ELEC ENG 4CL4:

Control System Design

Notes for Lecture #4

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High-order differential and difference-equation models

An alternative format to state-space equations is a high-order differential equation that directly relates inputs to outputs, commonly referred to as input-output models.

In the *continuous-time case*, these models have the form:

$$l\left(\frac{d^n y(t)}{dt^n}, \ldots, y(t), \frac{d^{n-1} u(t)}{dt^{n-1}}, \ldots, u(t)\right) = 0,$$

where l is some nonlinear function.

In the *discrete-time case*, we can write:

$$m\left(y[k+n], y[k+n-1], \ldots, y[k], u[k+n-1], \ldots, u[k]\right) = 0,$$

where m is a nonlinear function.

Modeling Errors

The so-called *additive modeling error* (AME) is defined by a transformation g_ϵ such that

$$y = y_o + g_\epsilon\langle u\rangle$$

A difficulty with the AME is that it is not scaled relative to the *size* of the nominal model. This is the advantage of the so-called *multiplicative modeling error* (MME), g_Δ , defined by

$$y = g_o\Big\langle u + g_\Delta\langle u\rangle\Big\rangle$$

Example 3.5

The output of a plant is assumed to be exactly described by

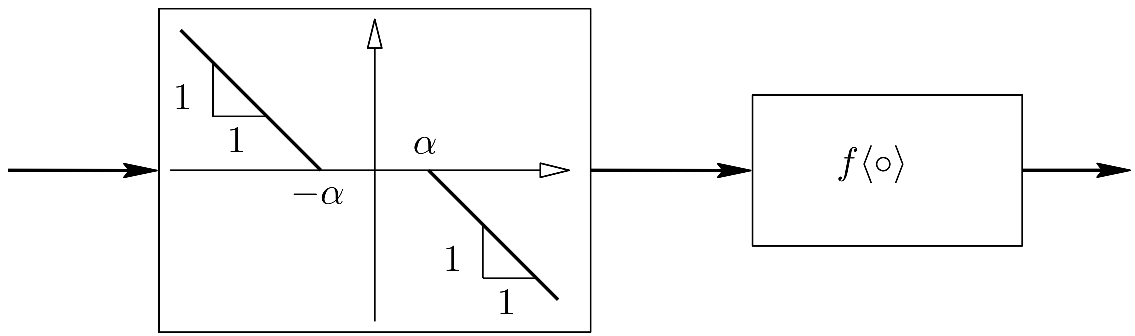
$$y = f\left\langle sat_\alpha \langle u \rangle \right\rangle$$

where $f\langle \circ \rangle$ *is a linear transformation and sat denotes the saturation operator, i.e.*

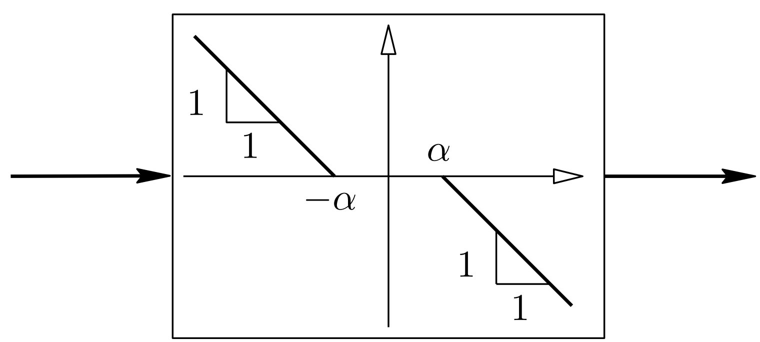
$$sat_\alpha\langle x \rangle = \begin{cases} \alpha & |x(t)| > |\alpha| \\ x & |x(t)| \leq |\alpha| \end{cases}$$

If the nominal model is chosen as $g_0\langle \circ \rangle = f\langle \circ \rangle$, *i.e. the saturation is ignored, determine the additive and the multiplicative modeling errors.*

Figure 3.3: *AME and MME due to saturation*



Additive modelling error



Multiplicative modelling error

Linearization

Although almost every real system includes nonlinear features, many systems can be reasonably described, at least within certain operating ranges, by linear models.

Linearization using a Taylor series expansion around an operating point

Consider two variables y and x related via the nonlinear function $f(\circ)$, i.e., $y = f(x)$.

The normal operating point is designated by x_0.

If the function $f(\circ)$ is continuous over the range of interest, the Taylor series expansion of the function is given by:

$$
\begin{aligned}
y &= f(x) \\
&= f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} \frac{(x - x_0)}{1!} + \left.\frac{d^2 f}{dx^2}\right|_{x=x_0} \frac{(x - x_0)^2}{2!} + \cdots
\end{aligned}
$$

The slope at the operating point:

$$\left.\frac{df}{dx}\right|_{x=x_0}$$

is a good approximation to the function $f(\circ)$ over a small range of $(x-x_0)$, the deviation from the operating point.

A reasonable linear approximation of the nonlinear function is then:

$$y = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0) = y_0 + m\,(x - x_0)\,,$$

where m is the slope at the operating point. This can be written as the linear equation:

$$(y - y_0) = m\,(x - x_0) \quad \text{or} \quad \Delta y = m\Delta x.$$

Thus consider

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

Say that $\{x_Q(t), u_Q(t), y_Q(t); t \in \mathbb{R}\}$ is a given set of trajectories that satisfy the above equations, i.e.

$$\dot{x}_Q(t) = f(x_Q(t), u_Q(t)); \qquad x_Q(t_o) \text{ given}$$
$$y_Q(t) = g(x_Q(t), u_Q(t))$$

$$\dot{x}(t) \approx f(x_Q, u_Q) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} (u(t) - u_Q)$$
$$y(t) \approx g(x_Q, u_Q) + \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} (u(t) - u_Q)$$

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) + \mathbf{E}$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) + \mathbf{F}$$

$$\mathbf{A} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} ; \qquad \mathbf{B} = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\mathbf{C} = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} ; \qquad \mathbf{D} = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\mathbf{E} = f(x_Q, u_Q) - \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} x_Q - \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} u_Q$$

$$\mathbf{F} = g(x_Q, u_Q) - \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} x_Q - \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}} u_Q$$

Example 3.6

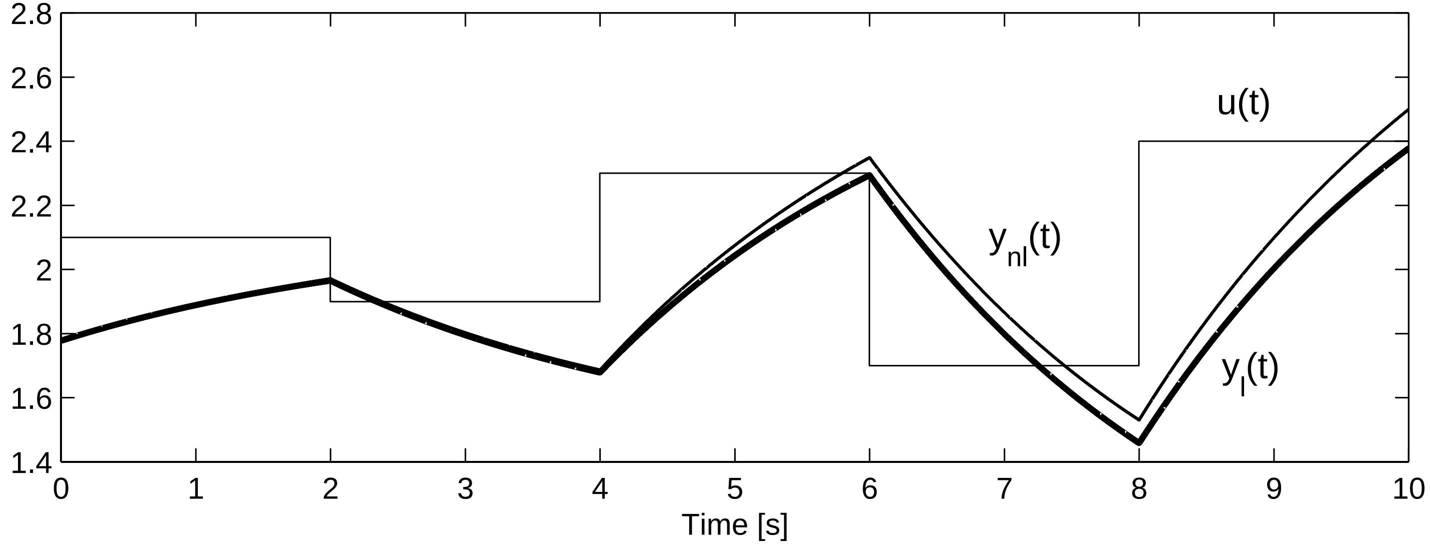
Consider a continuous time system with true model given by

$$\frac{dx(t)}{dt} = f(x(t), u(t)) = -\sqrt{x(t)} + \frac{(u(t))^2}{3}$$

Assume that the input $u(t)$ fluctuates around $u = 2$. Find an operating point with $u_Q = 2$ and a linearized model around it.

$$\frac{d\Delta x(t)}{dt} = -\frac{3}{8}\Delta x(t) + \frac{4}{3}\Delta u(t)$$

Figure 3.4: *Nonlinear system output, $y_{nl}(t)$, and linearized system output, $y_l(t)$, for a square wave input of increasing amplitude, $u(t)$.*



Example 3.7 (Inverted pendulum)

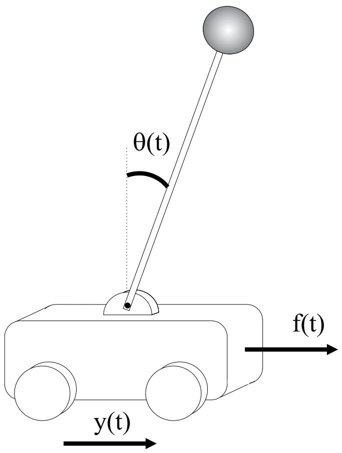


Figure 3.5: *Inverted pendulum*

In Figure 3.5, we have used the following notation:

$y(t)$	-	distance from some reference point
$\theta(t)$	-	angle of pendulum
M	-	mass of cart
m	-	mass of pendulum (assumed concentrated at tip)
ℓ	-	length of pendulum
$f(t)$	-	forces applied to pendulum

Example of an Inverted Pendulum

Application of Newtonian physics to this system leads to the following model:

$$\ddot{y} = \frac{1}{\lambda_m + \sin^2\theta(t)}\left[\frac{f(t)}{m} + \dot{\theta}^2(t)\ell\sin\theta(t) - g\cos\theta(t)\sin\theta(t)\right]$$

$$\ddot{\theta} = \frac{1}{\ell(\lambda_m + \sin^2\theta(t))}\left[-\frac{f(t)}{m}\cos\theta(t) + \dot{\theta}^2(t)\ell\sin\theta(t)\cos\theta(t) + (1 \overset{+}{\cancel{-}} \lambda_m)g\sin\theta(t)\right]$$

where $\lambda_m = (M/m)$

This is a linear state space model in which A, B and C are:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell} & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Summary

- In order to systematically design a controller for a particular system, one needs a formal - though possibly simple - description of the system. Such a description is called a model.
- A model is a set of mathematical equations that are intended to capture the effect of certain system variables on certain other system variables.

- The italicized expressions above should be understood as follows:
  - *Certain system variables*: It is usually neither possible nor necessary to model the effect of every variable on every other variable; one therefore limits oneself to certain subsets. Typical examples include the effect of input on output, the effect of disturbances on output, the effect of a reference signal change on the control signal, or the effect of various unmeasured internal system variables on each other.

- *Capture*: A model is never perfect and it is therefore always associated with a modeling error. The word capture highlights the existence of errors, but does not yet concern itself with the precise definition of their type and effect.

- *Intended*: This word is a reminder that one does not always succeed in finding a model with the desired accuracy and hence some iterative refinement may be needed.

- *Set of mathematical equations*: There are numerous ways of describing the system behavior, such as linear or nonlinear differential or difference equations.

- Models are classified according to properties of the equation they are based on. Examples of classification include:

Model Attribute	*Contrasting Attribute*	*Asserts whether or not …*
Single input Single output	Multiple input multiple output	… the model equations have one input and one output only
Linear	Nonlinear	… the model equations are linear in the system variables
Time varying	Time invariant	… the model parameters are constant
Continuous	Sampled	… model equations describe the behavior at every instant of time, or only in discrete *samples* of time
Input-output	State space	… the model equations rely on functions of input and output variables only, or also include the so called *state variables*.
Lumped parameter	Distributed parameter	… the model equations are ordinary or partial differential equations

- In many situations nonlinear models can be linearized around a user defined operating point.