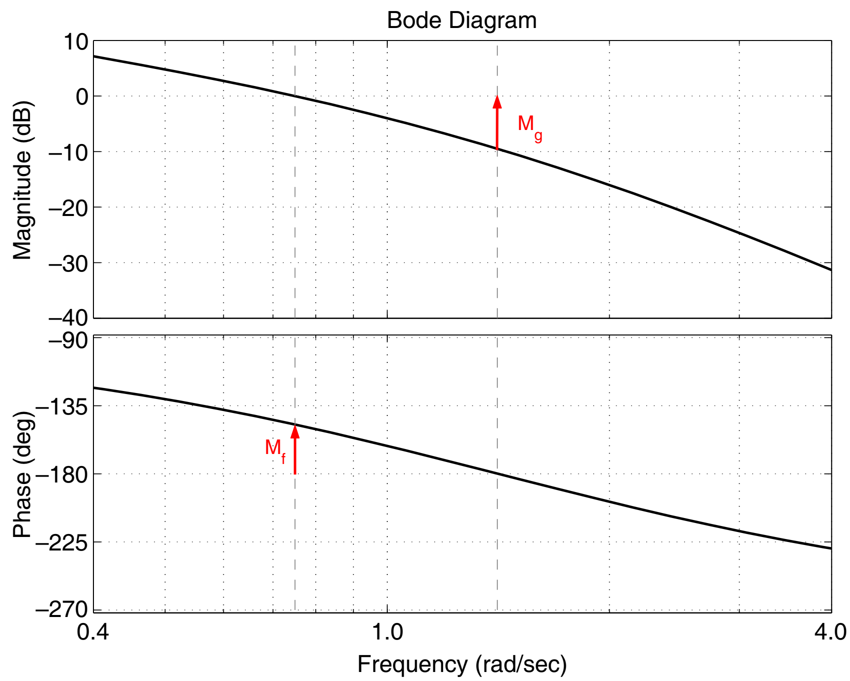
Electrical Engineering EE 4CL4

Solutions to Final Exam 2003

1. **The figure below shows the Bode diagram for the open-loop transfer function $C(s)G_o(s)$ of a system to be placed in a one-d.o.f. unity-feedback loop.**



a. **From the Bode diagram, estimate both the *gain margin* and the *phase margin* for stability.**

b. **Design a lead compensator to increase the phase margin for this system.**

a. The gain margin is defined as the increase in gain that would be required at the frequency for which the phase is −180° to bring the gain to 0 dB. For this system, the gain margin ≈ 9.5 dB, as indicated on the plot above. The phase margin is defined as the decrease in phase that would be required at the frequency for which the gain is 0 dB to bring the phase to −180°. For this system, the phase margin ≈ 33°, as indicated on the plot above.

b. A lead compensator is obtained from Eqn. (6.6.1) given on page 6 of the exam paper, in the case where $\tau_1 > \tau_2$. In order to increase the phase margin, τ_1 is set to the reciprocal of the frequency at which the phase margin is determined, which is ~ 0.75 rad/sec in the plot above. Therefore, $\tau_1 = 1/0.75 = 1.3333$, and τ_2 is set to some value much less than τ_1, e.g., $\tau_2 = \tau_1/10 = 0.1333$, giving the transfer function:

$$C(s) = \frac{1.3333s + 1}{0.1333s + 1}.$$

2. **Determine the PID controller parameters (K_p, T_r and T_d) for a plant with the nominal model:**

$$G_o(s) = \frac{-s+4}{(s+2)^2},$$

using the *Ziegler-Nichols oscillation method*.

The closed-closed characteristic polynomial for this nominal plant model in a one-d.o.f. unity-feedback loop with a proportional controller is:

$$1 + K_p G_o(s) = (s+2)^2 + K_p(-s+4) = 0 .$$

At the point of critical stability, $K_p = K_c$ and $s = j\omega_c$, such that:

$$(s+2)^2 + K_p(-s+4) = (j\omega_c + 2)^2 + K_c(-j\omega_c + 4) = 0$$

$$\Rightarrow K_c = \frac{-(j\omega_c + 2)^2}{(-j\omega_c + 4)} = \frac{8\omega_c^{\,2} - 16 + j(\omega_c^{\,3} - 20\omega_c)}{\omega_c^{\,2} + 16} .$$

The critical gain $K_c \in \Re$, so the complex term in the equation above must equal zero, which gives:

$$\omega_c^{\,3} - 20\omega_c = 0 \Rightarrow \omega_c = 2\sqrt{5} \Rightarrow P_c = \frac{2\pi}{\omega_c} = \frac{\pi}{\sqrt{5}} .$$

Substituting the value for ω_c into the equation for K_c yields:

$$K_c = \frac{8\omega_c^{\,2} - 16}{\omega_c^{\,2} + 16} = \frac{8 \cdot 20 - 16}{20 + 16} = 4 .$$

From Table 6.1 given on page 5 of the exam paper, the PID parameters are then:

$$K_p = 0.6K_c = 2.4 ,\ T_r = 0.5P_c = 0.7025 , \text{ and } T_d = P_c/8 = 0.1756 .$$

3. **Compare and contrast the requirements for *stability robustness* and *performance robustness*.**

Robust stability occurs when modelling errors are small enough that the control loop is stable for the true plant (as described by the calibration model) as well as for the nominal plant model. From Eqn. (5.9.6) given on page 5 of the exam paper, *robust stability* is assured if $|T_o(\mathrm{j}\omega)||G_\Delta(\mathrm{j}\omega)| < 1, \quad \forall \omega$.

Robust performance occurs when the achieved (true) sensitivity functions given in Eqns. (5.9.1)–(5.9.4) on page 5 of the exam paper are close to the nominal sensitivity functions. From Eqns. (5.9.15)–(5.9.19) on page 5 of the exam paper, this will occur when the error sensitivity $S_\Delta(s)$ is close to 1 + j0. The error sensitivity takes a value around 1 when $|T_o(\mathrm{j}\omega)G_\Delta(\mathrm{j}\omega)| << 1, \quad \forall \omega$.

Comparing these two requirements, it can be seen that both stability robustness and performance robustness require that the magnitude of the nominal complementary sensitivity $T_o(s)$ becomes very small at frequencies for which the multiplicative modelling error (MME) $G_\Delta(s)$ grows large. However, robust performance sets a more stringent requirement that the magnitude of the product of the nominal complimentary sensitivity and the MME be not only less than 1 but *much less* than 1.

4. **Find suitable PID controller parameters (K_p, T_r and T_d) for a plant with the open-loop unit-step response:**

$$y(t) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t} \qquad \textbf{for} \qquad t \geq 0,$$

using the *Cohen-Coon reaction curve method*.

The open-loop unit-step response given above is referred to as the *process reaction curve*. The slope of the reaction curve is found by taking its derivative with respect to time:

$$\frac{\mathrm{d}\,y(t)}{\mathrm{d}\,t} = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}.$$

The time of the maximal slope of the reaction curve is obtained by finding the time at which the derivative of the slope with respect to time is zero:

$$\frac{\mathrm{d}^2\,y(t)}{\mathrm{d}\,t^2} = -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} = 0$$

$$\Rightarrow \frac{3}{2}e^{-3t} = \frac{1}{2}e^{-t}$$

$$3e^{-3t} = e^{-t}$$

$$3e^{-2t} = 1$$

$$e^{-2t} = \frac{1}{3}$$

$$-2t = \ln\left(\frac{1}{3}\right)$$

$$t = \frac{1}{2}\ln(3)$$

$$\Rightarrow t \approx 0.5493.$$

The maximal slope is then found by evaluating the equation for the slope at $t = ½\ln(3)$:

$$\left.\frac{\mathrm{d}\,y(t)}{\mathrm{d}\,t}\right|_{t=\frac{1}{2}\ln(3)} = \frac{1}{2}e^{-\frac{1}{2}\ln(3)} - \frac{1}{2}e^{-\frac{3}{2}\ln(3)} = \frac{1}{2}3^{-\frac{1}{2}} - \frac{1}{2}3^{-\frac{3}{2}} = 0.2887 - 0.0962 = 0.1925,$$

and the reaction curve value at the same time is:

$$y(t)\Big|_{t=\frac{1}{2}\ln(3)} = \frac{1}{3} - \frac{1}{2}e^{-\frac{1}{2}\ln(3)} + \frac{1}{6}e^{-\frac{3}{2}\ln(3)} = 0.0767.$$

Consequently, the *maximum slope tangent* (*m.s.t.*) is given by:

$$m.s.t. = 0.1925(t - 0.5493) + 0.0767.$$

The *m.s.t.* is equal to $y_0 = 0$ at time $t_1 = 0.1509$ and is equal to $y_\infty = 1/3$ at $t_2 = 1.8825$. The unit step ($u_0 = 0$; $u_\infty = 1$) was applied at time $t_0 = 0$, giving the parameter model:

$$K_0 = \frac{y_\infty - y_0}{u_\infty - u_0} = \frac{1}{3}; \qquad \tau_0 = t_1 - t_0 = 0.1509; \qquad \nu_0 = t_2 - t_1 = 1.7316.$$

From Table 6.3 on page 6 of the exam paper, the *Cohen-Coon* PID parameters are:

$$K_p = \frac{\nu_0}{K_0\tau_0}\left[\frac{4}{3} + \frac{\tau_0}{4\nu_0}\right] = 46.651,$$

$$T_r = \frac{\tau_0\left[32\nu_0 + 6\tau_0\right]}{13\nu_0 + 8\tau_0} = 0.3583, \text{ and}$$

$$T_d = \frac{4\tau_0\nu_0}{11\nu_0 + 2\tau_0} = 0.0540.$$

5. **There are a number of fundamental design limitations that place upper or lower bounds on the desired closed-loop bandwidth of a feedback control system. Discuss any two (2) of these limitations.**

   Possible answers include:

- The effect of measurement noise on the plant output in a unity-feedback control loop is given by $Y_m(s) = -T_o(s)D_m(s)$, where $y_m(t)$ is the plant output component due to the measurement noise. From this equation, the effects of the measurement noise can be reduced by ensuring that $|T_o(j\omega)|$ is small at frequencies where $|D_m(j\omega)|$ is large. Measurement noise is normally dominated by high frequencies and thus sets an *upper limit* on the bandwidth of the control loop.

- The effect of actuator maximal movement constraints (i.e., saturation limits on actuator amplitude or slew-rate limits on the speed of actuator movements) can be understood by noting that the amplitude of the plant input is related to the reference and the output disturbance via the relationship $U(s) = S_{uo}(s)[R(s) - D_o(s)]$ and the rate-of-change of the plant input via the relationship $sU(s) = S_{uo}(s)[sR(s) - sD_o(s)]$. The nominal control sensitivity $S_{uo}(s) = T_o(s)/G_o(s)$, and consequently if the closed-loop bandwidth is much larger than that of the open-loop plant model $G_o(s)$, large and fast changes in the reference or the output disturbance will result in large and fast changes in the plant input. Therefore, to avoid actuator saturation or slew-rate problems, it is necessary to place an *upper limit* on the control loop bandwidth.

- The closed-loop bandwidth should be greater than the real part of any unstable open-loop poles. Otherwise, the integral constraint $\int_0^\infty e(t)e^{-\eta_0 t}dt = 0$ in Lemma 8.3 states, the overshoot in response to a step reference change will be very large. Consequently, this sets a *lower limit* on the control loop bandwidth.

- The closed-loop bandwidth should be less than the smallest nonminimum-phase zero. Otherwise, the integral constraint $\int_0^\infty y(t)e^{-z_0 t}dt = 0$ in Lemma 8.3 states, the undershoot in response to a step reference change will be very large. Consequently, this sets an *upper limit* on the control loop bandwidth.

6. **Consider the nominal plant model:**

$$G_o(s) = \frac{16s^2 + 1}{(s+2)^3},$$

which is to be controlled by a one-d.o.f. unity-feedback loop.

a. **Find a time-domain integral constraint on the controller error *e*(*t*) in response to a unit step reference.**

b. **From the integral constraint obtained in part a, what can be said about the trade-off between the system response time and large transient responses in this feedback system?**

a. This plant has a pair of zeros on the imaginary axis at $\omega_o = \pm 0.25\text{j}$, and therefore Eqn. (8.6.39) given on page 7 of the exam paper can be applied to give the time-domain integral constraint for a unit-step reference change:

$$\int_0^\infty e(t)\cos(0.25t)\,\mathrm{d}t = 0\,.$$

b. Two cases exist that will satisfy the integral constraint obtained in part a:

- If the error $e(t)$ changes very slowly with respect to the period of the cosine term $\cos(\omega_o t)$, then the areas of the positive and negative phases of the cosine term will cancel to give an integral equal to zero, as required.

- If the error changes very rapidly from a positive error to a negative error (such that the $\cos(0.25t)$ term is fairly constant over this brief time) and then quickly decays to zero, then the areas of the positive and negative errors will cancel to give an integral equal to zero, as required.

The first case corresponds to a slow system response and will produce small transient responses and the second case to a fast system response time with large transient responses. It is not possible to have both small transient responses and a fast response time for this system because of the imaginary-axis zeros.

7. **Consider a feedback control loop where the open-loop transfer function is given by:**

$$C(s)G_o(s) = \frac{s+5}{(s+1)(s+3)}.$$

In light of Bode's integral constraint on sensitivity:

a. **calculate the frequency ω_0 at which the log-magnitude of the nominal sensitivity is equal to zero, i.e., $\ln|S_o(\text{j}\omega_0)| = 0$;**

b. **determine the difference between the areas of $\ln|S_o(\text{j}\omega)|$ above and below the line $\ln|S_o(\text{j}\omega)| = 0$; and**

c. **based on the results from parts a and b, sketch the system's log-magnitude nominal sensitivity versus frequency, i.e., $\ln|S_o(\text{j}\omega)|$ versus ω.**

a. For the given open-loop transfer function, the nominal sensitivity is:

$$S_o(s) = \frac{1}{1 + C(s)G_o(s)} = \frac{1}{1 + \dfrac{s+5}{(s+1)(s+3)}} = \frac{(s+1)(s+3)}{(s+1)(s+3) + s + 5} = \frac{s^2 + 4s + 3}{s^2 + 5s + 8}.$$

The log-magnitude of the nominal sensitivity is equal to zero when the magnitude is equal to one, or alternatively when the magnitude squared is equal to one. The latter can be found easily via the relationship $|S_o(\mathrm{j}\omega)|^2 = S_o(\mathrm{j}\omega)S_o^*(\mathrm{j}\omega)$, where * indicates the complex conjugate, giving:

$$S_o(\mathrm{j}\omega_o)S_o^*(\mathrm{j}\omega_o) = \left(\frac{-\omega_o^2 + 3 + \mathrm{j}4\omega}{-\omega_o^2 + 8 + \mathrm{j}5\omega}\right)\left(\frac{-\omega_o^2 + 3 - \mathrm{j}4\omega_o}{-\omega_o^2 + 8 - \mathrm{j}5\omega_o}\right) = 1$$

$$\Rightarrow \frac{(\omega_o^2 - 3)^2 + (4\omega_o)^2}{(\omega_o^2 - 8)^2 + (5\omega_o)^2} = \frac{\omega_o^4 - 6\omega_o^2 + 9 + 16\omega_o^2}{\omega_o^4 - 16\omega_o^2 + 64 + 25\omega_o^2} = \frac{\omega_o^4 + 10\omega_o^2 + 9}{\omega_o^4 + 9\omega_o^2 + 64} = 1$$

$$\Rightarrow \omega_o^4 + 10\omega_o^2 + 9 = \omega_o^4 + 9\omega_o^2 + 64$$

$$\Rightarrow \omega_o^2 = 55$$

$$\Rightarrow \omega_o = \sqrt{55}$$

b. The difference between the areas of $\ln|S_o(\mathrm{j}\omega)|$ above and below the line $\ln|S_o(\mathrm{j}\omega)| = 0$ can be determined according to Bode's frequency-domain integral constraint as expressed in Eqn. (9.2.3) given on page 7 of the exam paper, for the case where $\tau = 0$. The value of κ can be found by evaluating:

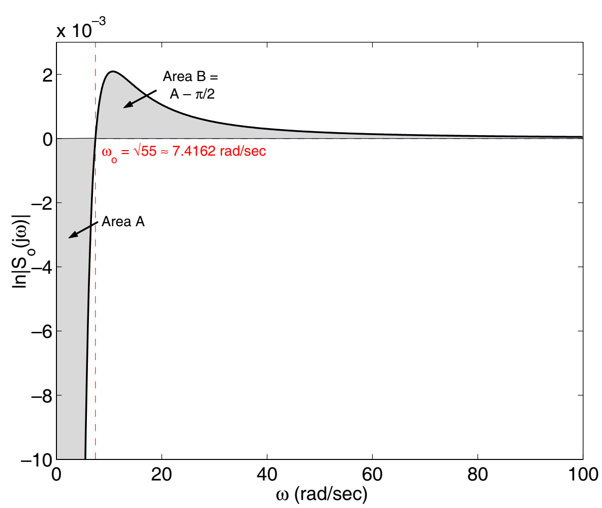
$$\lim_{s\to\infty} sC(s)G_o(s) = \lim_{s\to\infty} s\frac{s+5}{(s+1)(s+3)} = \lim_{s\to\infty} \frac{s^2 + 5s}{s^2 + 4s + 3} = 1,$$

giving:

$$\int_0^\infty \ln|S_o(\mathrm{j}\omega)|\,\mathrm{d}\,\omega = -\kappa\frac{\pi}{2} = -\frac{\pi}{2}.$$

Thus the area of $\ln|S_o(\mathrm{j}\omega)|$ below the line $\ln|S_o(\mathrm{j}\omega)| = 0$ exceeds the area above the line by π/2.

c. The main features of the system's log-magnitude nominal sensitivity versus frequency are illustrated in the plot below.



8. **A discrete-time (shift form) approximation of a continuous-time plant model is given by:**

$$G_{oq}(z) = \frac{0.0175(z+0.875)}{(z-0.819)^2}.$$

a. **Design a discrete-time (shift form) minimum-time dead-beat controller $C_q(z)$ for this plant.**

b. **Calculate the resulting nominal complimentary sensitivity $T_{oq}(z)$.**

a. From Eqns. (13.6.30) and (13.6.32) on page 9 of the exam paper, a discrete-time (shift form) minimum-time dead-beat controller $C_q(z)$ obtained from:

$$G_{oq}(z) = \frac{B_{oq}(z)}{A_{oq}(z)} = \frac{0.0175(z+0.875)}{(z-0.819)^2}$$

$$\Rightarrow \alpha = \frac{1}{B_{oq}(1)} = \frac{1}{0.0175(1+0.875)} = 30.476; \quad n = 2$$

$$C_q(z) = \frac{\alpha A_{oq}(z)}{z^n - \alpha B_{oq}(z)} = \frac{30.476(z-0.819)^2}{z^2 - 30.476 \cdot 0.0175(z+0.875)}$$

$$= \frac{30.476(z-0.819)^2}{z^2 - 0.5333z - 0.4667} = \frac{30.476(z-0.819)^2}{(z-1)(z+0.4667)}.$$

b. From part a, the resulting open-loop transfer function is:

$$C_q(z)G_{oq}(z) = \frac{0.0175(z+0.875)}{(z-0.819)^2}\frac{30.476(z-0.819)^2}{(z-1)(z+0.4667)}$$

$$= \frac{0.5333(z+0.875)}{(z-1)(z+0.4667)},$$

producing the nominal complimentary sensitivity:

$$T_{oq}(z) = \frac{C_q(z)G_{oq}(z)}{1+C_q(z)G_{oq}(z)} = \frac{\frac{0.5333(z+0.875)}{(z-1)(z+0.4667)}}{1+\frac{0.5333(z+0.875)}{(z-1)(z+0.4667)}} = \frac{\frac{0.5333(z+0.875)}{(z-1)(z+0.4667)}}{\frac{(z-1)(z+0.4667)+0.5333(z+0.875)}{(z-1)(z+0.4667)}}$$

$$= \frac{\frac{0.5333(z+0.875)}{(z-1)(z+0.4667)}}{\frac{(z-1)(z+0.4667)+0.5333(z+0.875)}{(z-1)(z+0.4667)}} = \frac{0.5333(z+0.875)}{(z-1)(z+0.4667)+0.5333(z+0.875)}$$

$$= \frac{0.5333(z+0.875)}{z^2 - 0.5333z - 0.4667 + 0.5333z + 0.4667}$$

$$= \frac{0.5333(z+0.875)}{z^2}.$$