COMP ENG 4TL4:

# Digital Signal Processing

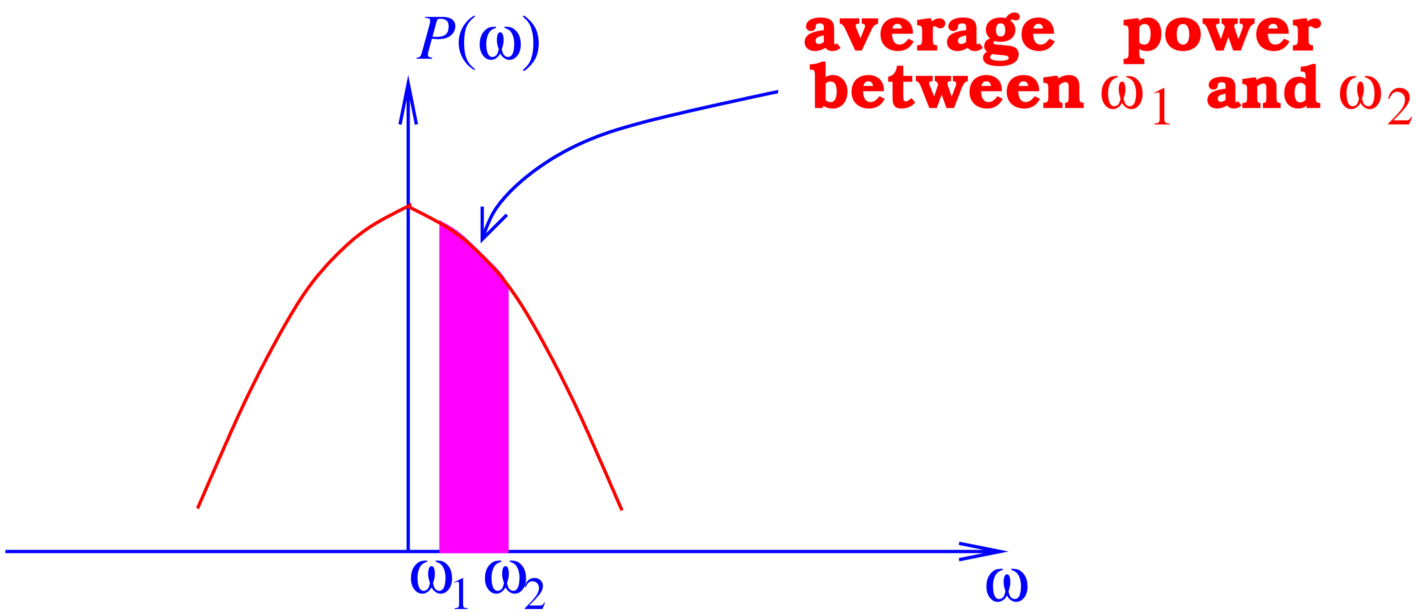
## Notes for Lecture #28

Wednesday, November 12, 2003

# 6.3 Spectral Estimation of Stationary Random Signals

Definition of the *power spectral density* (PSD) for a finite-power random signal:

$$P_{xx}(\omega) = \lim_{N\to\infty} \frac{1}{N} \mathrm{E}\left\{ \left| \sum_{n=-(N-1)/2}^{(N-1)/2} x[n]\, e^{-j\omega n} \right|^2 \right\}.$$



# The Periodogram:

In the task of *spectral estimation*, we wish to obtain an estimate of the PSD from a single sequence $x[n]$, i.e., without having to calculate an expected value $\mathrm{E}\{\cdot\}$ as is required for computing the PSD.

An obvious estimator of the PSD that can be obtained using the DTFT of a windowed sequence $x[n]w[n]$, where $w[n]$ is a rectangular window of length $L$, is the periodogram:

$$\widehat{P}_p(\omega) = \frac{1}{L}\left|\sum_{n=0}^{L-1} x[n]\, e^{-j\omega n}\right|^2 .$$

Properties of the periodogram:

– it can be computed for equally-spaced frequencies using the FFT with zero-padding

– its variance:

$$\mathrm{var}\left[\hat{P}_p(\omega)\right] \simeq P_{xx}^2(\omega)\,.$$

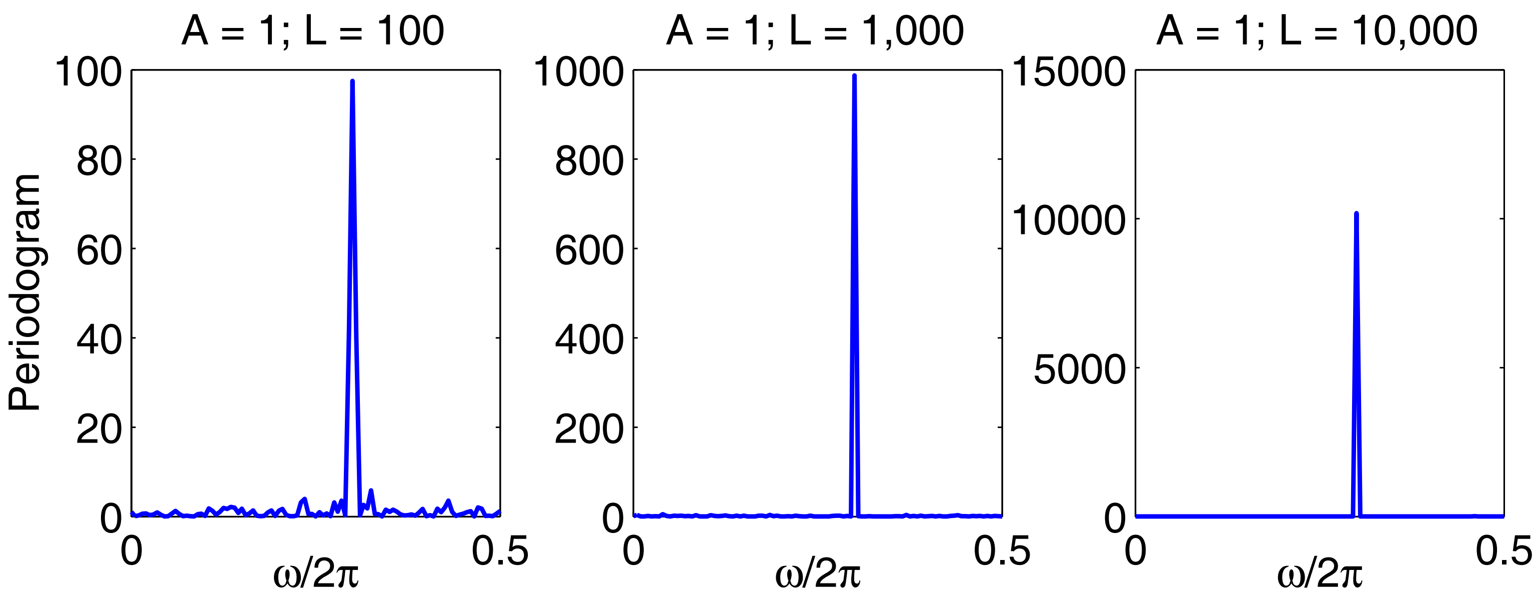
That is, its variance is quite large and it does not reduce with increasing $L$ for a stationary random signal!

$\Rightarrow$ this is our second case for which increasing the window length does not improve spectral estimation

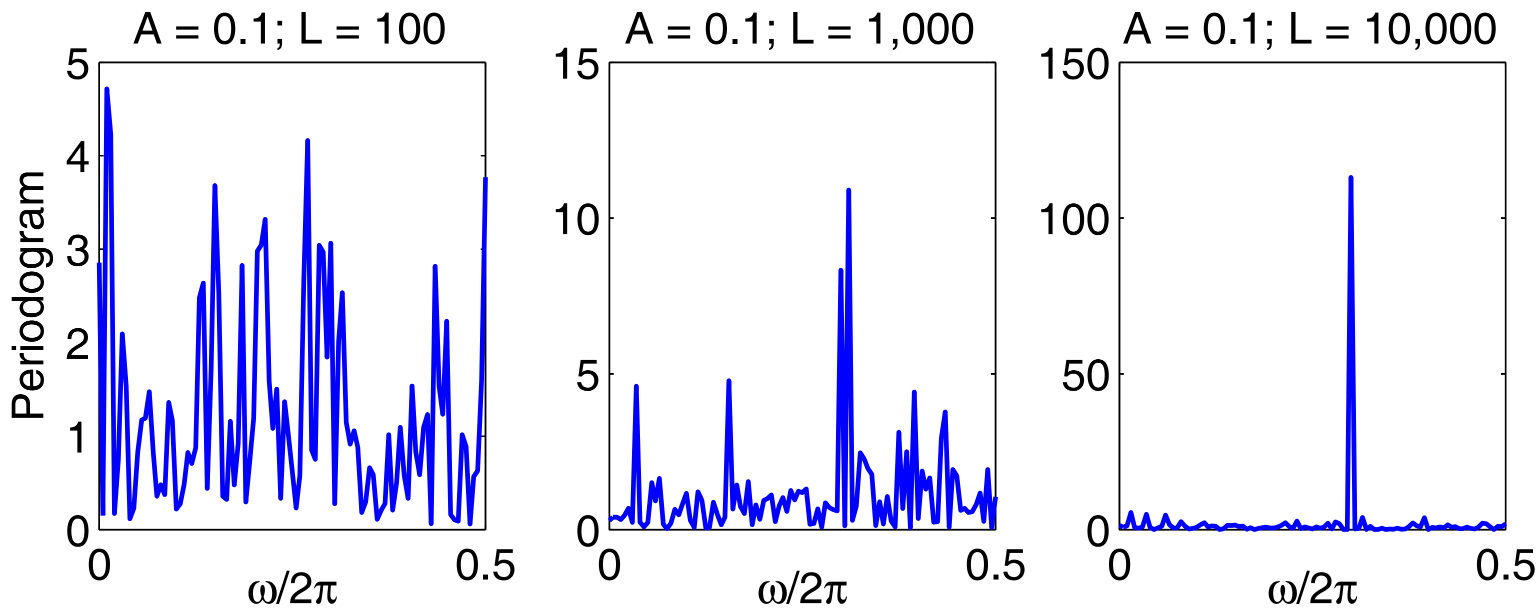
Example #1:

$$x[n] = A\ \exp(j2\pi f_0 n) + \xi[n]\,,$$

where $f_0 = 0.3$ and $\xi[n]$ is a zero-mean, unit-variance complex white Gaussian noise. Note that $x[n]$ consists of a *deterministic* (nonrandom) complex exponential and a white (flat-spectrum) *stationary noise*.



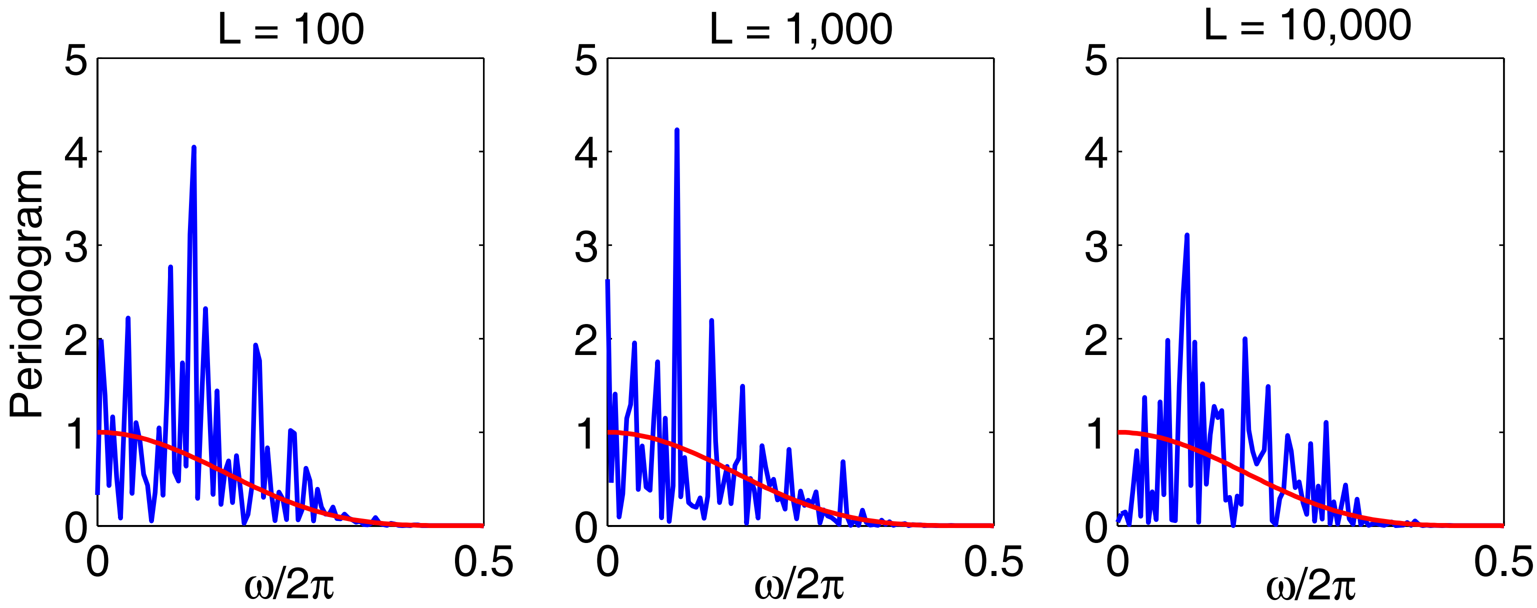
## Example #1 (cont.):




The contribution of the *deterministic* component of $x[n]$ to the periodogram increases with increasing $L$, but the contribution of the *stationary random* component does not increase.

$\Rightarrow$ spectral estimation of the deterministic component improves with increasing $L$

Example #2: Let $x[n]$, a zero-mean, unit-variance white Gaussian noise, be filtered by a lowpass filter with the magnitude-squared frequency response indicated by the red line in the plots below to give the lowpass Gaussian noise signal $y[n]$. The periodogram of one realization of $y[n]$ is:




Note that the PSD of $y[n]$ is equal to the filter's magnitude-squared frequency response (indicated by the red line), but the periodogram of $y[n]$ does not converge to the PSD with increasing $L$.

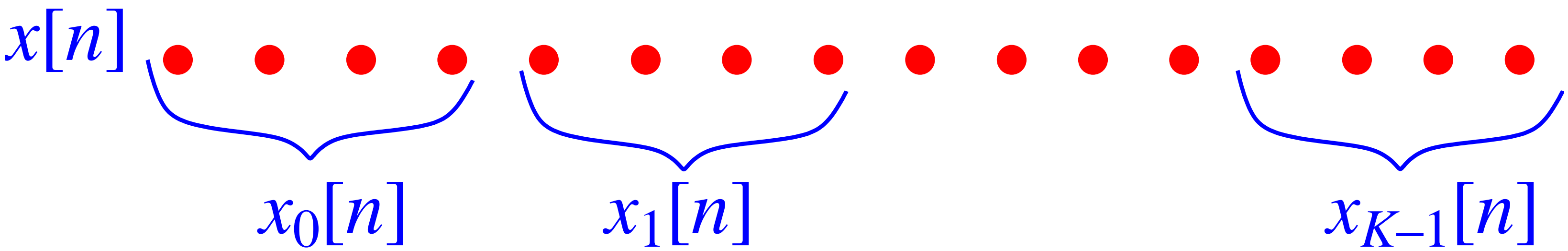
## Periodogram Averaging:

Comparing the equations for the PSD and the periodogram, we see that the problem with the variance of the periodogram arises because it does not include the expectation operation $\mathrm{E}\{\cdot\}$.

However, we can approximate this operation for a stationary random signal by breaking it up into a set of shorter segments, calculating the periodogram for each segment and then averaging the results. The basis of this methodology is:

- the periodogram of a short segment of the sequence will have a variance not much larger than the periodogram of the whole sequence
- the signal is stationary, so its PSD is identical for the different segments
- if the random signal is relatively uncorrelated, then the periodograms are relatively independent random variables, so the averaging process reduces the estimator's variance

# <u>The Bartlett periodogram method:</u>




Based on dividing the original sequence into $K = L/M$ <u>nonoverlapping</u> segments of length $M$, computing periodogram for each segment, and averaging the result:

$$\widehat{P}_B(\omega) = \frac{1}{K} \sum_{k=0}^{K-1} \widehat{P}_k(\omega)$$

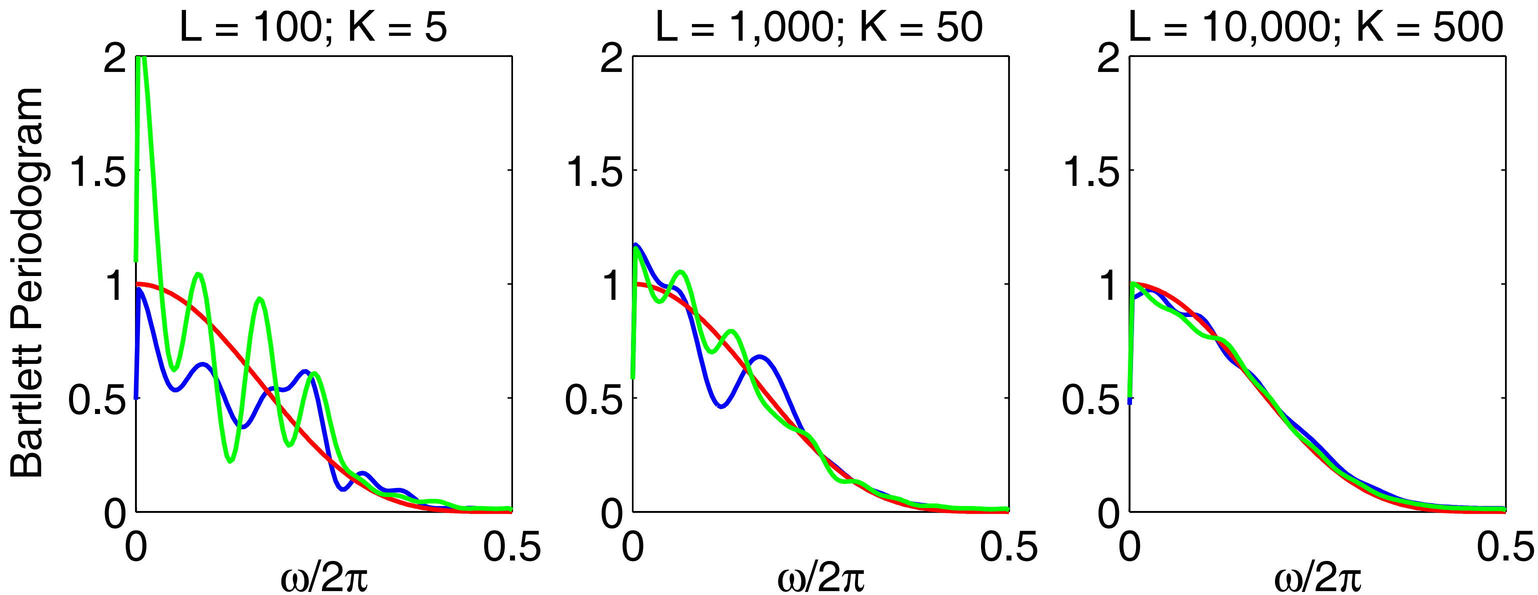
$$\widehat{P}_k(\omega) = \frac{1}{M} \left| \sum_{n=0}^{M-1} x_k[n]\, e^{-j\omega n} \right|^2 .$$

If the $K$ periodograms in the Bartlett method are independent, then the variance of the Bartlett average periodogram:

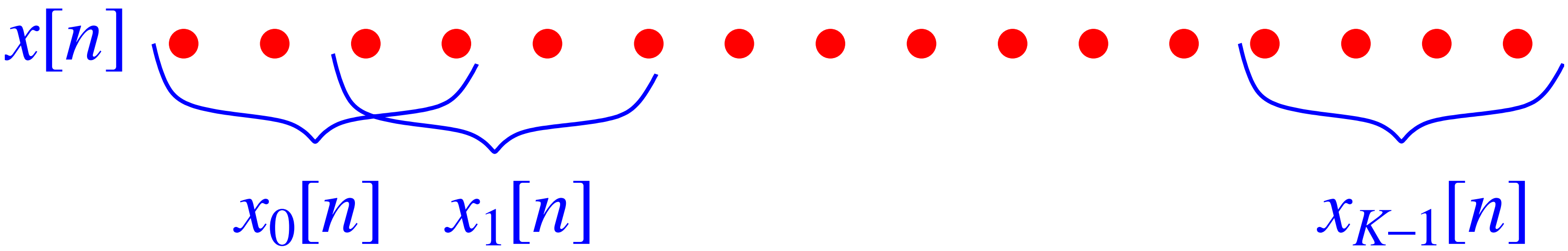
$$\text{var}\left[\widehat{P}_B(\omega)\right] \simeq \frac{1}{K} P_{xx}^2(\omega)\,.$$

That is, its variance decreases with increasing $K$!

<u>Example #3:</u> The Bartlett periodogram with $M = 20$ for the same signal as in Example #2, where the blue and green lines represent the periodograms for two different realizations of $y[n]$:



# The Welch periodogram method:

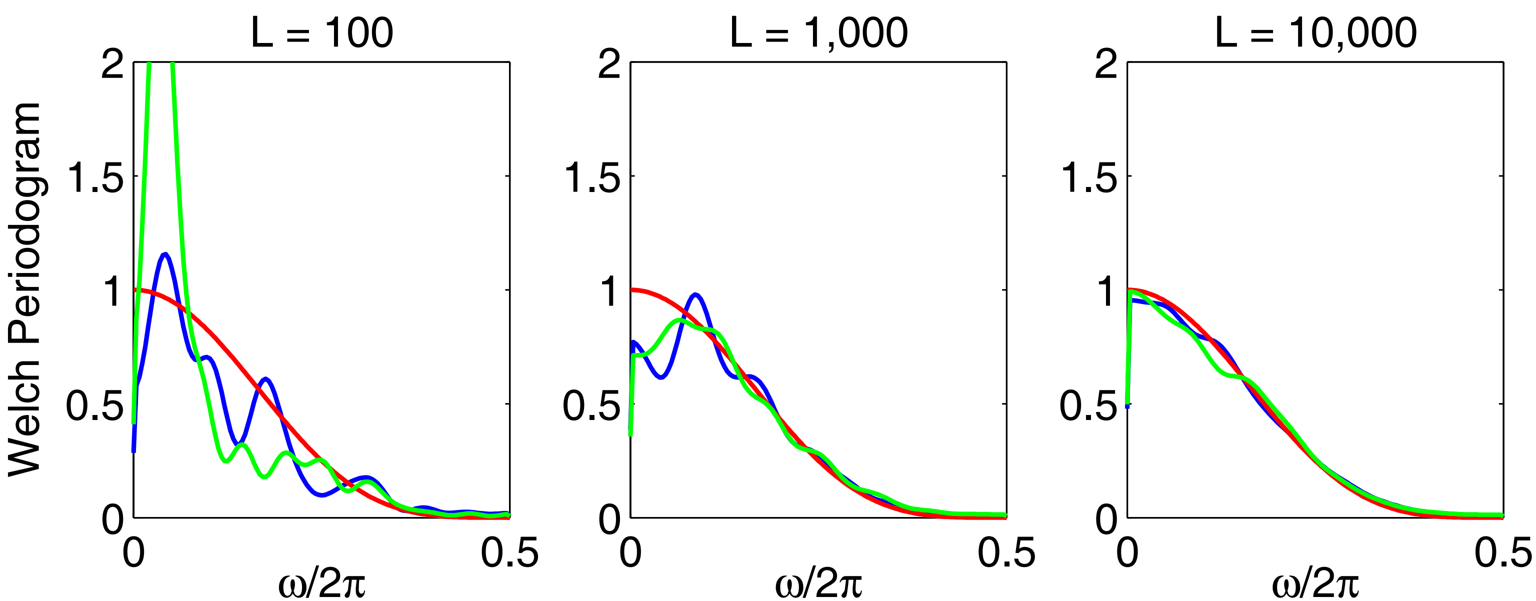



Refines the Bartlett method by dividing the original sequence into $K$ overlapping segments of length $M$.

Welch showed that:

- if the segments overlap by 50%, then the variance is reduce by almost of factor of 2 compared to the Bartlett method, because of the doubling in the number of sections
- increasing the overlap by more than 50% does not further reduce the variance, because the segments become less and less independent
- the variance still behaves the same if a nonrectangular window is used $\rightarrow$ the *modified periodogram*

Example #4: The Welch periodogram with $M = 20$ and 50% overlap for the same signal as in Example #2, where the blue and green lines represent the periodograms for two different realizations of $y[n]$:




Note that the variance has decreased only slightly from that of the Bartlett method because the lowpass noise is somewhat correlated.