ECE 796:

Models of the Neuron

Slides for Lecture #6

Tuesday, February 15, 2011

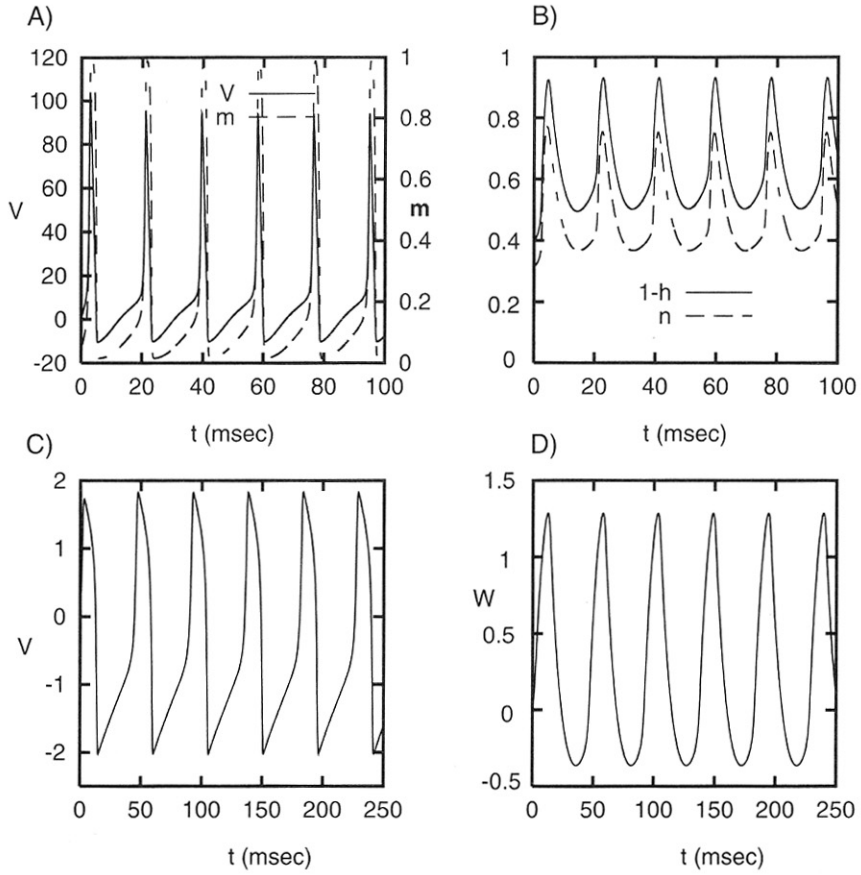
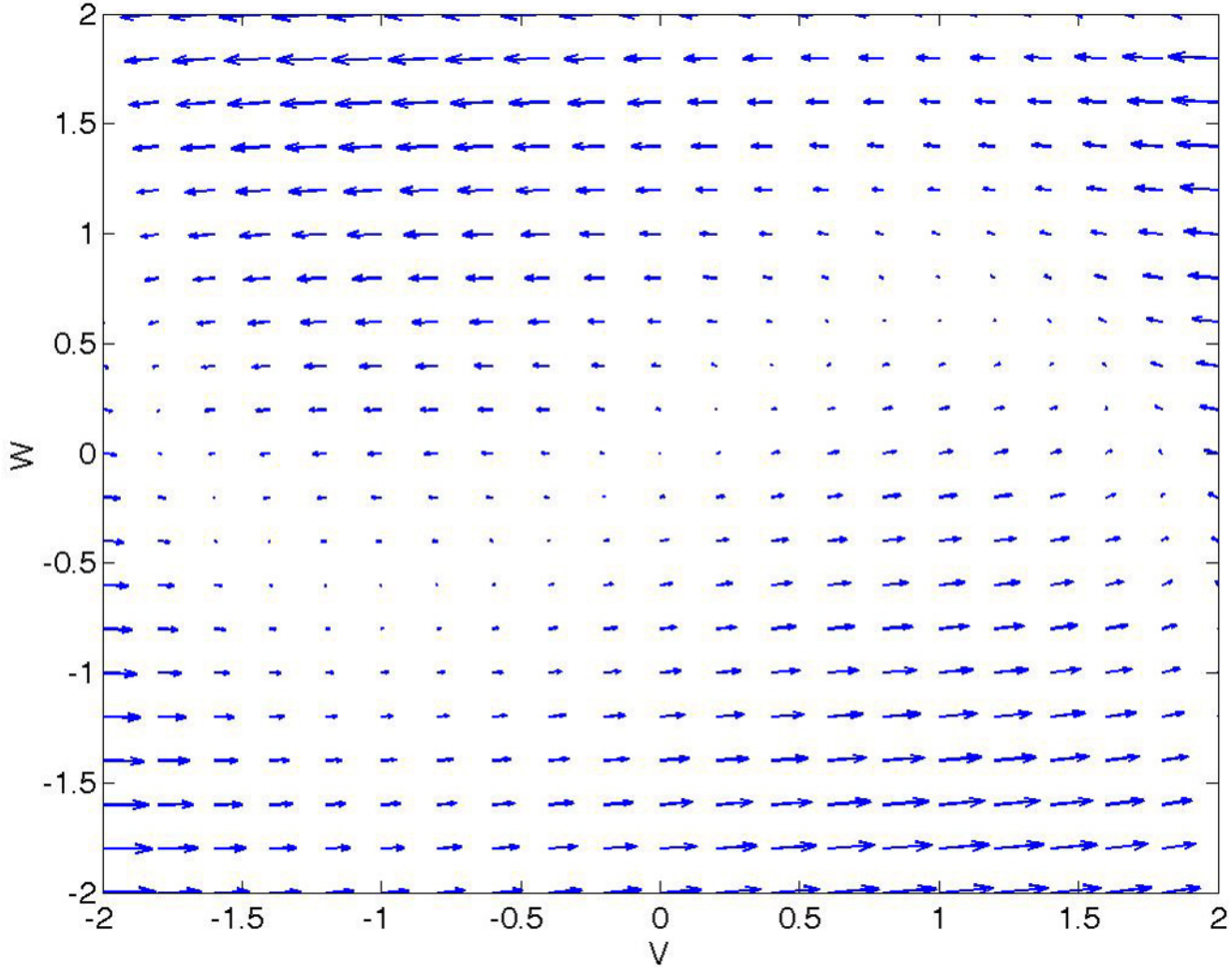


Fig. 7.1 Reducing the Hodgkin–Huxley Model to the FitzHugh–Nagumo System Evolution of the space-clamped Hodgkin–Huxley and the FitzHugh-Nagumo equations in response to a current step of amplitude 0.18 nA in A and B and of amplitude I = 0.35 in C and D. **(A)** Membrane potential $V(t)$ and sodium activation $m(t)$ (see also Fig. 6.8). Sodium activation closely follows the dynamics of the membrane potential. **(B)** Sodium inactivation $1 - h$ and potassium activation n of the Hodgkin–Huxley system. **(C)** "Excitability" $V(t)$ of the two-dimensional FitzHugh–Nagumo equations (Eqs. 7.1) with constant parameters has a very similar time course to V and m of the squid axon (notice the different scaling). **(D)** The "accommodation" variable W shows modulations similar to $1 - h$ and n of the Hodgkin–Huxley equations.


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
W
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
V

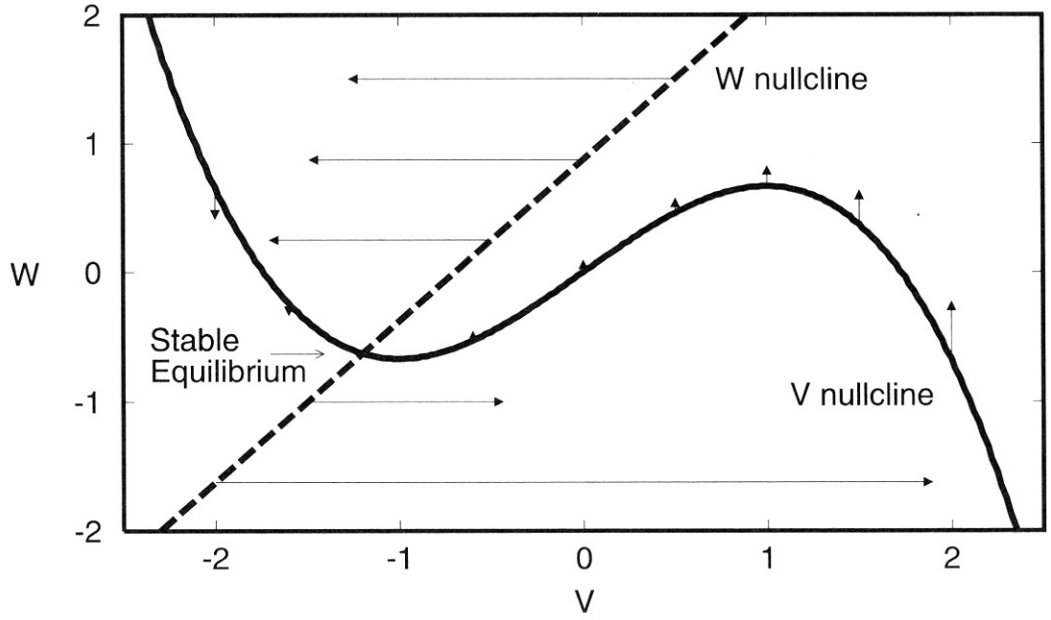
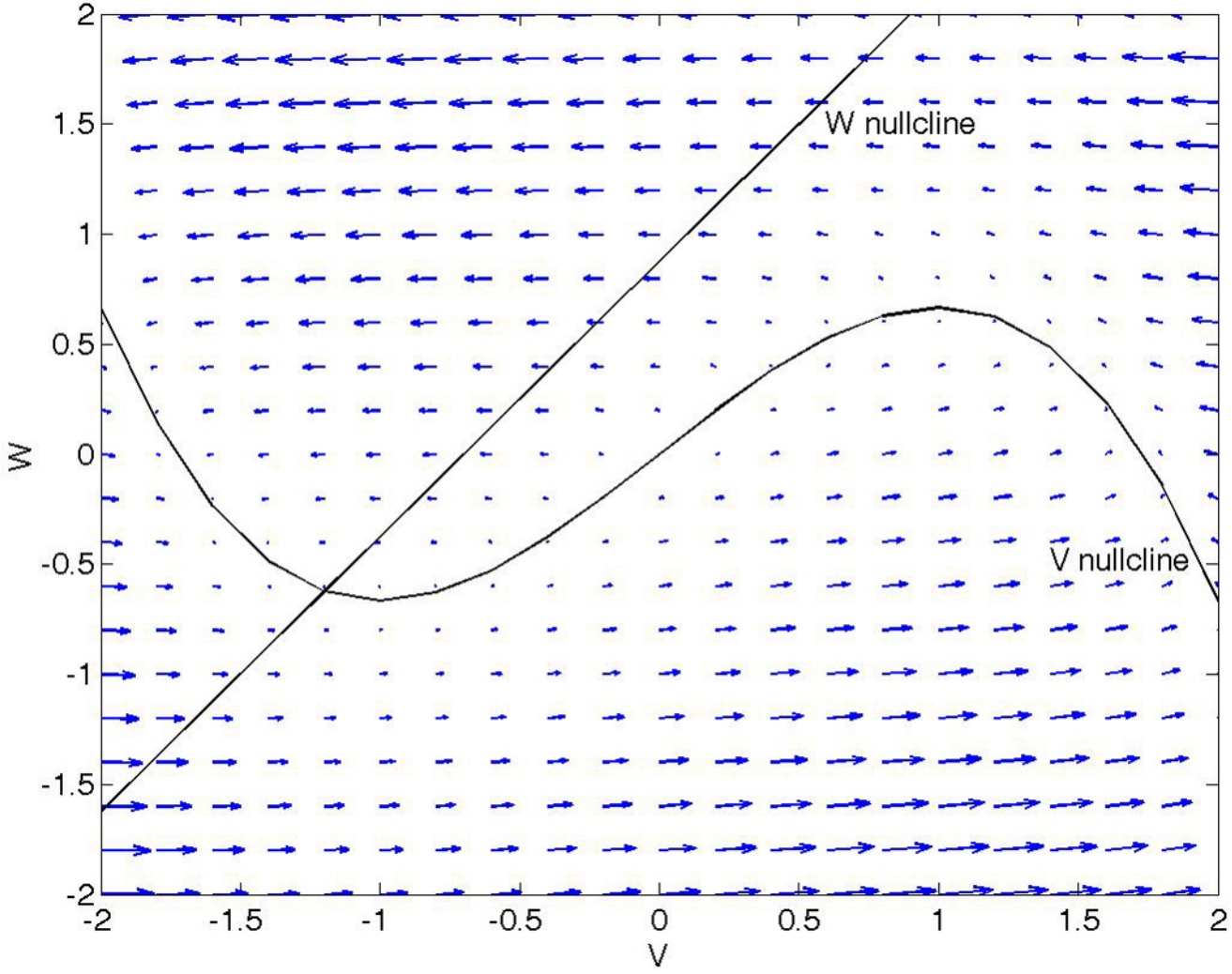


Fig. 7.2 PHASE PLANE PORTRAIT OF THE FITZHUGH–NAGUMO MODEL Phase plane associated with the FitzHugh–Nagumo Eqs. 7.1 for $I = 0$. The fast variable V corresponds to membrane excitability while the slower variable W can be visualized as the state of membrane accommodation. The nullcline for the V variable, that is, all points with $\dot{V} = 0$, is a cubic polynomial, and the W nullcline (all points with $\dot{W} = 0$) is a straight line. The system can only exist in equilibrium at the intersection of these curves. For our choice of parameters and for $I = 0$, a single equilibrium point exists: $(\overline{V}, \overline{W}) = (-1.20, -0.625)$. The arrows are proportional to $(\dot{V}, \dot{W})$ and indicate the direction and rate of change of the system: V usually changes much more rapidly than W.


W nullcline
V nullcline
W
V
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

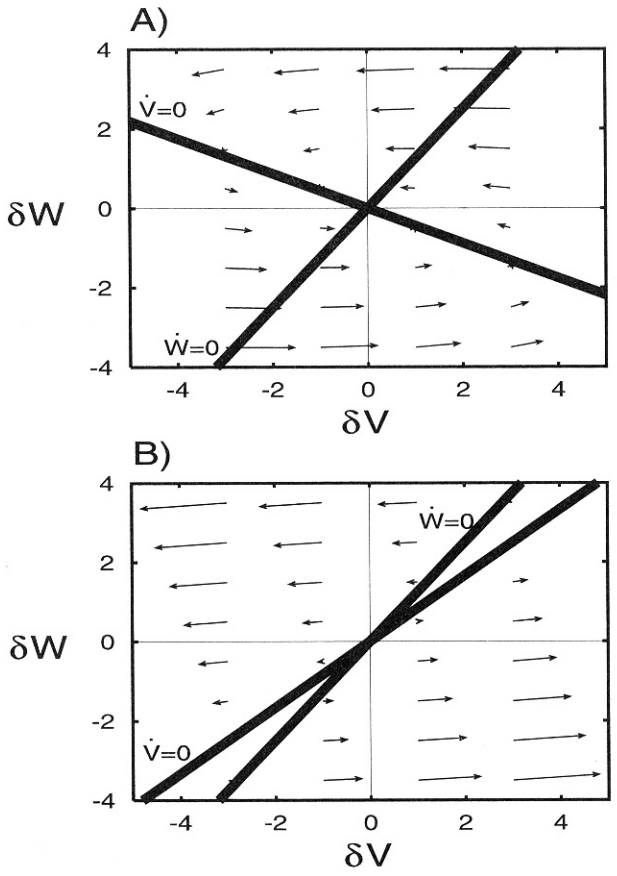
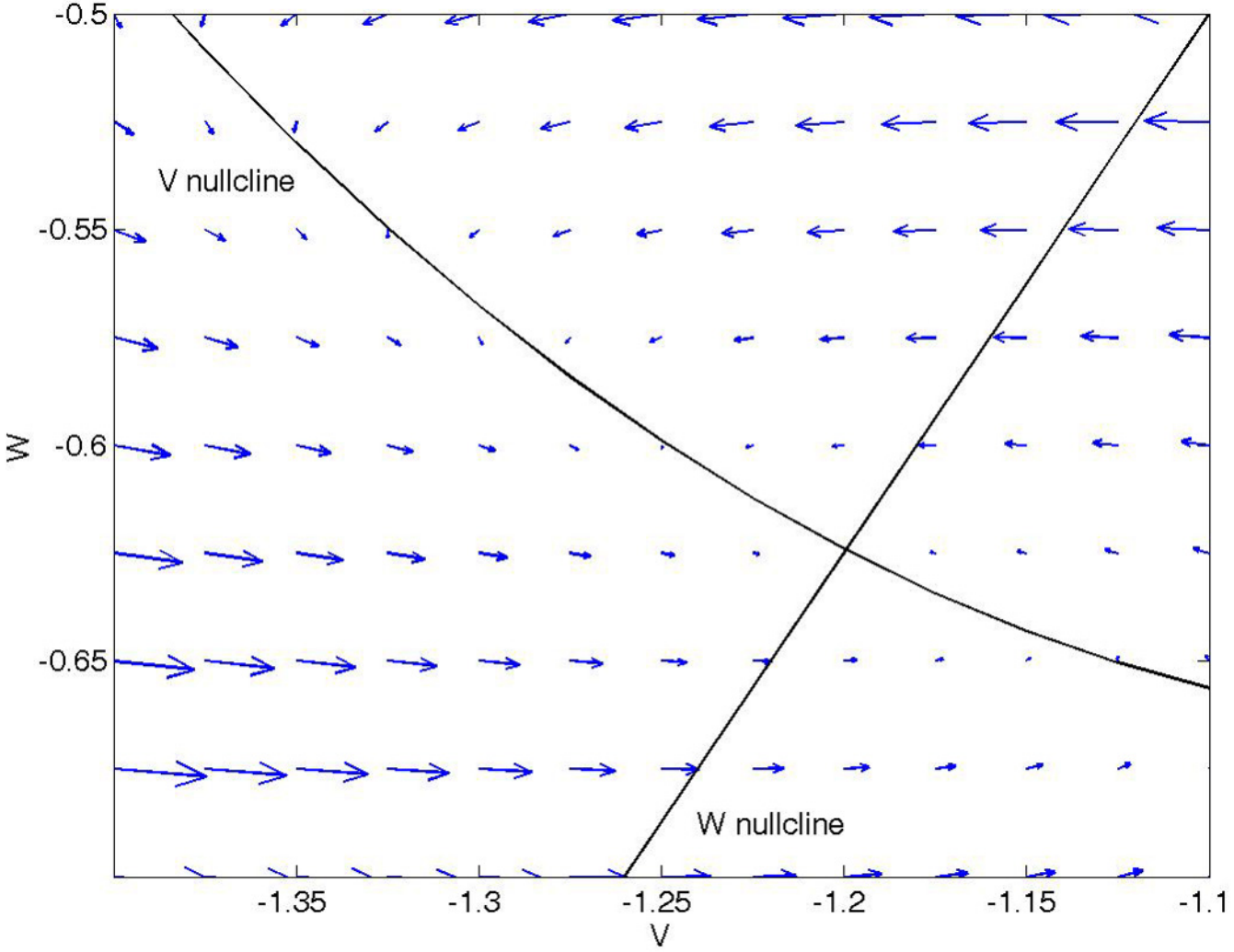


Fig. 7.3 Behavior around the Equilibrium Point The behavior of the FitzHugh–Nagumo equations in a small neighborhood around the fixed point is determined by linearizing these equations around their fixed point and computing the associated pair of eigenvalues. **(A)** Evolution of $(\delta V, \delta W)$ in a coordinate system centered at the equilibrium point $\mathbf{r}^*$ for $I = 0$. Because the real part of both eigenvalues is negative, a small perturbation away from the fixed point will decay to zero, rendering this point asymptotically stable. Any point in this plane will ultimately converge to the fixed point at the origin. **(B)** Similar analysis for the equilibrium point $\mathbf{r}^{*\prime}$ for a sustained input with $I = 1$ (Fig. 7.5A). The fixed point is unstable.


-0.5
-0.55
-0.6
-0.65
W
V nullcline
W nullcline
-1.35
-1.3
-1.25
-1.2
-1.15
-1.1
V

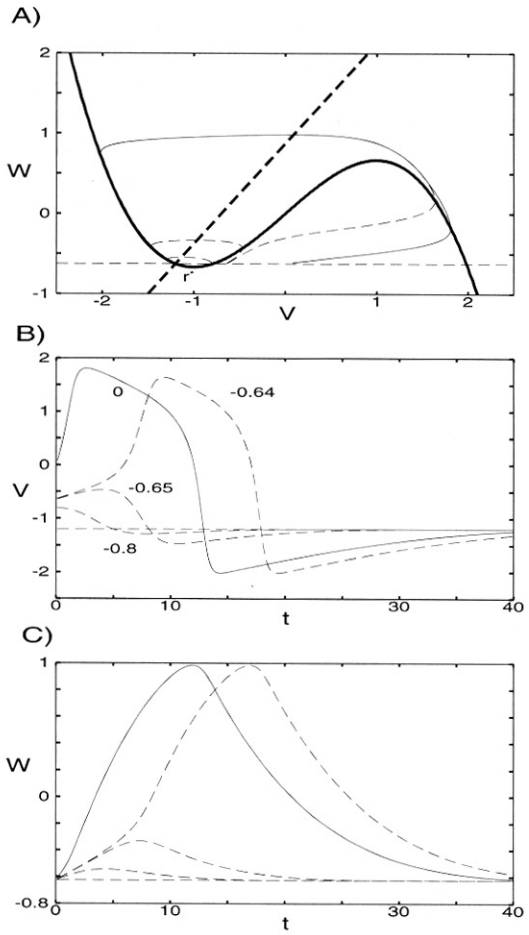
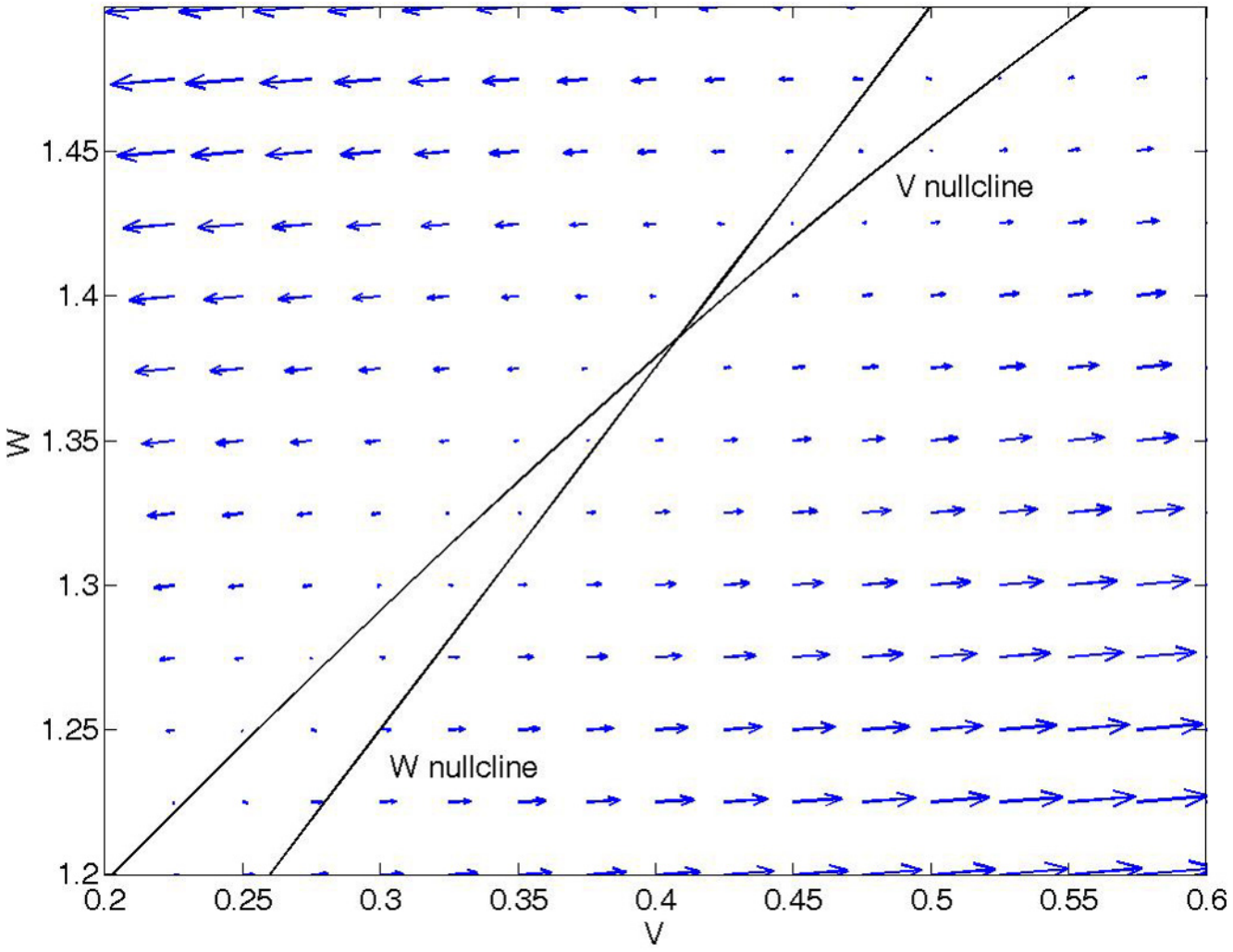


Fig. 7.4 Response of the FitzHugh–Nagumo Model to Current Pulses **(A)** The quiescent system is excited by a current pulse of different amplitudes $I = Q\delta(t)$, displacing the system from its resting state $\mathbf{r}^*$ along the dashed horizontal line. Following Eqs. 7.1, this briefly increases $\dot{V}$, in agreement with physical intuition, since a brief current pulse will cause a transient capacitive current $C\,dV/dt$ to flow. The evolution of the voltage V and of the adaptation variable W is plotted in **(B)** and **(C)**. Changing V from its initial value of -1.2 to -0.8 or -0.65 only causes quick excursions of the voltage around the equilibrium point with the system rapidly returning to rest (the oscillatory manner in which the system does so is not readily apparent at the scale of these panels). If the current pulse is large enough so that V exceeds -0.64, a stereotyped "all-or-none" sequence is triggered: V rapidly increases to positive values but then dives below its resting value $\overline{V}$ before finally coming to rest again at $\mathbf{r}^*$. Notice that the trajectory in this case consists of "fast" segments, where V changes rapidly but W remains essentially constant (upper and lower segments), interconnected by "slow" segments, where the system changes so slowly that V is always in equilibrium (the "slow" segments closely coincide with the V nullcline).


V nullcline
W nullcline
W
V
1.45
1.4
1.35
1.3
1.25
1.2
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6

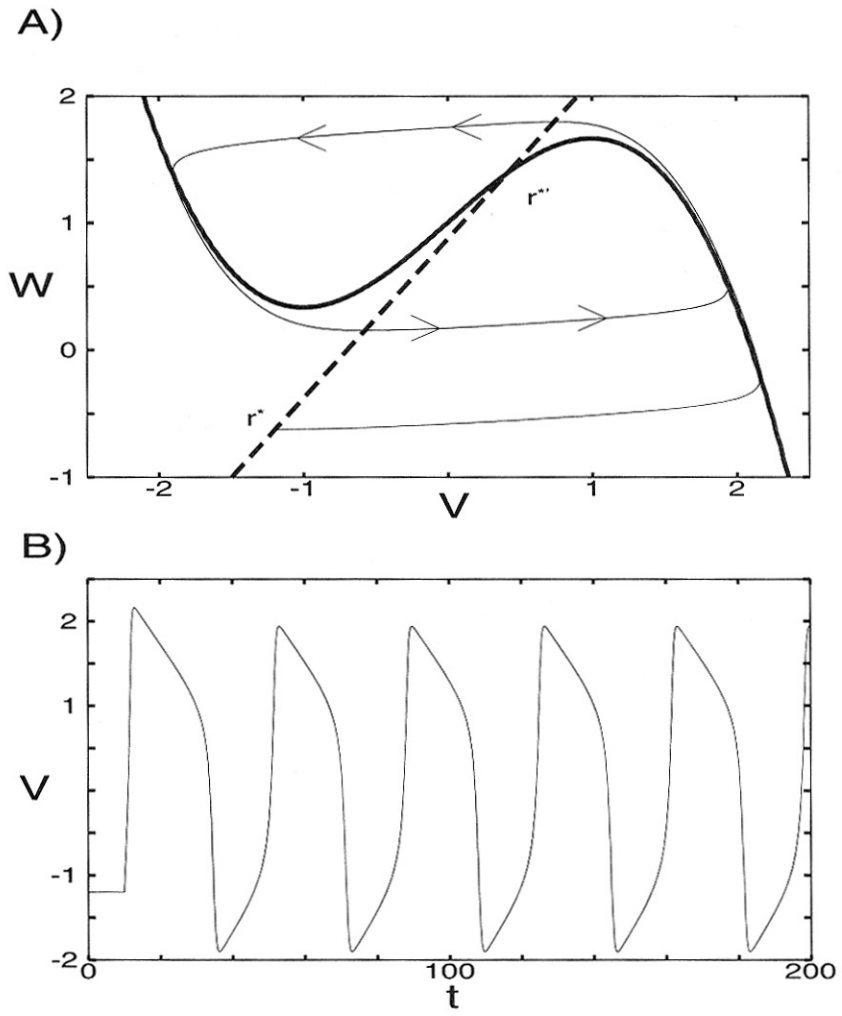
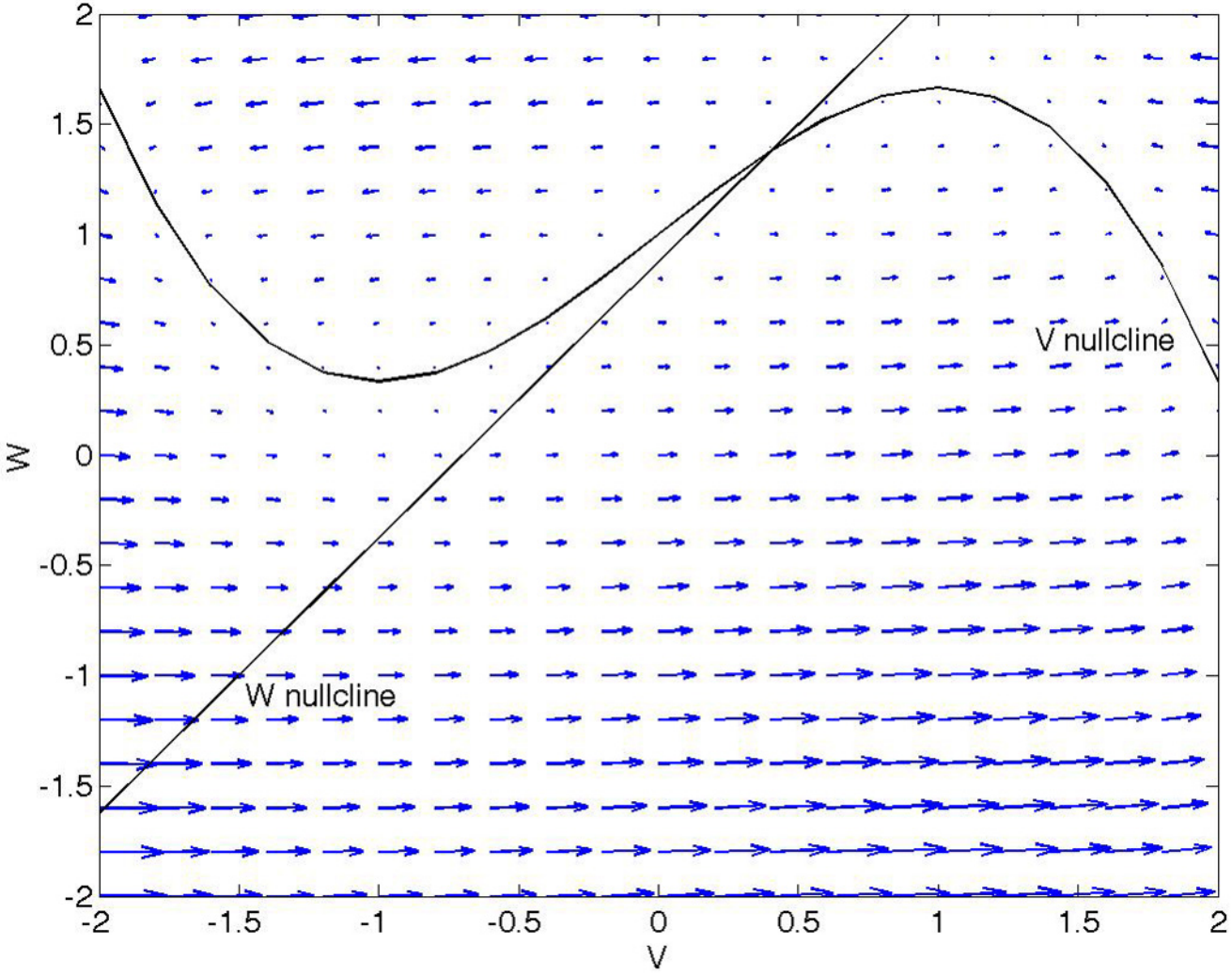


Fig. 7.5 Response of the Fitz-Hugh–Nagumo Model to Current Steps **(A)** The quiescent system is subject to a current step of amplitude I. For $I < 0.32$, the new equilibrium $\mathbf{r}^{*\prime}$ is stable: the system depolarizes but remains subthreshold. For larger steps (here $I = 1$), the new equilibrium point lies along the middle portion of the V nullcline and is unstable. Because the system has a stable limit cycle, it will not diverge; rather, it generates a train of "action potentials" whose time course is shown in **(B)**. Regardless of the initial state of the system, it will always converge rapidly onto this limit cycle.


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
W
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
V
V nullcline
W nullcline

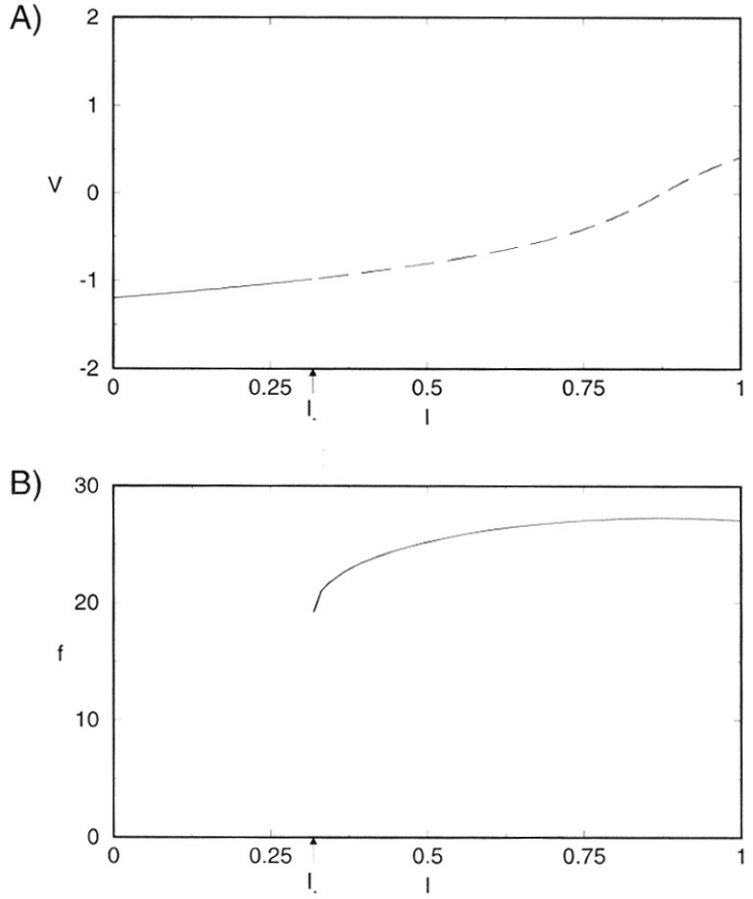


Fig. 7.6 Discharge Curve of the FitzHugh–Nagumo Model **(A)** Steady-state membrane potential and **(B)** oscillation frequency of the limit cycle for the FitzHugh–Nagumo equations as a function of a sustained current I. As the membrane is depolarized in response to increasing injection of current, it loses stability at $I_- = 0.33$ (arrow) and moves on a stable limit cycle. An important property of this type of bifurcation phenomenon, known as hard excitation or subcritical Hopf bifurcation, is that oscillations occur with nonzero frequency. This behavior is also characteristic for the Hodgkin–Huxley equations. Between I_- and $I_+ = 1.42$, the system moves along the limit cycle. The frequency of the oscillation is a function of I (dashed line in panel A). Beyond I_+, the equilibrium point becomes stable again, and the system remains "locked" at a depolarized level (not shown).