Transient response of low order systems

- In addition to the steady-state response, we are also interested in the transient response of the closed loop
- This is more difficult to obtain directly, so we usually approximate the system by a first or second-order system in order to get a "feel" for the transient response.
- The material in this section is essentially a review from 2CJ4
- Transient response is difficult to interpret for general signals, so usually we just consider the step response. The unit step function has a fast transient and is easily repeatable for testing purposes

First-order System

$$T(s) = \frac{b_0}{s+a_0} = \frac{T(0)}{\tau s+1}$$

where $T(0) = b_0/a_0$ is the DC gain
$\tau = 1/a_0$

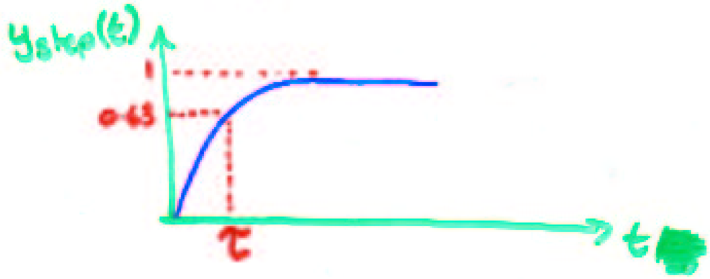
This system has a pole at $s = -1/\tau$

The step response of this system is

$$y_{step}(t) = \mathcal{L}_t^{-1}\left\{ T(s) \cdot \frac{1}{s} \right\}$$

Assuming $T(0) = 1$ we have

$$y_{step}(t) = \left(1 - e^{-t/\tau}\right) u(t)$$



SECOND-ORDER SYSTEMS

$$T(s) = \frac{b_0}{s^2 + a_1 s + a_0} = \frac{T(0)\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where $T(0) = \frac{b_0}{a_0}$, $\omega_n^2 = a_0$, $2\zeta\omega_n = a_1$

$T(0)$ is the DC gain

ζ is the damping ratio

ω_n is the natural frequency.

This system has poles at $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

if $0 < \zeta < 1$ poles form a conjugate pairs

if $\zeta = 1$ poles coincide at $s = -\omega_n$

if $\zeta > 1$ poles are real and distinct

- The step response is once again

$$y_{step}(t) = \mathcal{L}_t^{-1}\left\{ T(s) \cdot \frac{1}{s} \right\}$$

- As you may recall, the response depends on the damping factor.

① If $0 < \zeta < 1$ the system is said to be underdamped. Assuming $T(0) = 1$ we have

$$y(t) = \left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t + \phi\right) \right] u(t)$$

where $\phi = \operatorname{atan}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \operatorname{acos}(\zeta)$

This an oscillating decaying signal

rate of decay / time constant is $\tau = \frac{1}{\zeta\omega_n}$

frequency of oscillation (damped natural frequency) is $\omega_n\sqrt{1-\zeta^2}$

② If $\zeta > 1$ the system is said to be overdamped

$$y_{step}(t) = \left[1 + k_1 e^{-t/\tau_1} + k_2 e^{-t/\tau_2}\right] u(t)$$

where the time constants are

$$\tau_{1,2} = \frac{1}{\zeta \omega_n \mp \omega_n \sqrt{\zeta^2 - 1}}$$

and scaling factors are

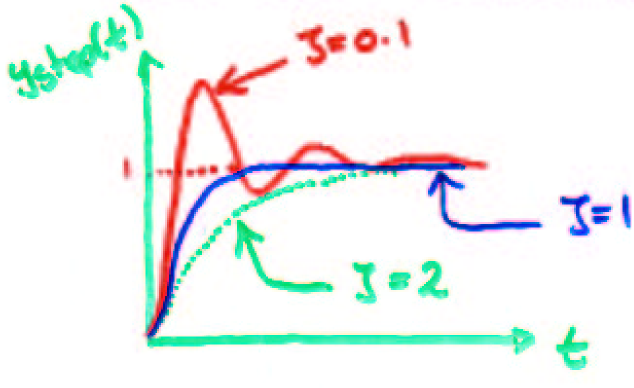
$$k_{1,2} = \frac{1}{2}\left(1 \pm \frac{\zeta}{\sqrt{\zeta^2 - 1}}\right)$$

③ If $\zeta = 1$ the system is said to be critically damped

$$y_{step}(t) = \left[1 - e^{-t/\tau} - t e^{-t/\tau}\right] u(t)$$

where $\tau = 1/\omega_n$ is the time constant

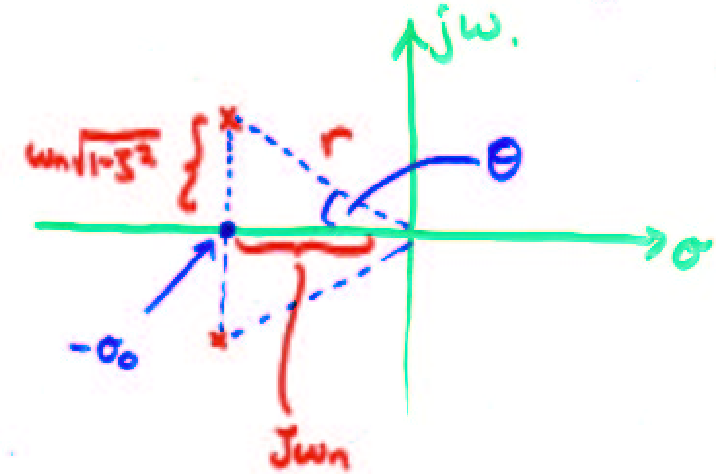
Typical examples of these three cases are:



Recall that for an underdamped second-order system

poles are at

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



Note that $r = \sqrt{\zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2)}$

$= \omega_n$

Hence $\cos\Theta = \dfrac{\zeta\omega_n}{\omega_n}$

$\Rightarrow \quad \zeta = \cos\Theta$

$\Rightarrow \quad \omega_n = \dfrac{\sigma_0}{\cos\Theta}$

$\Rightarrow$ we can determine ζ and ω_n directly from a pole plot using simple geometry

Very helpful in intuitively analysing second order systems