarrow \quad y(t) \xleftrightarrow{FT} H(j\omega) X(j\omega)$$

That is convolution in time $\xleftrightarrow{FT}$ multiplication in frequency

Hence, using FTs we have an alternative method for computing the output of a system

$$
\begin{tikzcd}
x(t) \arrow[r, "FT"] \arrow[d, "\text{convolution}"'] & X(j\omega) \arrow[d, "\text{multiplication}"] \\
y(t) & Y(j\omega) = H(j\omega)X(j\omega) \arrow[l, "\text{Inverse}", "FT"']
\end{tikzcd}
$$

EXAMPLE 3.29.

$x(t) = \text{sinc}(t)$

$h(t) = 2\,\text{sinc}(2t)$

Find $x(t) * h(t)$.

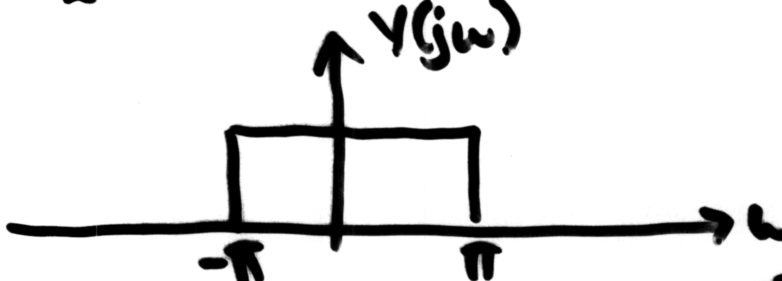
This is very difficult in the time domain

However,

$$x(t) \overset{FT}{\longleftrightarrow} X(jw) = \begin{cases} & |w| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

$$h(t) \overset{FT}{\longleftrightarrow} H(jw) = \begin{cases} & |w| \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$






$$Y(jw) = \begin{cases} 1 & |w| \leq \pi \\ 0 & o/w \end{cases}$$

$\Rightarrow y(t) = \text{sinc}(t)$

A similar result applies in discrete-time

$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(e^{j\Omega})$$
$$= H(e^{j\Omega}) X(e^{j\Omega})$$

Multiplication of Non-periodic signals

- if $x(t)$ and $z(t)$ are non-periodic, and $x(t) \xleftrightarrow{FT} X(j\omega)$ and $z(t) \xleftrightarrow{FT} Z(j\omega)$ what is $Y(j\omega)$ if $y(t) = x(t)z(t)$ ?

$$y(t) = x(t)z(t)$$
$$= \frac{1}{4\pi^2} \int X(j\lambda) e^{j\lambda t} d\lambda \int Z(j\mu) e^{j\mu t} d\mu$$
$$= \frac{1}{4\pi^2} \iint X(j\lambda) Z(j\mu) e^{j(\lambda+\mu)t} d\lambda d\mu$$

let $\lambda = \omega \quad \mu$

$$\frac{1}{2\pi} \int \frac{1}{2\pi} \int X(j(\omega-\mu)) Z(j\mu) d\mu \, e^{j\omega t} d\omega$$
$$= \frac{1}{2\pi} \int Y(j\omega) e^{j\omega t} d\omega$$

But what is $Y(j\omega)$?

$$Y(j\omega) = \frac{1}{2\pi}\int X(j(\omega-\mu))\,Z(j\mu)\,d\mu$$

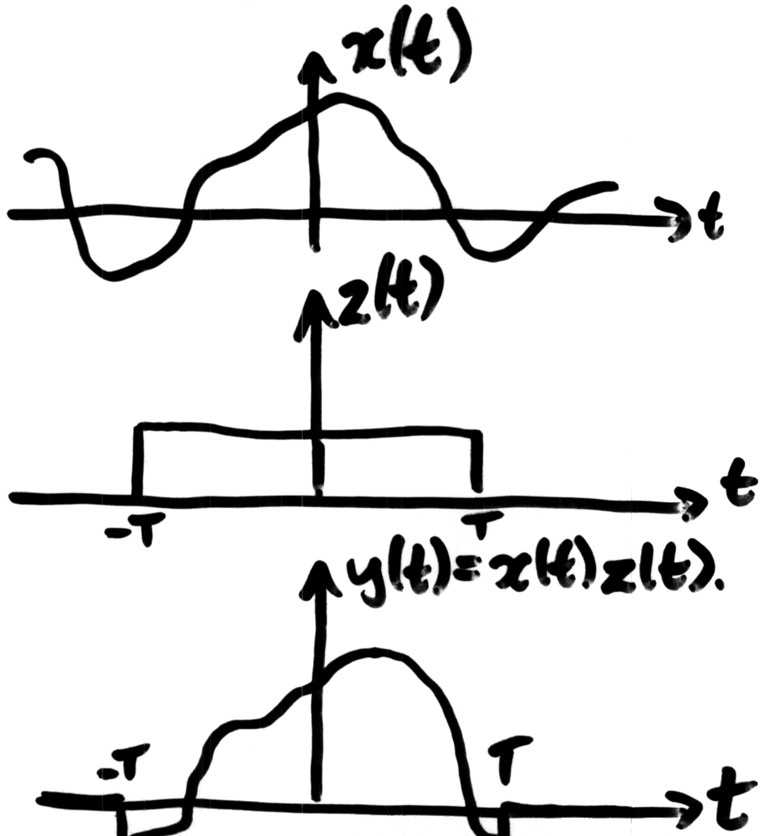
$\int x(\tau)h(t-\tau)d\tau$

$$\frac{1}{2\pi}\; X(j\omega) * Z(j\omega).$$

That is

multiplication in time $\xleftrightarrow{FT}$ convolution in frequency

This result is most often used in "windowing" where we attempt to ~~ad~~ approximate $X(j\omega)$ by taking the FT of a segment of $x(t)$.




$Y(j\omega) = X(j\omega) * Z(j\omega)$

as T gets large.

$Z(j\omega) \rightarrow$ narrow 'sinc'

$\Rightarrow Y(j\omega) \approx X(j\omega)$

# Parseval relations

- Fourier transforms are orthogonal.
- Hence we can measure the energy of a signal in the time or frequency domain, and we'll get the same answer

- $$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt.$$

$$x(t) = \frac{1}{2\pi}\int X(j\omega) e^{j\omega t} d\omega$$

$$x^*(t) = \frac{1}{2\pi}\int X^*(j\omega) e^{-j\omega t} d\omega.$$

$$\Rightarrow \quad E_x = \int x(t) \frac{1}{2\pi}\int X^*(j\omega) e^{-j\omega t} d\omega \; dt.$$

$$= \frac{1}{2\pi}\int X^*(j\omega) \int x(t) e^{-j\omega t} dt \; d\omega.$$

$$= \frac{1}{2\pi}\int X^*(j\omega)\, X(j\omega)\, d\omega$$

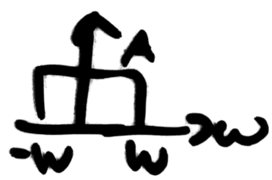
$$= \frac{1}{2\pi}\int |X(j\omega)|^2 d\omega$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.$$

A similar result holds in ~~discrete-time~~

$$\sum_{n} |x[n]|^2 = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} |X(e^{j\Omega})|^2 \, d\Omega$$

- Similar relationships also hold for periodic signals, but recall that periodic signals have infinite energy.
- Therefore, the relations apply to power not energy

sinc $\leftrightarrow$ 


$$\int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{t^2} \, dt = \frac{1}{2\pi}\int_{-W}^{W} A^2 \, d\omega.$$

$$= \frac{A^2 \, 2W}{2\pi}$$

# Duality

We have seen that

Rect pulse in time $\overset{FT}{\longleftrightarrow}$ sinc in frequency

sinc in time $\overset{FT}{\longleftrightarrow}$ rect. pulse in frequency

$\delta(t) \overset{FT}{\longleftrightarrow} 1$

$\overset{FT}{\longleftrightarrow} \delta(\omega)$

convolution in time $\overset{FT}{\longleftrightarrow}$ multiplication in frequency

multiplication in time $\overset{FT}{\longleftrightarrow}$ convolution in frequency

We now attempt to formalize these relationships

If $f(t) \overset{FT}{\longleftrightarrow} F(j\omega)$

what is the FT of $F(jt)$?

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

let $\lambda = \omega$, $\mu = t$

$$\Rightarrow \quad F(j\lambda) \quad \int f(-\mu) e^{j\lambda\mu} d\mu$$

Recall that $x(t) \overset{FT}{\longleftrightarrow} X(j\omega)$

$$\Updownarrow$$

$$x(t) = \frac{1}{2\pi}\int X(j\omega)e^{j\omega t}d\omega$$

So set $\lambda = t$, $\mu = \omega$

$$\Rightarrow F(jt) = \int f(-\omega)e^{j\omega t}d\omega$$

$$\Rightarrow \quad F(jt) \overset{FT}{\longleftrightarrow} 2\pi f(-\omega)$$

see Fig 3.43.

- This "doubles" the size of our FT table
- Duality for other FTs is possible, but more difficult
- End of Chapter 3.