

Solutions to Final Exam

1. The *step response*  $y[n]$  of a causal, stable LTI system is:

$$y[n] = \left(\frac{1}{2}\right)^n u[n],$$

where  $u[n]$  is the unit step function.

a. Find the  $z$ -domain transfer function  $H(z)$  of this system.

b. Find the impulse response  $h[n]$  of this system.

c. Find the linear constant coefficient difference (LCCD) equation that describes this system.

a. Taking the  $z$ -transform of  $y[n]$  gives:

$$Y(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}.$$

The  $z$ -transform  $X(z)$  of the input unit step function is:

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1.$$

The transfer function  $H(z)$  is then:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}.$$

b. The impulse response of this system is found by computing the inverse  $z$ -transform of  $H(z)$ :

$$\begin{aligned} H(z) &= \frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}} = \frac{1}{1 - \frac{1}{2}z^{-1}} - z^{-1} \frac{1}{1 - \frac{1}{2}z^{-1}} \\ \Rightarrow h[n] &= \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{2}\right)^{n-1} u[n-1]. \end{aligned}$$

c. The LCCD equation that describes this system is:

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}} \\ \Rightarrow Y(z) - \frac{1}{2}z^{-1}Y(z) &= X(z) - z^{-1}X(z) \quad |z| > \frac{1}{2}. \\ \Rightarrow y[n] - \frac{1}{2}y[n-1] &= x[n] - x[n-1]. \end{aligned}$$

2. Consider the causal stable LTI system with the transfer function:

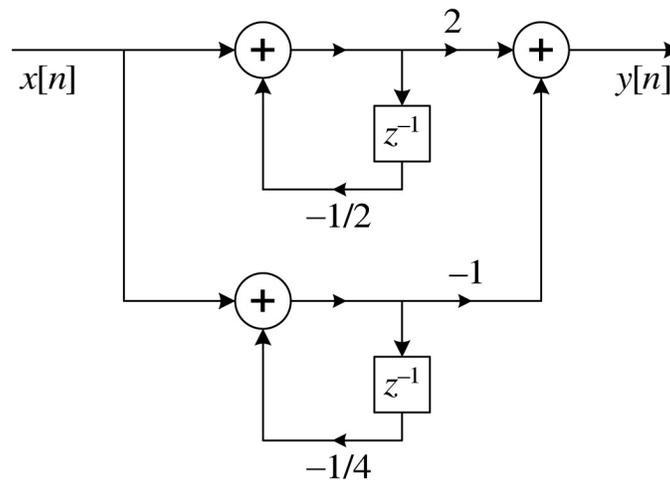
$$H(z) = \frac{1}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)}.$$

Draw the block diagram for this system implemented in *parallel form* with 1<sup>st</sup>-order subsystems.

To obtain the parallel form, we compute the partial fraction expansion of  $H(z)$  :

$$H(z) = \frac{1}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{4}z^{-1}\right)} = \frac{2}{1 + \frac{1}{2}z^{-1}} - \frac{1}{1 + \frac{1}{4}z^{-1}}.$$

The block diagram is then:



3. A causal LTI system has the transfer function:

$$H(z) = \frac{(1 + 0.2z^{-1})(1 - 9z^{-2})}{1 + 0.81z^{-2}}.$$

a. Find transfer functions for a *minimum-phase* system  $H_1(z)$  and an *all-pass* system  $H_{ap}(z)$  such that:

$$H(z) = H_1(z)H_{ap}(z).$$

b. Sketch the pole-zero plots of  $H(z)$ ,  $H_1(z)$  and  $H_{ap}(z)$ , clearly show their ROCs by shading them on the pole-zero plots.

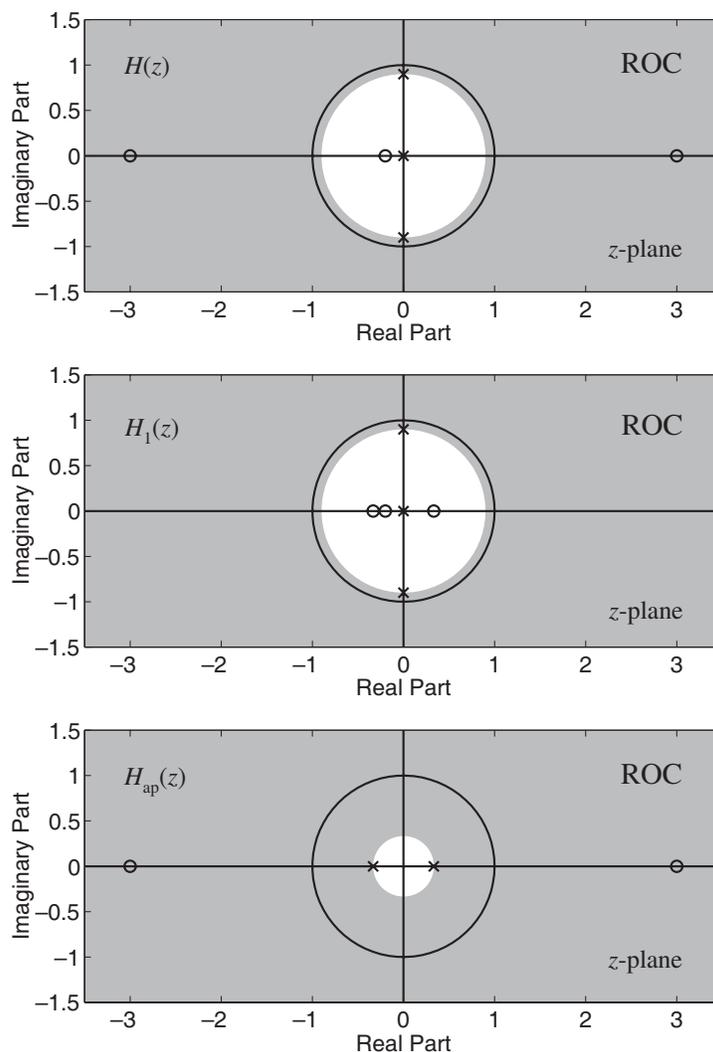
- a. The system is nonminimum-phase because of the pair of zeros at  $z = \pm 3$ . To form the minimum-phase system  $H_1(z)$ , these zeros must be reflected to their (conjugate) reciprocal locations at  $z = \pm \frac{1}{3}$ , giving:

$$H_1(z) = \frac{(1 + 0.2z^{-1})(1 - \frac{1}{9}z^{-2})}{1 + 0.81z^{-2}}, \quad |z| > 0.9.$$

In order for  $H_1(z)H_{ap}(z)$  to equal  $H(z)$ , the all-pass system  $H_{ap}(z)$  must have the pair of nonminimum-phase zeros at  $z = \pm 3$  from  $H(z)$  and a pair of poles at the reciprocal locations  $z = \pm \frac{1}{3}$ , giving:

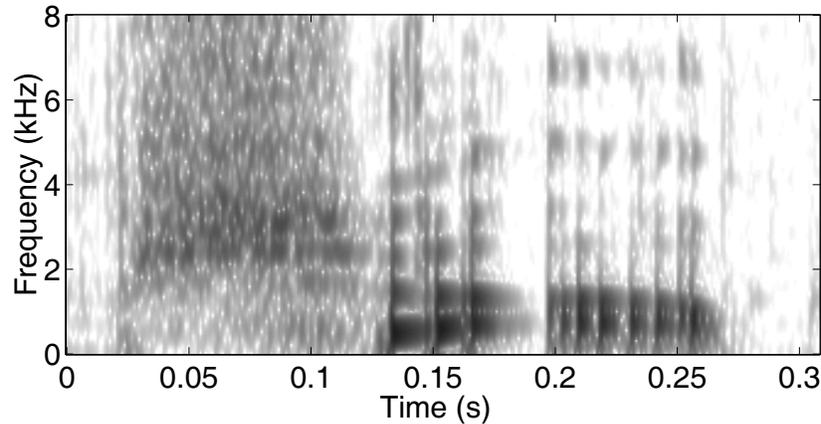
$$H_{ap}(z) = \frac{1 - 9z^{-2}}{1 - \frac{1}{9}z^{-2}}, \quad |z| > \frac{1}{3}.$$

- b. The pole-zero plots of  $H(z)$ ,  $H_1(z)$  and  $H_{ap}(z)$ , with their ROCs, are given below.



4. The spectrogram of a speech signal is shown below.

- a. Is this a *narrowband* spectrogram or a *wideband* spectrogram? Explain your reasoning.
- b. This signal is a single word, either “shop” or “posh”. From the spectrogram, which of these two words does it appear to be? Explain your reasoning.



- a. This is a wideband spectrogram because the temporal resolution is good and the frequency resolution is poor, such that vertical striations are seen in the voiced segment of speech and the harmonics are unresolved (i.e., no horizontal striations are observed).
- b. The word is shop, which is evident in the spectrogram because of the long noise-like spectrum at the beginning, produced by the unvoiced fricative “sh”.

5. Two alternative methods of digital IIR filter design make use of the:

- i. impulse invariance transformation and
- ii. bilinear transformation.

**Explain:**

- a. which of these two methods is applicable to a wider range of filter frequency response types (i.e., lowpass, highpass, bandpass, bandstop, multiband) and why, and
  - b. which of these two methods is simpler and why.
- a. The bilinear transform is applicable to a wider range of filter frequency responses because it maps the entire analog frequency axis into the digital frequency range  $(-\pi, \pi)$ , such that frequency aliasing is avoided. The impulse invariance method does not warp the frequency axis, so aliasing is produced except in cases of strictly bandlimited frequency responses.
  - b. The impulse invariance method is somewhat simpler because it just requires sampling and scaling the impulse response of the desired analog filter. However, there is no simple mapping of the  $s$ -domain transfer function to the  $z$ -domain. In contrast, the bilinear transform requires prewarping of the desired digital filter properties before design of the analog filter, application of the bilinear transformation and simplification of the  $z$ -domain transfer function, but there is a direct mapping of the  $s$ -domain transfer function to the  $z$ -domain.

6. Four values of a stationary random process  $x[n]$  have been observed, giving the discrete-time sequence  $v[n] = \{0.38, 0.66, 1.17, 0.86\}$ .

- a. Use the *autocorrelation-based method* of spectral estimation to estimate the PSD of  $x[n]$ , using a triangular correlation window  $w_c[m] = \{0.5, 1, 0.5\}$ .
- b. Does  $x[n]$  appear to be lowpass, bandpass or highpass?

- a. First we compute the estimate of autocorrelation function of the random process. Note that we will be applying a correlation window with nonzero values only at lags of  $m = -1, 0$  and  $1$ , so we only need to evaluate the autocorrelation estimator at those lags:

$$c_{vv}[m] = v[m] * v[-m] \Rightarrow$$

$$c_{vv}[-1] = 0.38 \times 0 + 0.66 \times 0.38 + 1.17 \times 0.66 + 0.86 \times 1.17 + 0 \times 0.86 = 2.0292$$

$$c_{vv}[0] = 0.38 \times 0.38 + 0.66 \times 0.66 + 1.17 \times 1.17 + 0.86 \times 0.86 = 2.6885$$

$$c_{vv}[1] = c_{vv}[-1] = 2.0292$$

$$\hat{\phi}_{vv}[m] = \frac{1}{Q} c_{vv}[m] = \frac{1}{4} c_{vv}[m].$$

Applying the triangular correlation window gives:

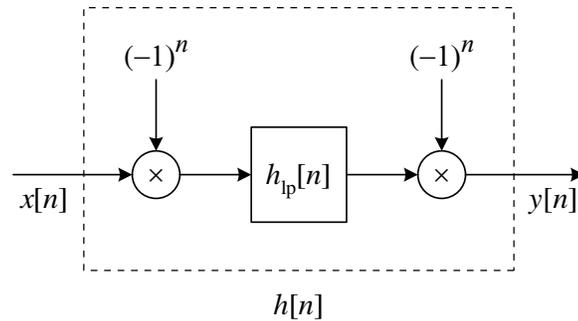
$$\begin{aligned} \hat{\phi}_{vv}[m] w_c[m] &= \frac{1}{4} \{2.0292 \delta[m+1] + 2.6885 \delta[m] + 2.0292 \delta[m-1]\} \times \{0.5 \delta[m+1] + \delta[m] + 0.5 \delta[m-1]\} \\ &= \{0.2536 \delta[m+1] + 0.6721 \delta[m] + 0.2536 \delta[m-1]\}. \end{aligned}$$

Taking the Fourier transform of the windowed autocorrelation estimate gives the PSD estimator:

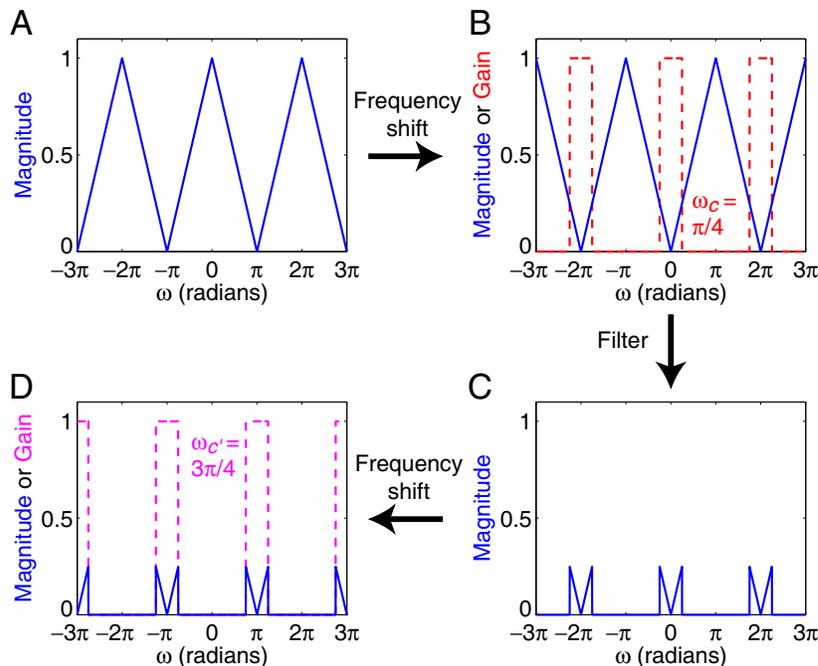
$$\begin{aligned} S(\omega) &= \sum_{m=-1}^1 \hat{\phi}_{vv}[m] w_c[m] e^{-j\omega m} \\ &= \sum_{m=-1}^1 \{0.2536 \delta[m+1] + 0.6721 \delta[m] + 0.2536 \delta[m-1]\} e^{-j\omega m} \\ &= 0.2536 e^{j\omega} + 0.6721 + 0.2536 e^{-j\omega} \\ &= 0.6721 + 0.5072 \cos \omega. \end{aligned}$$

- b. From the final expression for  $S(\omega)$  we would estimate that  $x[n]$  is a lowpass signal.

7. Let  $h_{lp}[n]$  denote the impulse response of an ideal lowpass filter with unity passband gain and cutoff frequency  $\omega_c = \pi/4$ . The figure below shows an ideal LTI frequency-selective filter  $h[n]$  that incorporates  $h_{lp}[n]$  as a subsystem. Sketch  $|H(e^{j\omega})|$ , the magnitude frequency response of  $h[n]$ , indicating explicitly the band-edge frequencies in terms of  $\omega_c$  and specify whether the system is a lowpass, highpass, bandpass, bandstop or multiband filter.



Consider a lowpass discrete-time signal  $x[n]$  bandlimited to  $(-\pi, \pi)$ , shown by the blue line in panel A of the figure below. Multiplication of  $x[n]$  by the sequence  $(-1)^n$  in the time domain corresponds to a frequency shift of  $\pi$  radians in the frequency domain, from Property 3 of Table 2.2, where  $\omega_0 = \pi$  in this case. This frequency shift turns the lowpass signal  $x[n]$  into a highpass signal, shown in panel B. After lowpass filtering (shown by the red dashed line in panel B), the signal (shown in panel C) is again frequency shifted by  $\pi$  radians to give the blue line in panel D. The net filtering effect of these operations is a highpass filter with cutoff frequency  $\omega_c' = \pi - \omega_c = \pi - \pi/4 = 3\pi/4$ , as illustrated by the magenta line in panel D.



8. Consider the finite sequence  $x[n] = \{1, 0.75, 0.09, -0.06, -1, -0.85\}$ . The DFT of this sequence  $X[k]$  has been calculated for  $0 \leq k \leq 5$  and lossy compression has been applied to the sequence of DFT coefficients  $X[k]$  by ignoring (i.e., setting to zero) the coefficients for  $k = 2, 3$  and  $4$ , to obtain the sequence of compressed DFT coefficients:

$$\tilde{X}[k] = \{-0.07, 1.465 - j2.3296, 0, 0, 0, 1.465 + j2.3296\}.$$

- a. Compute the reconstructed sequence  $\tilde{x}[n]$  by taking the inverse DFT of  $\tilde{X}[k]$ .
- b. Calculate the root mean squared (RMS) error between  $\tilde{x}[n]$  and  $x[n]$ . Express this RMS error as a percentage of the RMS of  $x[n]$ .

- a. The inverse DFT of  $\tilde{X}[k]$  is:

$$\tilde{x}[n] = \frac{1}{6} \sum_{k=0}^5 \tilde{X}[k] e^{j2\pi kn/6} \Rightarrow$$

$$\tilde{x}[0] = \frac{1}{6} \sum_{k=0}^5 \tilde{X}[k] e^0$$

$$= \frac{1}{6} \{-0.07 + 1.465 - j2.3296 + 1.465 + j2.3296\} = 0.4767$$

$$\tilde{x}[1] = \frac{1}{6} \sum_{k=0}^5 \tilde{X}[k] e^{j2\pi k/6}$$

$$= \frac{1}{6} \{-0.07 + (1.465 - j2.3296) e^{j2\pi/6} + (1.465 + j2.3296) e^{j2\pi 5/6}\} = 0.9050$$

$$\tilde{x}[2] = \frac{1}{6} \sum_{k=0}^5 \tilde{X}[k] e^{j2\pi k 2/6}$$

$$= \frac{1}{6} \{-0.07 + (1.465 - j2.3296) e^{j2\pi 2/6} + (1.465 + j2.3296) e^{j2\pi 10/6}\} = 0.4167$$

$$\tilde{x}[3] = \frac{1}{6} \sum_{k=0}^5 \tilde{X}[k] e^{j2\pi k 3/6}$$

$$= \frac{1}{6} \{-0.07 - (1.4267 + j0.8237) e^{j2\pi 3/6} + (1.465 + j2.3296) e^{j2\pi 15/6}\} = -0.5000$$

$$\tilde{x}[4] = \frac{1}{6} \sum_{k=0}^5 \tilde{X}[k] e^{j2\pi k 4/6}$$

$$= \frac{1}{6} \{-0.07 - (1.4267 + j0.8237) e^{j2\pi 4/6} + (1.465 + j2.3296) e^{j2\pi 20/6}\} = -0.9283$$

$$\tilde{x}[5] = \frac{1}{6} \sum_{k=0}^5 \tilde{X}[k] e^{j2\pi k 5/6}$$

$$= \frac{1}{6} \{-0.07 + (1.465 - j2.3296) e^{j2\pi 5/6} + (1.465 + j2.3296) e^{j2\pi 25/6}\} = -0.4400,$$

giving the sequence  $\tilde{x}[n] = \{0.4767, 0.9050, 0.4167, -0.5000, -0.9283, -0.4400\}$ .

b. The RMS error is:

$$\begin{aligned}\text{RMSE} &= \sqrt{E\{(x[n] - \tilde{x}[n])^2\}} \\ &= \sqrt{\frac{1}{6} \left\{ (1 - 0.4767)^2 + (0.75 - 0.9050)^2 + (0.09 - 0.4167)^2 \right. \\ &\quad \left. + (-0.06 + 0.5)^2 + (-1 + 0.9283)^2 + (-0.85 + 0.4400)^2 \right\}} \\ &= 0.3586.\end{aligned}$$

The RMS of  $x[n]$  is:

$$\begin{aligned}\text{RMS} &= \sqrt{E\{(x[n])^2\}} \\ &= \sqrt{\frac{1}{6} \left\{ 1^2 + 0.75^2 + 0.09^2 + (-0.06)^2 + (-1)^2 + (-0.85)^2 \right\}} \\ &= 0.7412,\end{aligned}$$

giving a percentage RMS error of:

$$\frac{\text{RMSE}}{\text{RMS}} \times 100 = \frac{0.3586}{0.7412} \times 100 \approx 48\%$$