EE 4CL4 – Control System Design

Solutions to Homework Assignment #4

1. **The input-output model for a system is given by:**

$$\ddot{y}(t)+7\dot{y}(t)+12y(t)=3u(t), \qquad \textbf{where } \ddot{y}(t)=\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} \textbf{ and } \dot{y}(t)=\frac{\mathrm{d}y(t)}{\mathrm{d}t}.$$

   a. **Determine the system transfer function.**

   b. **Compute the unit step response with zero initial conditions.**

   c. **Repeat with initial conditions** $y(0) = -1$ **and** $\dot{y}(0) = 2$. **(25 pts)**

a. The system transfer function $H(s)$ can be determined by taking the Laplace transform of the differential equation with zero initial conditions:

$$\mathcal{L}[\ddot{y}(t)+7\dot{y}(t)+12y(t)=3u(t)]=s^2Y(s)-sy(0)-\dot{y}(0)+7(sY(s)-y(0))+12Y(s)=3U(s),$$

$$\Rightarrow H(s)=\frac{Y(s)}{U(s)}=\frac{3}{s^2+7s+12}=\frac{3}{(s+3)(s+4)}.$$

b. To compute the unit step response $y(t)$ with zero initial conditions, we compute the inverse Laplace transform for $Y(s)=H(s)U(s)=H(s)/s$, giving:

$$y(t)=\frac{1}{4}-\mathrm{e}^{-3t}+\frac{3}{4}\mathrm{e}^{-4t}.$$

c. To compute the unit step response $y(t)$ with initial conditions $y(0) = -1$ and $\dot{y}(0) = 2$, we can compute the Laplace transform as for part a., but including the values for the initial conditions. The inverse Laplace transform of $Y(s)$ can then be calculated for $U(s)=1/s$, giving:

$$s^2Y(s)+s-2+7(sY(s)+1)+12Y(s)=3U(s)\Rightarrow Y(s)=\frac{-s^2-5s+3}{s(s+3)(s+4)},$$

$$y(t)=\frac{1}{4}-3\mathrm{e}^{-3t}+\frac{7}{4}\mathrm{e}^{-4t}.$$

A different approach is to use the result from part b. where we have computed the system's natural modes. The new output has the general form:

$$y(t)=\frac{1}{4}+K_1\mathrm{e}^{-3t}+K_2\mathrm{e}^{-4t},$$

where the constants K_1 and K_2 are chosen to satisfy the initial conditions, i.e.:

$$y(0)=-1=\frac{1}{4}+K_1+K_2 \qquad \text{and} \qquad \dot{y}(0)=2=-3K_1-4K_2.$$

The above equations are satisfied for $K_1 = -3$, $K_2 = 7/4$.

2. **Analyze, for $\beta \in \Re$, the frequency response of the AME and the MME when the true and the nominal models are given by:**

$$G(s) = \frac{\beta s + 2}{(s+1)(s+2)} \quad \textbf{and} \quad G_o(s) = \frac{2}{(s+1)(s+2)},$$

respectively.

Is the AME low-pass, band-pass or high-pass? What about the MME? (25 pts)

From the given equations, the AME is:

$$G_\epsilon(s) = G(s) - G_o(s) = \frac{\beta s}{(s+1)(s+2)},$$

and the MME is:

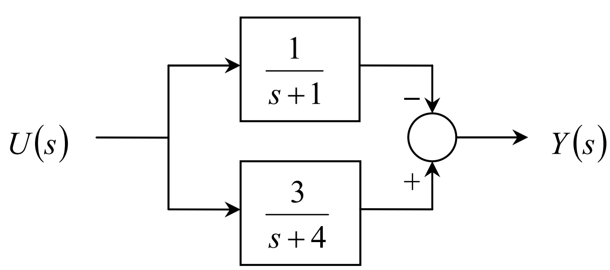
$$G_\Delta(s) = \frac{G_\epsilon(s)}{G_o(s)} = \frac{\beta s}{2}.$$

To obtain the frequency response of the AME and MME we substitute $s = j\omega$ into the above equations, giving:

$$G_\epsilon(j\omega) = \frac{\beta j\omega}{(j\omega+1)(j\omega+2)} \quad \text{and} \quad G_\Delta(j\omega) = \frac{\beta j\omega}{2}.$$

From these equations it can be seen that the AME is band-pass, while the MME is high-pass.

3. **A parallel connection of 2 systems is illustrated by the following block diagram:**



   a. **What is the transfer function from *u* to *y*?**

   b. **What are the system poles and zero?**

   c. **Calculate the system step response. How does the system zero influence the shape of the step response? (25 pts)**

a. The transfer function $H(s)$ from u to y is:

$$H(s)=\frac{3}{s+4}-\frac{1}{s+1}=\frac{2s-1}{(s+1)(s+4)}.$$

b. The system poles are the combination of the poles of the individual systems, i.e., $s=-1$ and $s=-4$. The system zero is located at $s = 0.5$, i.e. it is a non-minimum-phase zero.

c. The step response is given by:

$$y(t)=\mathcal{L}^{-1}[H(s)/s]=\mathcal{L}^{-1}\left[\frac{2s-1}{s(s+1)(s+4)}\right]=-\frac{1}{4}\left(1-4\,\mathrm{e}^{-t}+3\,\mathrm{e}^{-4t}\right).$$

The initial movement of $y(t)$ is in the positive direction $(\dot{y}(0)=2)$, but the steady state value of y is negative $(y(t\to\infty)=-1/4)$. That is, the non-minimum-phase zero creates an undershoot in the step response.

4. **Calculate the steady-state responses when a unit step is applied to the following systems, commenting on the differences observed.**

$$G(s)=\frac{1}{s^3+3s^2+3s+1},$$ **(Hint: you will need to use the Laplace-transform property** $\mathcal{L}[t^k y(t)]=(-1)^k\frac{\mathrm{d}^k Y(s)}{\mathrm{d}s^k}$ **for one inverse Laplace transform)**

$$G(s)=\frac{s^2+2s}{s^3+3s^2+3s+1}.$$ **(25 pts)**

Partial fraction decomposition of $Y(s)=G(s)U(s)=G(s)/s$ for each of the above transfer function gives:

$$Y(s)=\frac{1}{s(s+1)^3}=\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}-\frac{1}{(s+1)^3} \quad \text{and} \quad Y(s)=\frac{s+2}{(s+1)^3}=\frac{1}{(s+1)^2}+\frac{1}{(s+1)^3},$$

respectively. The inverse Laplace transform of each of these equations is:

$$y(t)=1-\mathrm{e}^{-t}-t\,\mathrm{e}^{-t}-\tfrac{1}{2}t^2\,\mathrm{e}^{-t} \quad \text{and} \quad y(t)=t\,\mathrm{e}^{-t}+\tfrac{1}{2}t^2\,\mathrm{e}^{-t},$$

respectively. Note that $\mathcal{L}^{-1}\left[\frac{1}{(s+1)^3}\right]=\mathcal{L}^{-1}\left[\tfrac{1}{2}(-1)^2\frac{\mathrm{d}^2F(s)}{\mathrm{d}s^2}\right]=\tfrac{1}{2}t^2\,\mathrm{e}^{-t}$ if $F(s)=\frac{1}{s+1}$.

Alternatively, the steady-state step responses can be calculated via the final value theorem.

The main difference between these two systems lies in the d.c. gains. The second system has a zero at the origin, thus it has zero d.c. gain, and the unit step response vanishes asymptotically.