COMP ENG 4TL4:

Digital Signal Processing

Notes for Lecture #15

Friday, October 10, 2003

5.2 The Effects of Windowing and Frequency Sampling on Spectral Resolution

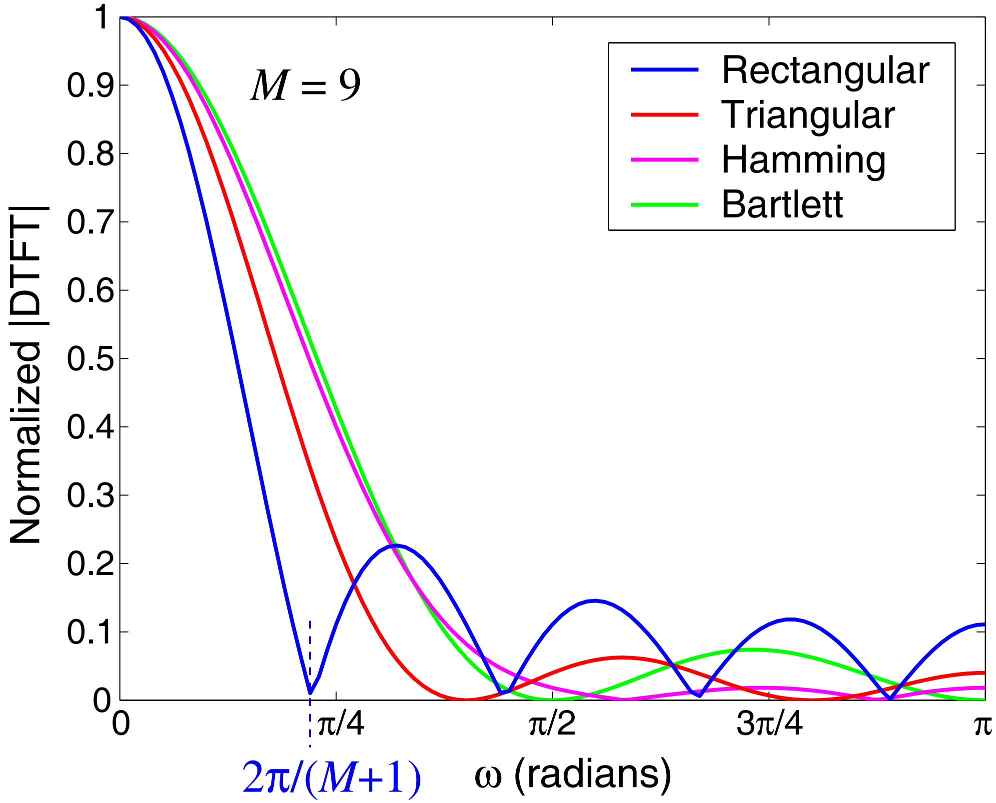
The effects of windowing on the DFT spectral resolution:

An N-point DFT samples the discrete-time spectrum in frequency steps of $\Delta\omega = 2\pi/N$.

However, we saw previously that windowing a discrete-time sequence (multiplication in the time-domain) is equivalent to (periodic) convolution in the frequency domain – see slides 12 and 13 from Lecture #9.

Consequently, the effective spectral resolution of the DFT is not determined by the number of points N of the DFT but rather by *the spectrum of the windowing function*, which is governed largely by the window length M.

Of the commonly used windows, the rectangular window has the narrowest mainlobe.

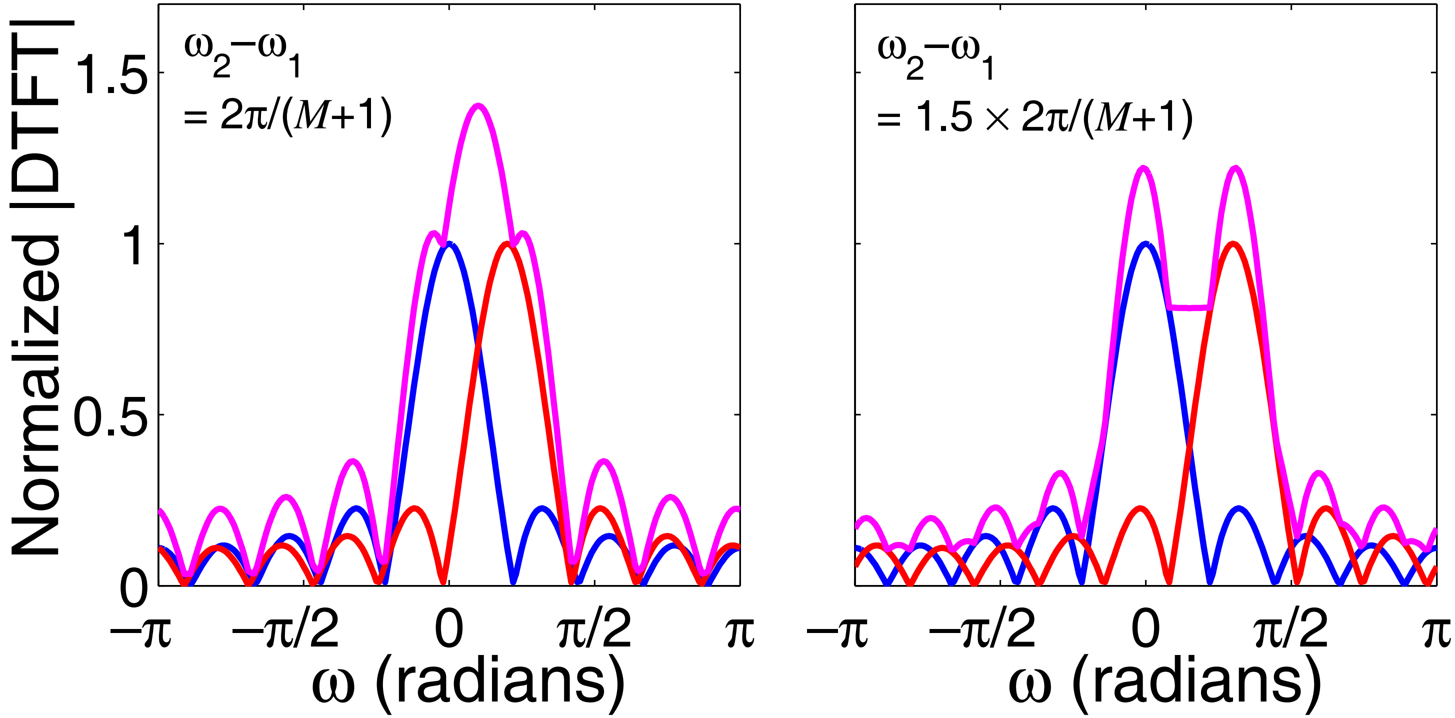


From the shape of the rectangular window's spectrum, it is reasonable to assume that we should be able to resolve two frequency components that are separated by approximately half the mainlobe width, i.e., $2\pi/(M+1)$.

Consider the sum of two complex exponentials with frequencies ω_1 $(= 0)$ and ω_2, respectively.

If the frequency difference $(\omega_2 - \omega_1)$ and is less than or equal to $2\pi/(M+1)$, then only one peak appears in the DTFT of the summed sequences.

However, if the frequency difference is somewhat greater than $2\pi/(M+1)$, then two peaks appear in the DTFT.



Half the rectangular window's mainlobe width is:

$$\begin{aligned} \omega &= \frac{2\pi}{M+1} && \text{radians} \\ \Rightarrow \Omega &= \frac{2\pi}{(M+1)\,T} && \text{radians/s} \\ \Rightarrow f &= \frac{1}{(M+1)\,T} && \text{Hz} \end{aligned}$$

Consequently, two frequency components should be resolved by the DFT if their frequency difference is greater than $\sim 1/(MT)$ Hz. That is, the spectral resolution in Hz is the reciprocal of the observation time MT seconds.

<u>Note:</u> If $N = M$, then the DFT frequency step size is exactly $1/(MT)$ Hz.

Question: Does zero-padding alter the effective DFT frequency resolution?

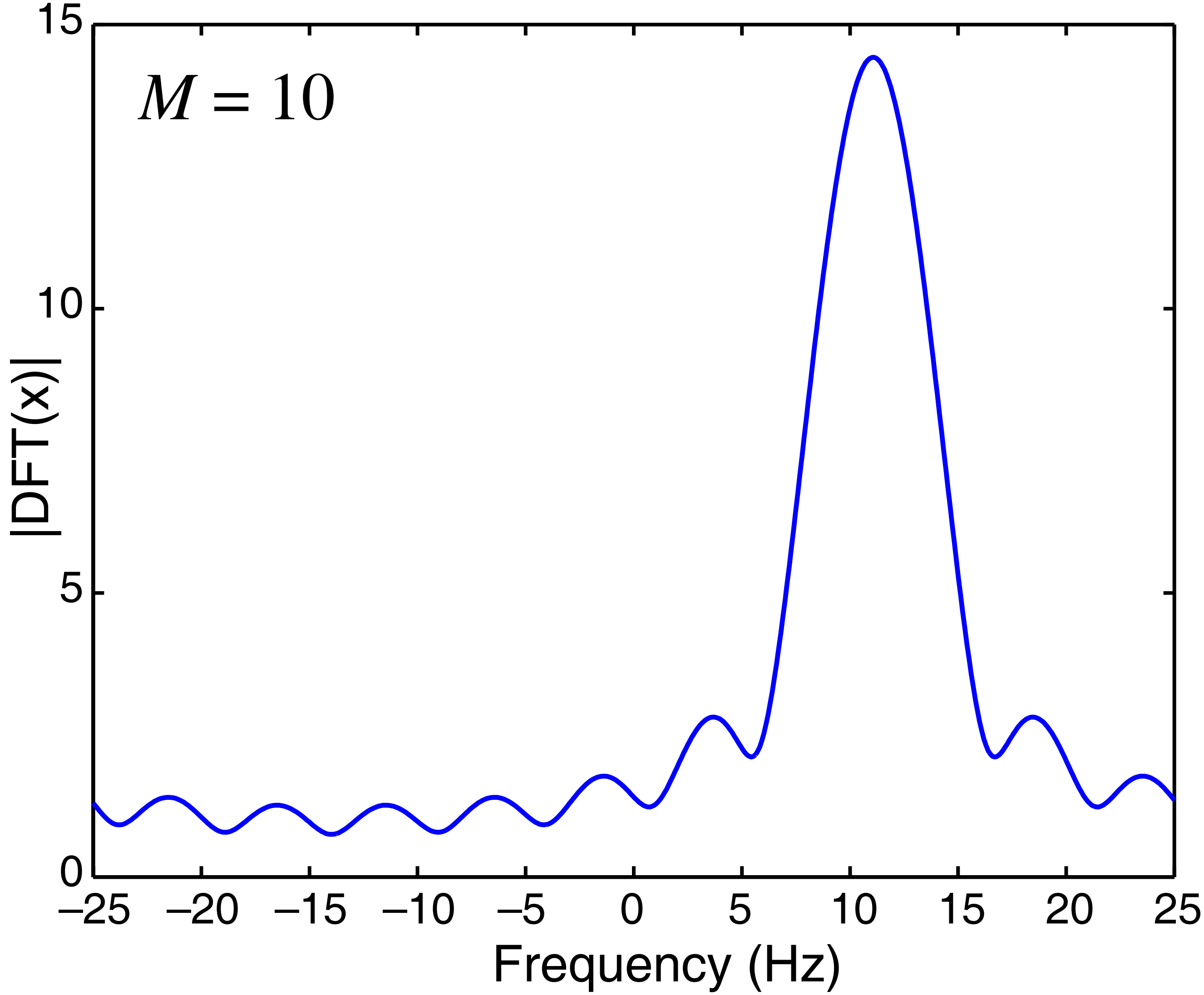
NO! Zero-padding only increases the number of points in the DFT. This improves the DFT's approximation of the DTFT of the windowed sequence, but the spectral resolution of the DTFT of the windowed sequence sets the effective spectral resolution of the DFT.

Consequently, the only way to increase the spectral resolution is to increase the window length M.

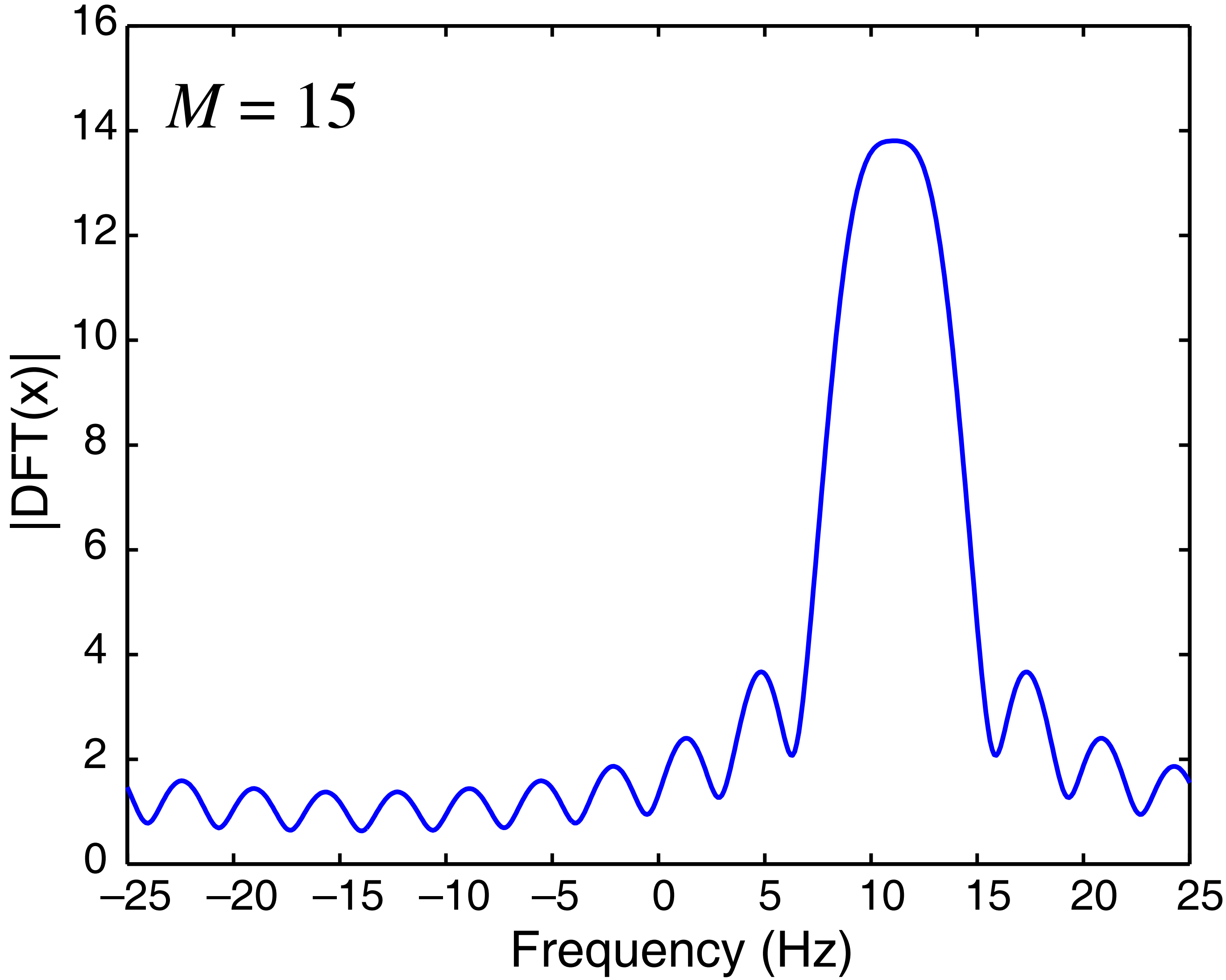
Examples:

Let two complex exponentials with the nearby frequencies $f_1 = 10 \text{ Hz}$ and $f_2 = 12 \text{ Hz}$ be sampled with the sampling interval $T = 0.02 \text{ seconds}$ and let us consider various rectangular window lengths $M = 10, 15, 30, 100, 300$ with zero-padding of each sequence to give $N = 512$ points.

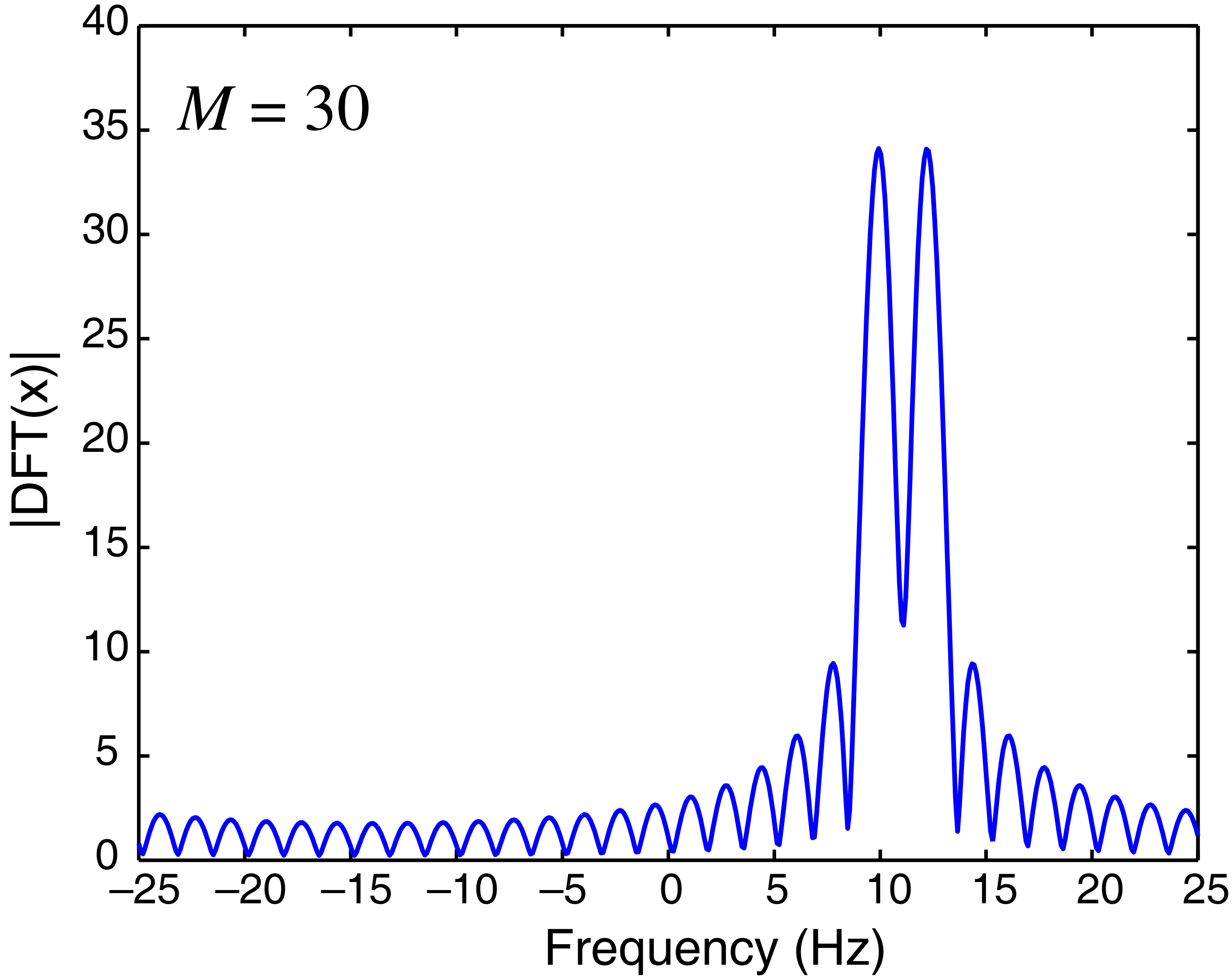
DFT with $M = 10$ and zero-padding to $N = 512$ points. The signals are <u>unresolved</u> because $f_2 - f_1 = 2\ \mathrm{Hz} < 1/(MT) = 5\ \mathrm{Hz}$.



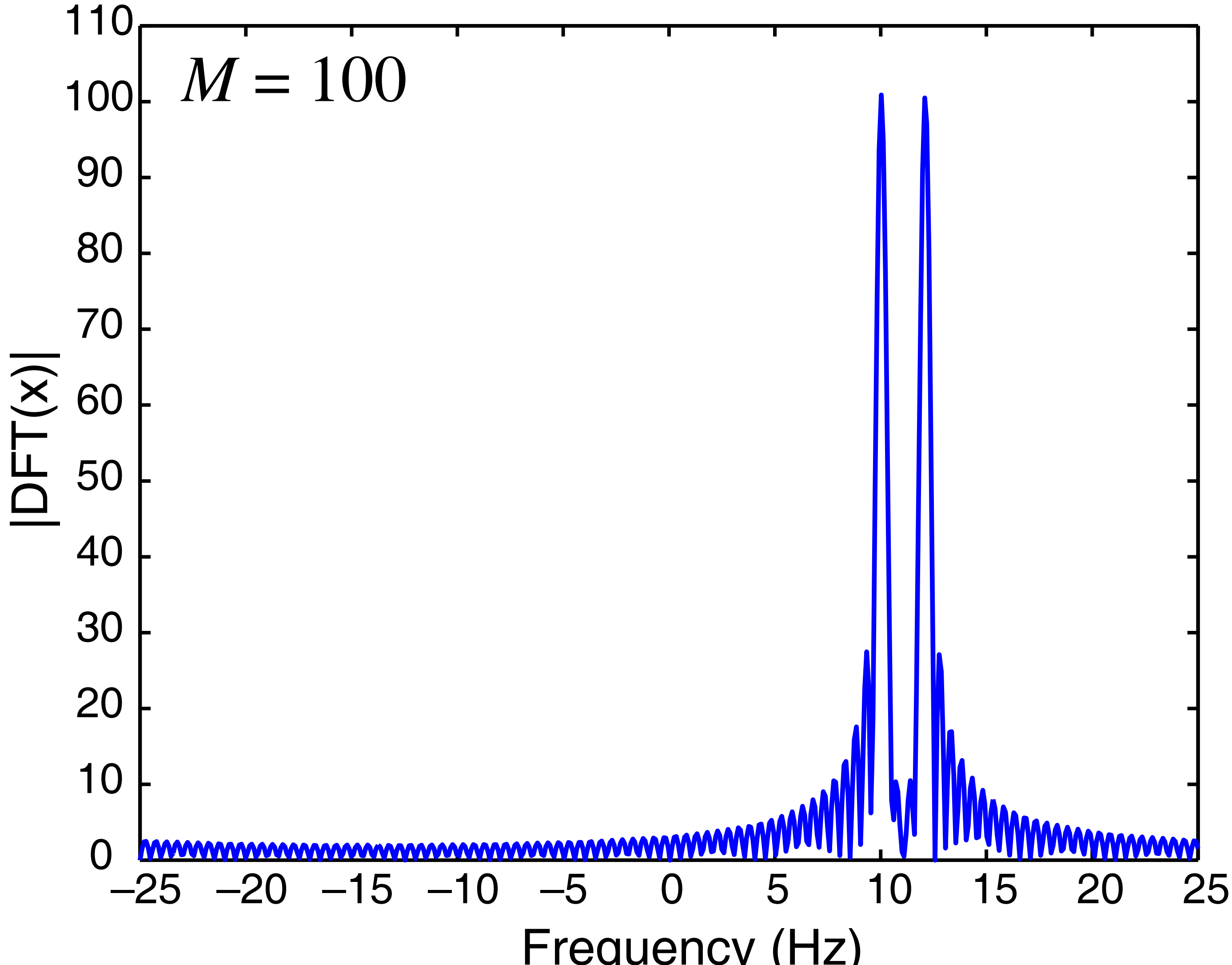
DFT with $M = 15$ and zero-padding to $N = 512$ points. The signals are still unresolved because $f_2 - f_1 = 2\ \mathrm{Hz} < 1/(MT) \approx 3.3\ \mathrm{Hz}$.



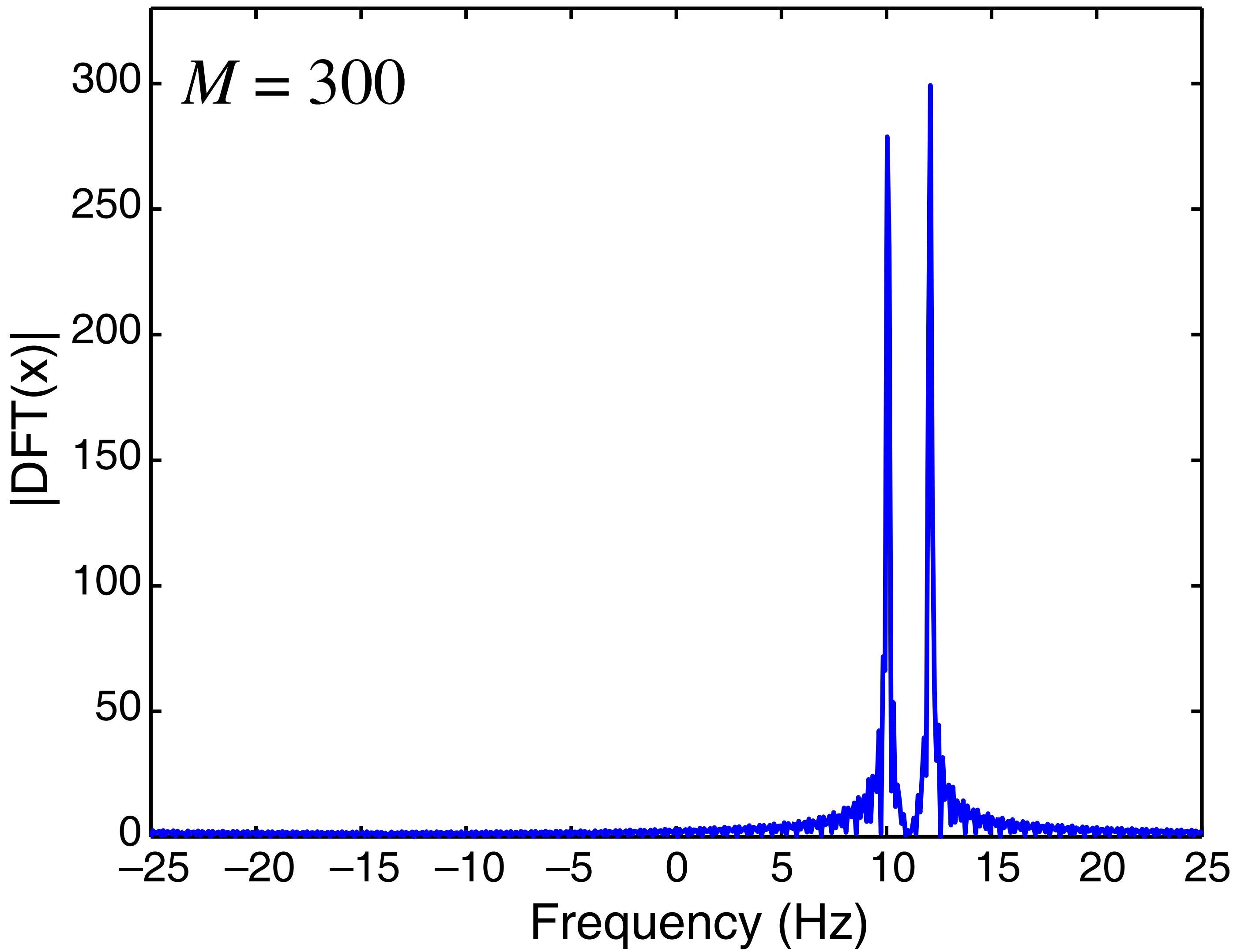
DFT with $M = 30$ and zero-padding to $N = 512$ points. The signals are now resolved because $f_2 - f_1 = 2\ \mathrm{Hz} > 1/(MT) \approx 1.7\ \mathrm{Hz}$.



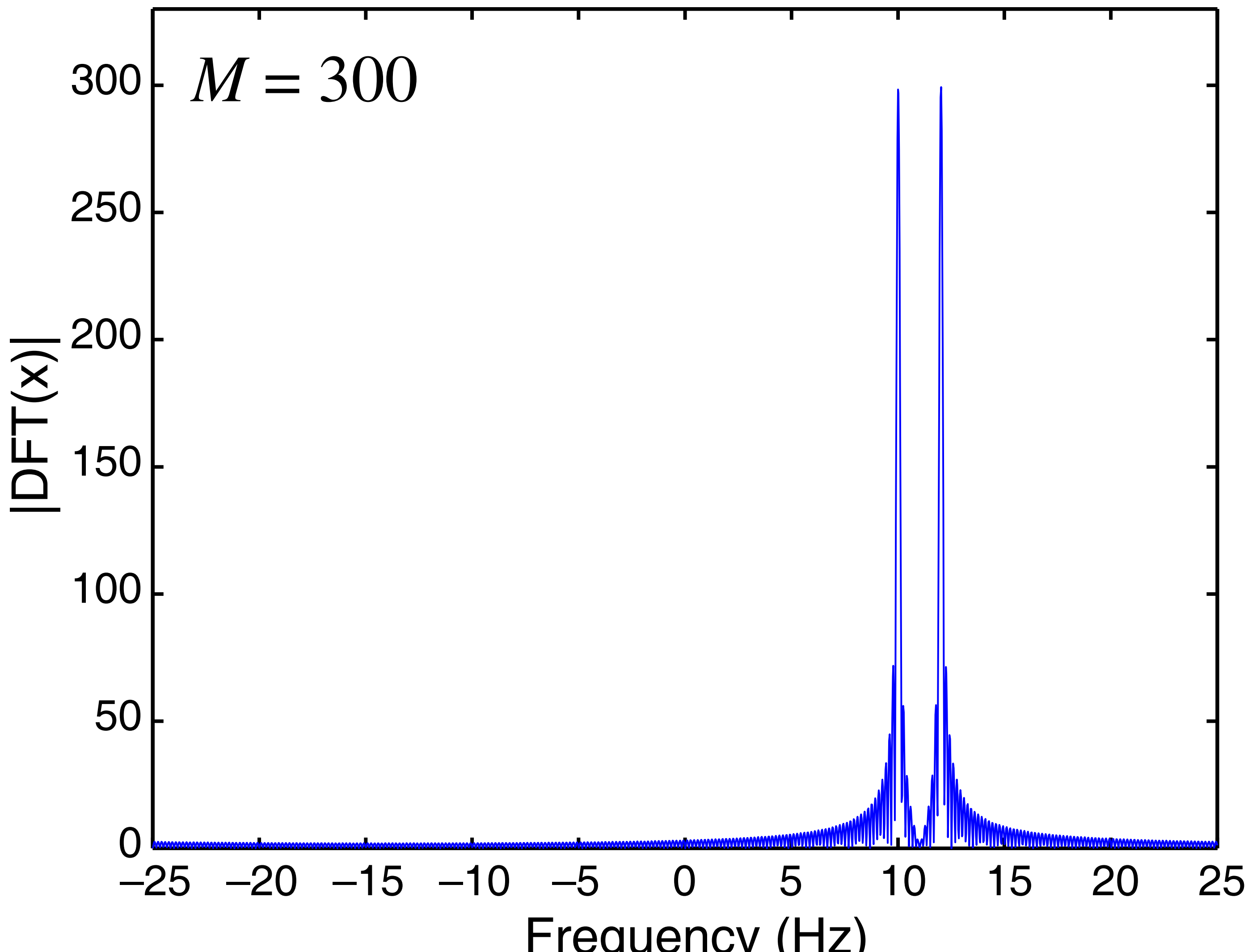
DFT with $M = 100$ and zero-padding to $N = 512$ points. The signals are now well resolved because $f_2 - f_1 = 2 \text{ Hz} > 1/(MT) = 0.5 \text{ Hz}$.



DFT with $M = 300$ and zero-padding to $N = 512$ points. The signals are now very well resolved because $f_2 - f_1 = 2 \text{ Hz} > 1/(MT) \approx 0.17 \text{ Hz}$.



A better representation of the previous DFT plot can be obtained by zero-padding to $N = 2048$ points.



The effects of frequency sampling on the DFT spectral resolution:

An N-point DFT samples the discrete-time spectrum in frequency steps of $\Delta\omega = 2\pi/N$. If a discrete-time sequence has a frequency component that does not fall exactly at one of the frequency sampling points, then the frequency, amplitude and phase of that component will not be well represented by the DFT.

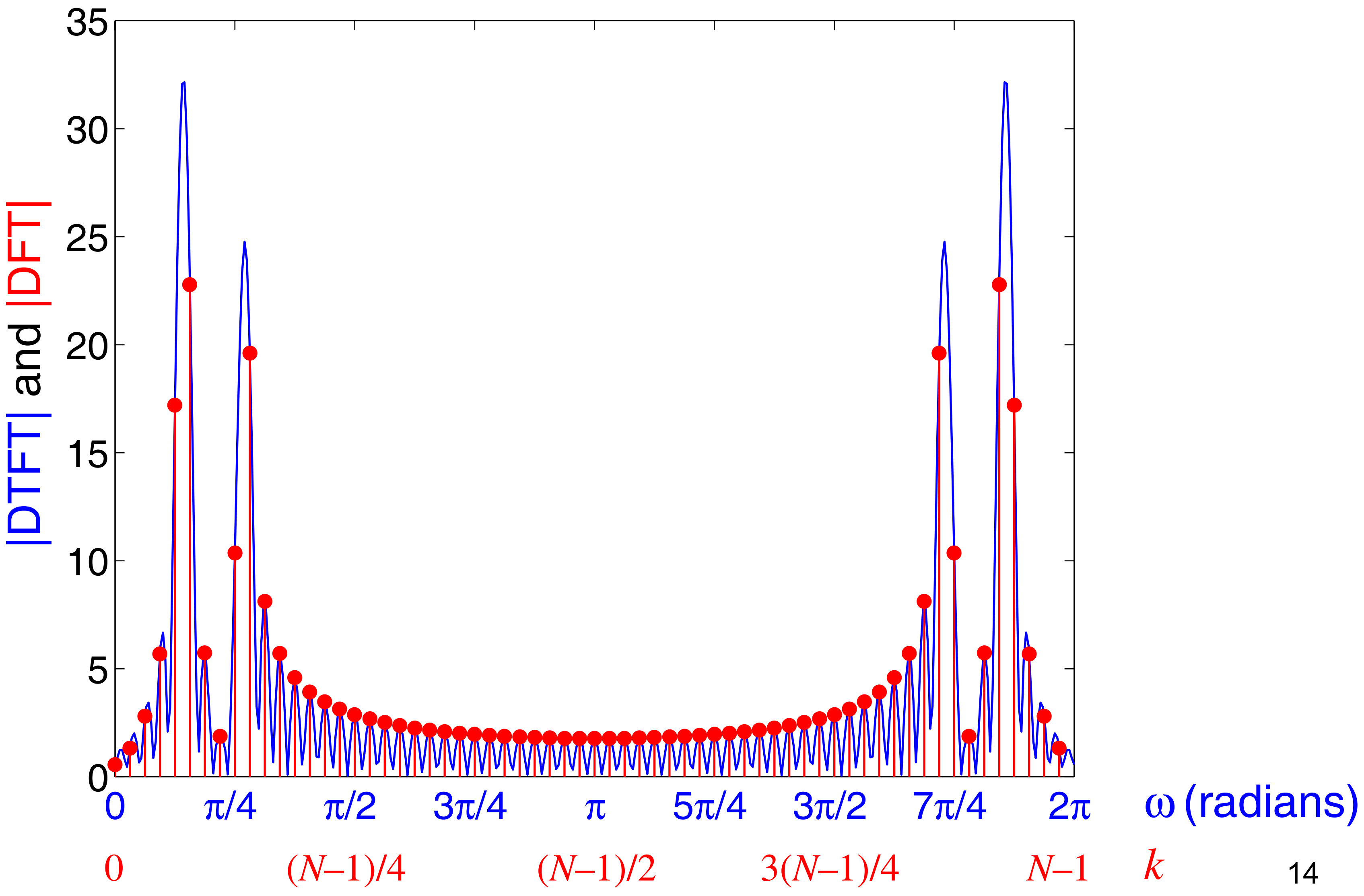
Example 1:

Consider the windowed discrete-time sequence:

$$x_1[n] = \begin{cases} \cos\left(\frac{2\pi}{14}n\right) + 0.75\cos\left(\frac{4\pi}{15}n\right), & 0 \le n \le 63, \\ 0, & \text{otherwise.} \end{cases}$$

The magnitude spectra for the DTFT and the 64-point DFT of $x_1[n]$ are shown on the next slide.

Note that the amplitudes and frequencies of the two cosines are not well represented by the DFT.



Example 2:

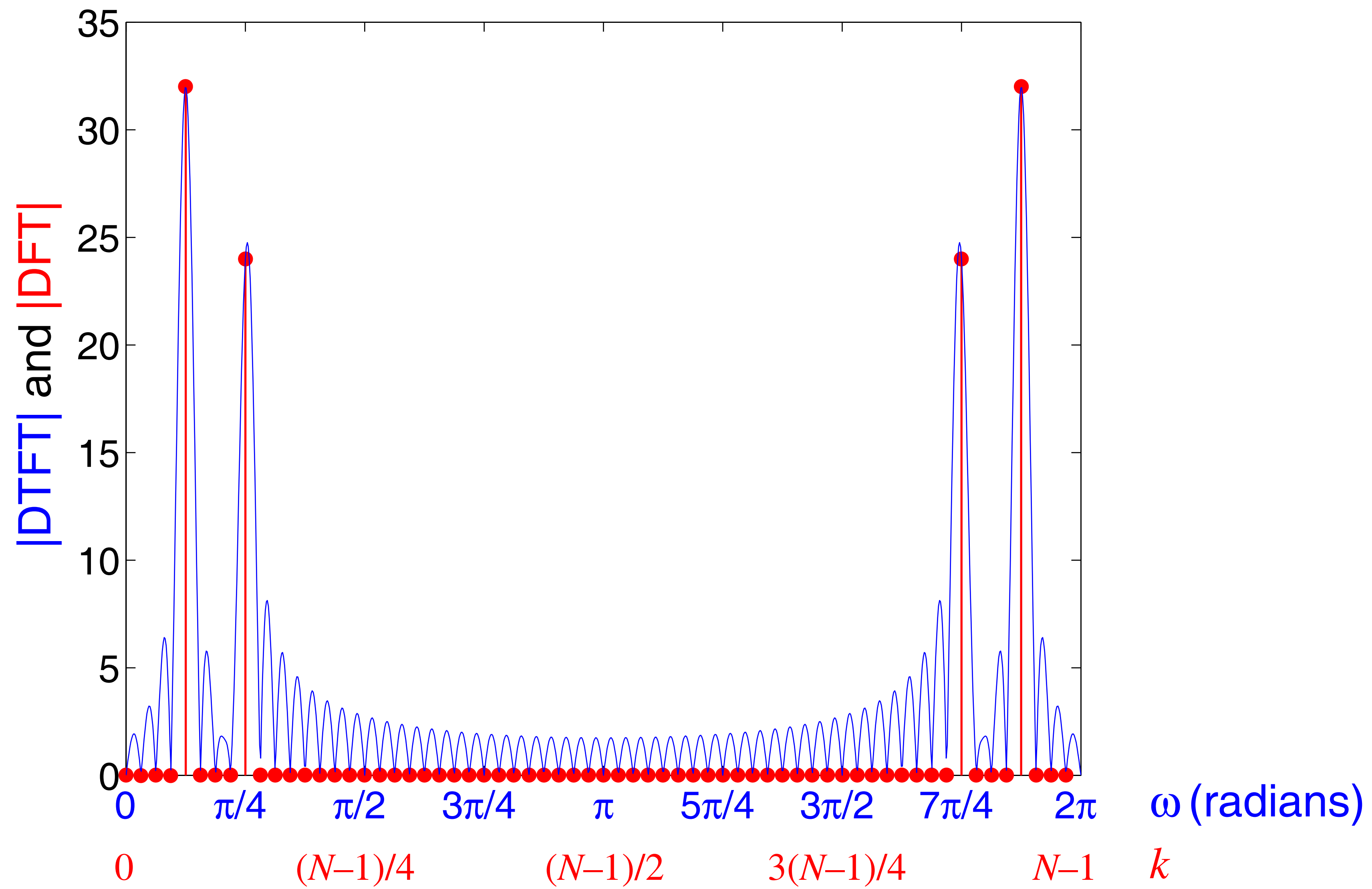
Now consider a second windowed discrete-time sequence:

$$x_2[n] = \begin{cases} \cos\left(\frac{2\pi}{16}n\right) + 0.75\cos\left(\frac{2\pi}{8}n\right), & 0 \le n \le 63, \\ 0, & \text{otherwise.} \end{cases}$$

The magnitude spectra for the DTFT and the 64-point DFT of $x_2[n]$ are shown on the next slide.

Note that the amplitudes and frequencies of the cosines are now well represented by the DFT, because the two cosine frequencies fall exactly on two frequency sampling points in the DFT.

In addition, all other frequency sampling points of the DFT fall on frequencies for which the amplitude spectrum of the DTFT is zero.


35
30
25
20
15
10
5
0
|DTFT| and |DFT|
0
π/4
π/2
3π/4
π
5π/4
3π/2
7π/4
2π
ω (radians)
0
(N–1)/4
(N–1)/2
3(N–1)/4
N–1
k