COMP ENG 4TL4 – Digital Signal Processing

Solutions to Homework Assignment #3

1. **Consider an LTI system that is stable and for which $H(z)$, the z-transform of the impulse response is given by:**

$$H(z) = \frac{3}{1 + \frac{1}{3} z^{-1}}.$$

Suppose $x[n]$, the input to the system, is the unit step sequence $u[n]$.

a. Find the output $y[n]$ by directly evaluating the discrete convolution of $x[n]$ and $h[n]$ in the time domain.

b. Find the output $y[n]$ by calculating the inverse z-transform of $Y(z)$. **(20 pts)**

a. For the system to be stable, the region of convergence (ROC) for the transfer function must be $|z| > \frac{1}{3}$, so that it includes the unit circle. Taking the inverse z-transform of $H(z)$ gives the impulse response:

$$h[n] = 3\left(-\tfrac{1}{3}\right)^n u[n].$$

Evaluating the convolution directly we obtain:

$$\begin{aligned}
y[n] = h[n] * x[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} 3\left(-\tfrac{1}{3}\right)^k u[k]u[n-k] \\
&= \sum_{k=0}^{n} 3\left(-\tfrac{1}{3}\right)^k u[n] \\
&= \left\{\sum_{k=0}^{\infty} 3\left(-\tfrac{1}{3}\right)^k - \sum_{k=n+1}^{\infty} 3\left(-\tfrac{1}{3}\right)^k\right\} u[n] \\
&= \left\{\sum_{k=0}^{\infty} 3\left(-\tfrac{1}{3}\right)^k - 3\left(-\tfrac{1}{3}\right)^{n+1} \sum_{k=0}^{\infty} 3\left(-\tfrac{1}{3}\right)^k\right\} u[n] \\
&= \left\{3\frac{1}{1+\frac{1}{3}} - 3\frac{\left(-\frac{1}{3}\right)^{n+1}}{1+\frac{1}{3}}\right\} u[n] \\
&= \tfrac{9}{4}\left(1 - \left(-\tfrac{1}{3}\right)^{n+1}\right) u[n].
\end{aligned}$$

b. The z-transform of $x[n]$ is:

$$X(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1,$$

giving the z-transform of the output:

$$Y(z) = H(z)X(z) = \frac{3}{1+\frac{1}{3}z^{-1}} \frac{1}{1-z^{-1}} = \frac{\frac{3}{4}}{1+\frac{1}{3}z^{-1}} + \frac{\frac{9}{4}}{1-z^{-1}}, \quad |z| > 1.$$

Taking the inverse z-transform of $Y(z)$ we obtain:

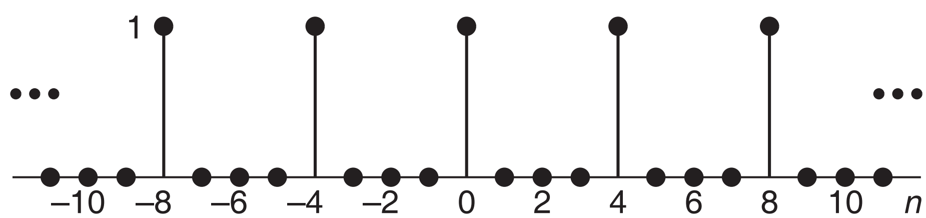
$$\begin{aligned} y[n] &= \tfrac{3}{4}\left(-\tfrac{1}{3}\right)^n u[n] + \tfrac{9}{4}u[n] \\ &= -\tfrac{9}{4}\left(-\tfrac{1}{3}\right)^{n+1} u[n] + \tfrac{9}{4}u[n] \\ &= \tfrac{9}{4}\left(1-\left(-\tfrac{1}{3}\right)^{n+1}\right)u[n]. \end{aligned}$$

2. Sketch each of the following sequences and determine their z-transforms, including the region of convergence (ROC).

a. $\sum_{k=-\infty}^{\infty} \delta[n-4k]$

b. $\frac{1}{2}\left[e^{j\pi n} + \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{2} + 2\pi n\right)\right]u[n]$ **(20 pts)**

a. The sequence $\sum_{k=-\infty}^{\infty} \delta[n-4k]$ is shown below:



The z-transform of this sequence is:

$$\mathcal{Z}\left\{\sum_{k=-\infty}^{\infty} \delta[n-4k]\right\} = \sum_{k=-\infty}^{\infty} \mathcal{Z}\{\delta[n-4k]\} = \sum_{k=-\infty}^{\infty} z^{-4k}, \quad 0 < |z| < \infty.$$

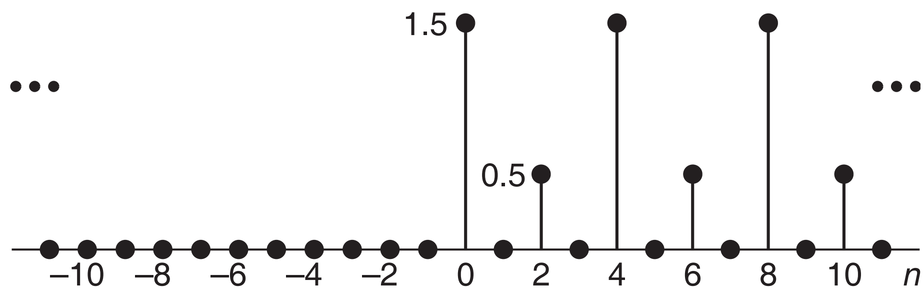
b. The sequence:

$$\frac{1}{2}\left[e^{j\pi n}+\cos\left(\frac{\pi}{2}n\right)+\sin\left(\frac{\pi}{2}+2\pi n\right)\right]u[n]$$

can be simplified to:

$$\frac{1}{2}\left[(-1)^n+\cos\left(\frac{\pi}{2}n\right)+1\right]u[n]=\begin{cases}\frac{3}{2}, & n=4k, \quad k\geq 0,\\ \frac{1}{2}, & n=4k+2, \quad k\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

This sequence is shown below.



The z-transform of this sequence is:

$$\sum_{n=0}^{\infty}\tfrac{3}{2}z^{-4n}+\tfrac{1}{2}\sum_{n=0}^{\infty}z^{-(4n+2)}=\frac{\frac{3}{2}}{1-z^{-4}}+\frac{\frac{1}{2}z^{-2}}{1-z^{-4}}=\frac{\frac{3}{2}+\frac{1}{2}z^{-2}}{1-z^{-4}},\quad |z|>1.$$

3. When the input to an LTI system is:

$$x[n]=\left(\frac{1}{2}\right)^n u[n]+2^n u[-n-1],$$

the output is:

$$y[n]=6\left(\frac{1}{2}\right)^n u[n]-6\left(\frac{3}{4}\right)^n u[n].$$

a. Find the transfer function $H(z)$ of the system. Plot the poles and zeros of $H(z)$ and indicate the ROC.

b. Find the impulse response $h[n]$ of the system.

c. Write the LCCD equation that characterizes the system.

d. Is the system stable? Is it causal? (40 pts)

a. The z-transforms of $x[n]$ and $y[n]$ are:

$$X(z)=\frac{1}{1-\frac{1}{2}z^{-1}}-\frac{1}{1-2z^{-1}}=\frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)},\quad \tfrac{1}{2}<|z|<2,$$

and

$$Y(z)=\frac{6}{1-\frac{1}{2}z^{-1}}-\frac{6}{1-\frac{3}{4}z^{-1}}=\frac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)},\quad |z|>\tfrac{3}{4},$$

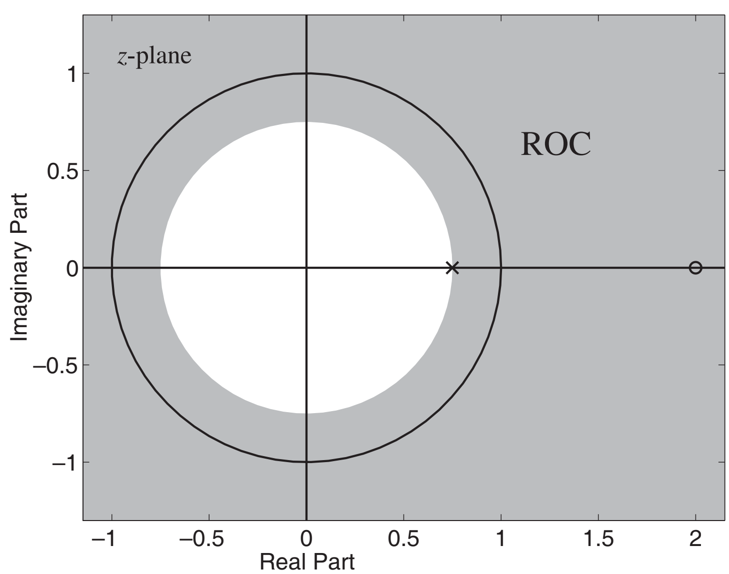
respectively.

Thus the transfer function $H(z)$ is:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\dfrac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{3}{4}z^{-1}\right)}}{\dfrac{-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}}=\frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}},\quad |z|>\tfrac{3}{4}.$$

Note that the ROC of $\dfrac{1}{X(z)}=\dfrac{\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)}{-\frac{3}{2}z^{-1}}$ is $0<|z|<\infty$, so the intersection of the ROCs of $Y(z)$ and $1/X(z)$ is $\frac{3}{4}<|z|<\infty$. However, the zero in $Y(z)$ at $z=\infty$ cancels the pole in $1/X(z)$ at $z=\infty$, so the ROC of $H(z)$ includes $z=\infty$.

The pole-zero plot of $H(z)$ with its ROC is shown below.



b. The impulse response $h[n]$ is found by taking the inverse z-transform of $H(z)$, giving:

$$\begin{aligned} h[n] &= \mathcal{Z}^{-1}\left\{\frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}\right\} \\ &= \mathcal{Z}^{-1}\left\{\frac{1}{1-\frac{3}{4}z^{-1}} - 2z^{-1}\frac{1}{1-\frac{3}{4}z^{-1}}\right\} \\ &= \left(\tfrac{3}{4}\right)^n u[n] - 2\left(\tfrac{3}{4}\right)^{n-1} u[n-1]. \end{aligned}$$

c. The LCCD equation that characterizes this system can be found from:

$$\begin{aligned} &H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}} \\ &\Rightarrow Y(z) - \tfrac{3}{4}z^{-1}Y(z) = X(z) - 2z^{-1}X(z). \end{aligned}$$

Taking the inverse z-transform of the equation above gives:

$$y[n] - \tfrac{3}{4}y[n-1] = x[n] - 2x[n-1].$$

d. The system is stable because the ROC includes the unit circle and is causal because it includes $z = \infty$. We can also deduce that the system is stable and causal because the impulse response $h[n]$ is absolutely summable and $h[n] = 0$ for $n < 0$.

4. Consider a real finite-length sequence $x[n]$ with Fourier transform $X\left(e^{j\omega}\right)$ and DFT $X[k]$. If:

$$\mathcal{Im}\{X[k]\} = 0, \quad k = 0,1,\ldots N-1,$$

can we conclude that:

$$\mathcal{Im}\left\{X\left(e^{j\omega}\right)\right\} = 0, \quad -\pi \le \omega \le \pi?$$

State your reasoning if your answer is yes. Give a counterexample if your answer is no. (10 pts)

No, we <u>cannot</u> conclude that a sequence with a real-valued DFT must have a real-valued DTFT, because the DFT is only sampling the DTFT at certain equally spaced frequencies, and the phase of the DTFT at those frequencies may just happen to be an integer multiple of π, in which case the imaginary part of the DFT is zero.

For example, consider a signal which consists of an impulse centered at $n = 1$:

$$x[n] = \delta[n-1].$$

The DTFT is:

$$\begin{aligned} X\left(e^{j\omega}\right) &= e^{-j\omega} \\ \mathfrak{Re}\left\{X\left(e^{j\omega}\right)\right\} &= \cos(\omega) \\ \mathfrak{Im}\left\{X\left(e^{j\omega}\right)\right\} &= -\sin(\omega) \end{aligned}$$

Note that the imaginary part is *not* equal to zero for all ω between – π and π.

Now, suppose we take the 2-point DFT:

$$\begin{aligned} X[k] &= e^{-jk\pi} \\ &= \begin{cases} 1, & k = 0, \\ -1, & k = 1. \end{cases} \end{aligned}$$

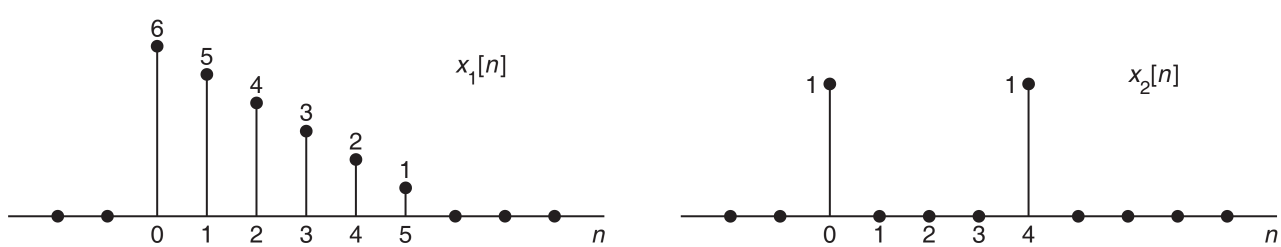
So $\mathcal{Im}\{X[k]\} = 0, \quad \forall k$, but $\mathcal{Im}\left\{X\left(e^{j\omega}\right)\right\} \neq 0, \quad -\pi \leq \omega \leq \pi$.

Note that the size of the DFT plays a crucial role. If we were to take the 3-point DFT instead, then we would obtain $\mathcal{Im}\{X[k]\} \neq 0$ for $k = 1$ or $k = 2$.

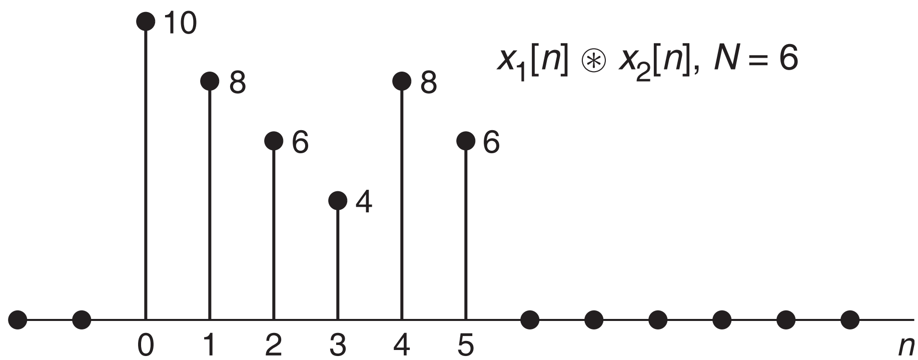
5. Two finite-length sequences $x_1[n]$ and $x_2[n]$ are shown in the figure below. Sketch their *N*-point circular convolution for:

a. *N* = 6, and

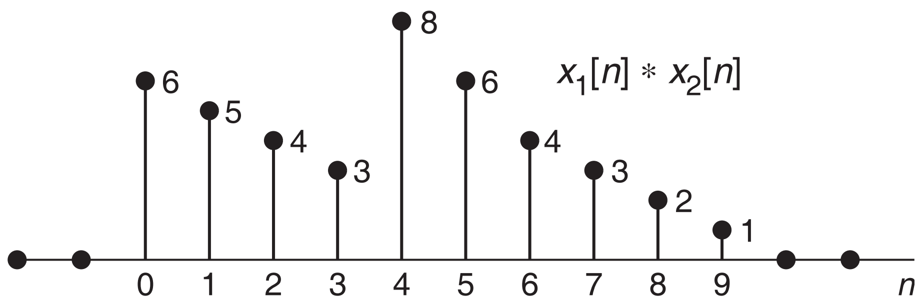
b. *N* = 10. **(10 pts)**



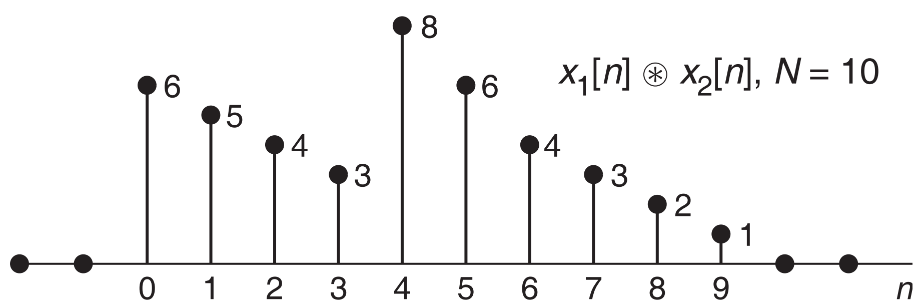
a. The 6-point circular convolution gives:



If L and P are the lengths of $x_1[n]$ and $x_2[n]$, respectively, we note that $N < L + P - 1$ (= 10), so the circular convolution exhibits time-aliasing, i.e., it is not identical to the linear convolution of these sequences, shown below.



b. The 10-point circular convolution gives:



We note that in this case $N > L + P - 1$ (= 10), so the circular convolution does not exhibits time-aliasing, i.e., it is identical to the linear convolution of these sequences.