COMP ENG 4TL4 – Digital Signal Processing

Solutions to Homework Assignment #1

1. **Using the definition of *linearity* given in Lecture #5, show that the following two systems are both linear.**

   a. **The ideal-delay system, i.e., $y[n] = \mathcal{T}_a\{x[n]\} = x[n-n_d]$, where n_d is a fixed positive integer called the delay of the system.**

   b. **The moving-average system described by** $y[n] = \mathcal{T}_b\{x[n]\} = \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2} x[n-k]$ **.**

(20 pts)

a. Using the definition of linearity:

$$\begin{aligned} T\{ax_1[n]+bx_2[n]\} &= ax_1[n-n_d]+bx_2[n-n_d] \\ &= ay_1[n]+by_2[n]. \end{aligned}$$

⇒ The ideal delay system is linear.

b. Using the definition of linearity:

$$\begin{aligned} T\{ax_1[n]+bx_2[n]\} &= \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2}\left(ax_1[n-k]+bx_2[n-k]\right) \\ &= \frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2}ax_1[n-k]+\frac{1}{M_1+M_2+1}\sum_{k=-M_1}^{M_2}bx_2[n-k] \\ &= ay_1[n]+by_2[n]. \end{aligned}$$

⇒ The moving average system is linear.

2. **Determine which of the following signals are periodic. If a signal is periodic, determine its period.**

   a. $x[n] = \mathrm{e}^{\mathrm{j}(2\pi n/5)}$

   b. $x[n] = \sin(\pi n/19)$

   c. $x[n] = n\,\mathrm{e}^{\mathrm{j}\pi n}$

   d. $x[n] = \mathrm{e}^{\mathrm{j}n}$

(20 pts)

a. $x[n]$ is periodic with period 5:

$$\mathrm{e}^{\mathrm{j}\left(\frac{2\pi}{5}n\right)} = \mathrm{e}^{\mathrm{j}\left(\frac{2\pi}{5}\right)(n+N)} = \mathrm{e}^{\mathrm{j}\left(\frac{2\pi}{5}n+2\pi k\right)}$$

$$\Rightarrow 2\pi k = \frac{2\pi}{5}N, \quad \text{for integers } k,\ N$$

Making $k = 1 \Rightarrow N = 5$, so that $x[n]$ has a period of 5 samples.

b. $x[n]$ is periodic with period 38:

$$x[n+38] = \sin(\pi(n+38)/19) = \sin(\pi n/19 + 2\pi) = x[n].$$

c. This sequence is not periodic, because the linear term n is not periodic.

d. Again, this sequence is not periodic, because no integer value of n is divisible by 2π.

3. The sequence:

$$x[n] = \sin\left(\frac{\pi}{2}n\right), \qquad -\infty < n < \infty,$$

was obtained by sampling a continuous-time signal:

$$x_c(t) = \sin(\Omega_0 t), \qquad -\infty < t < \infty,$$

at a sampling frequency f_s = 2 kHz.

a. What are two possible *positive* values of Ω_0 that could have resulted in the sequence $x[n]$?

b. Explain these frequency values in light of the derivation of the Nyquist sampling theorem given in Lecture #2. (20 pts)

a. The lowest possible positive frequency is obtained when:

$$\Omega_0 nT = \frac{\pi}{2}n \quad \Rightarrow \quad \Omega_0 = \frac{\pi}{2T} = \frac{\pi}{2}f_s = 1000\pi \text{ radians/s} \quad \left(f_0 = \frac{\Omega_0}{2\pi} = 500 \text{ Hz}\right)$$

The next-lowest positive frequency is obtained when:

$$\Omega_0 nT = \frac{\pi}{2}n + 2\pi n \quad \Rightarrow \quad \Omega_0 = \frac{5\pi}{2T} = \frac{5\pi}{2}f_s = 5000\pi \text{ radians/s} \quad \left(f_0 = \frac{\Omega_0}{2\pi} = 2500 \text{ Hz}\right)$$

b. From the derivation of the Nyquist theorem given in Lecture #2, we expect a continuous-time spectral component at f_0 = 500 Hz to exist at the corresponding discrete-time frequency and to be replicated at $f_0 = 500 + kf_s$, where k is an integer. If $k = 1$, then f_0 = 500 + 2000 = 2500 kHz, the second-lowest positive frequency found in part a above.

4. An ideal lowpass filter has been implemented via the cascade of an A/D converter, a discrete-time ideal lowpass filter, and a D/A converter. The discrete-time ideal lowpass filter is known to have a discrete-time cutoff frequency $\omega_c = \pi/5$ radians.

a. If $x_c(t)$ is bandlimited to 3 kHz, what is the minimum sampling frequency f_s required by the A/D converter to avoid aliasing?

b. If f_s = 10 kHz, what will be the effective continuous-time cutoff frequency Ω_c of the ideal lowpass filter? (20 pts)

a. According to the Nyquist sampling theorem, the minimum sampling frequency f_s to avoid aliasing is 2 times the signal bandwidth $\Rightarrow f_s = 6$ kHz.

b. If $f_s = 10$ kHz $\rightarrow T = 1/10000$ seconds, then the effective continuous-time cutoff frequency Ω_c of the ideal lowpass filter is can be found from:

$$\omega_c = \Omega_c T \quad \Rightarrow \quad \frac{\pi}{5} = \frac{\Omega_c}{10000} \quad \Rightarrow \quad \Omega_c = 2000\pi \text{ radians/s.}$$

5. A particular digital communication channel is capable of transmitting 19200 bits per second. We wish to use the channel to transmit a band-limited analog signal $x_c(t)$, by sampling and digitizing. The magnitude of the analog signal is limited to $|x_c(t)| \leq X_m$. The error between the digitized signal and $x_c(t)$ must not exceed $\pm 10^{-4} X_m$.

a. What is the required number of bits in the A/D, assuming a uniform rounding quantizer?

b. What is the maximum bandwidth of the analog signal for which the channel can be used? (20 pts)

a. The step size Δ for a uniform quantizer is equal to the full range of the quantizer ($= 2X_m$) divided by the number of quantization levels ($= 2^{nbits}$), where *nbits* is the number of bits representing the quantization levels. That is:

$$\Delta = \frac{2X_m}{2^{nbits}}$$

In a uniform rounding quantizer for which the input is scaled so as to avoid peak clipping, the magnitude of the error $e[n]$ is less than or equal to half the quantizer step size. Given the design constraint specified above:

$$|e[n]| \leq \frac{\Delta}{2} \leq \frac{X_m}{2^{nbits}} \leq 10^{-4} X_m$$

$$\Rightarrow \frac{1}{2^{nbits}} \leq 10^{-4} \Rightarrow 2^{nbits} \geq 10^4 \Rightarrow nbits \geq \log_2\left(10^4\right) \geq 13.2877$$

The actual number of bits must take an integer value, therefore a 14-bit A/D converter is required to meet the design specification.

b. If the digital communication channel can transmit 19200 bits per second and each sample is coded by 14 bits, then the maximum sampling rate $f_s = 19200/14 = 1371.4$ Hz. According to the Nyquist sampling theorem, in order to avoid aliasing an analog signal must be bandlimited to half of the sampling frequency $\Rightarrow$ this channel can only be used for analog signals with bandwidths up to 685.7 Hz.