ELEC ENG 4CL4:

Control System Design

Notes for Lecture #7

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Frequency Response

We next study the system response to a rather special input, namely a sine wave. The reason for doing so is that the response to sine waves also contains rich information about the response to other signals.

Let the transfer function be

$$H(s) = K\frac{\sum_{i=0}^{m} b_i s^i}{s^n + \sum_{k=1}^{n-1} a_k s^k}$$

Then the steady state response to the input $\sin(wt)$ is

$$y(t) = |H(jw)|\sin(wt + \phi(w))$$

where

$$H(j\omega) = |H(j\omega)|e^{j\phi(\omega)}$$

In summary:

A sine wave input forces a sine wave at the output with the same frequency. Moreover, the amplitude of the output sine wave is modified by a factor equal to the magnitude of H(jw) and the phase is shifted by a quantity equal to the phase of H(jw).

Bode Diagrams

Bode diagrams consist of a pair of plots. One of these plots depicts the magnitude of the frequency response as a function of the angular frequency, and the other depicts the angle of the frequency response, also as a function of the angular frequency.

Usually, Bode diagrams are drawn with special axes:

- The abscissa axis is linear in log(w) where the log is base 10. This allows a compact representation of the frequency response along a wide range of frequencies. The unit on this axis is the decade, where a *decade* is the distance between w_1 and $10w_1$ for any value of w_1.

- The magnitude of the frequency response is measured in *decibels* [dB], i.e. in units of $20\log|H(jw)|$. This has several advantages, including good accuracy for small and large values of $|H(jw)|$, facility to build simple approximations for $20\log|H(jw)|$, and the fact that the frequency response of cascade systems can be obtained by adding the individual frequency responses.
- The angle is measured on a linear scale in radians or degrees.

Drawing Approximate Bode Diagrams

- A simple gain K has constant magnitude and phase Bode diagram. The magnitude diagram is a horizontal line at $20\log|K|$[dB] and the phase diagram is a horizontal line at 0[*rad*] (when $K \in \mathbb{R}^-$).
- The factor s^k has a magnitude diagram which is a straight line with slope equal to $20k$[dB/*decade*] and constant phase, equal to $k\pi/2$. This line crosses the horizontal axis (0[dB]) at $w = 1$.

- The factor $as + 1$ has a magnitude Bode diagram which can be asymptotically approximated as follows:
  - for $|aw| << 1$, $20 \log|ajw + 1| \approx 20 \log(1) = 0$[dB], i.e. for low frequencies, this magnitude is a horizontal line. This is known as *the low frequency asymptote*.
  - For $|aw| >> 1$, $20 \log|ajw + 1| \approx 20 \log(|aw|)$ i.e. for high frequencies, this magnitude is a straight line with a slope of 20[dB/*decade*] which crosses the horizontal axis (0[dB]) at $w = |a|^{-1}$. This is known as the *high frequency asymptote*.

- the phase response is more complex. It roughly changes over two decades. One decade below $|a|^{-1}$ the phase is approximately zero. One decade above $|a|^{-1}$ the phase is approximately sign(a)0.5π[rad]. Connecting the points $(0.1|a|^{-1}, 0)$ and $(10|a|^{-1}, 0)$ by a straight line, gives sign(a)0.25 π for the phase at $w = |a|^{-1}$. This is a very rough approximation.

❖ For $a = a_1 + ja_2$, the phase Bode diagram of the factor $as + 1$ corresponds to the angle of the complex number with real part $1 - wa_2$ and imaginary part a_1w.

Example

Consider a transfer function given by

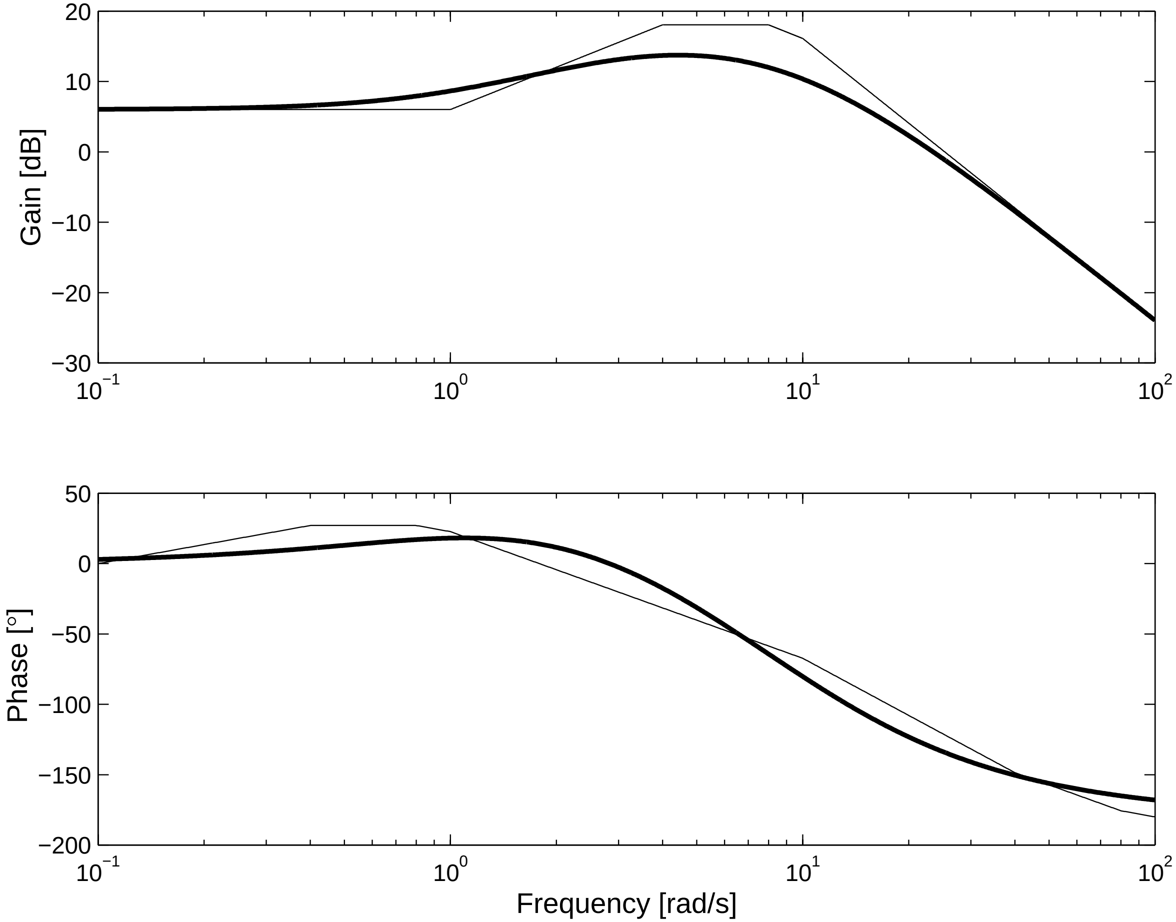
$$H(s) = 640\frac{(s+1)}{(s+4)(s+8)(s+10)}$$

To draw the asymptotic behavior of the gain diagram we first arrange *H*(*s*) into a form where the poles and zeros are designated, i.e.

$$H(s) = 2\frac{(s+1)}{(0.25s+1)(0.125s+1)(0.1s+1)}$$

Then using the approximate rules gives the result below:

Figure 4.7: *Exact (thick line) and asymptotic (thin line) Bode plots*



Filtering

In an ideal amplifier, the frequency response would be $H(jw) = K$, constant $\forall w$, i.e. every frequency component would pass through the system with equal gain and no phase.

We define:

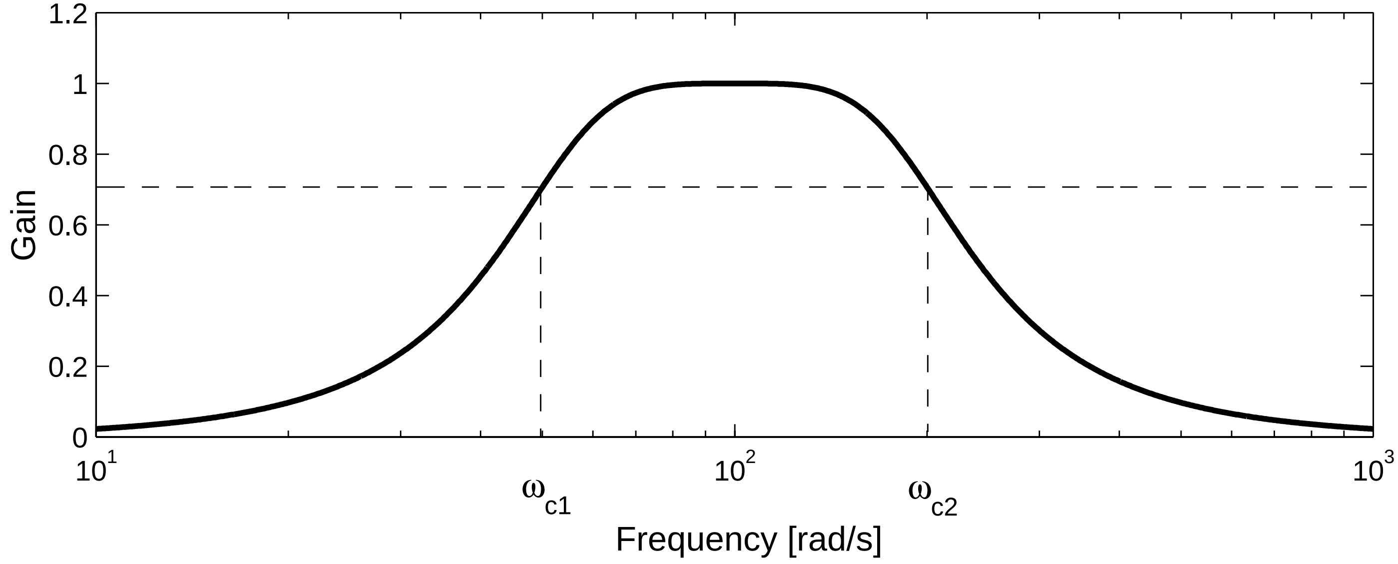
- The *pass band* in which all frequency components pass through the system with approximately the same amplification (or attenuation) and with a phase shift which is approximately proportional to w.

- The *stop band*, in which all frequency components are stopped. In this band $|H(jw)|$ is small compared to the value of $|H(jw)|$ in the pass band.
- The *transition band(s)*, which are intermediate between a pass band and a stop band.

- Cut-off frequency w_c. This is a value of w, such that $\left|H\left(jw_c\right)\right| = \hat{H}/\sqrt{2},$ where $\hat{H}$ is respectively
  - $|H(0)|$ for low pass filters and band reject filters
  - $|H(\infty)|$ for high pass filters
  - the maximum value of $|H(jw)|$ in the pass band, for band pass filters

- Bandwidth B_w. This is a measure of the frequency width of the pass band (or the reject band). It is defined as $B_w = w_{c2} - w_{c1}$, where $w_{c2} > w_{c1} \geq 0$. In this definition, w_{c1} and w_{c2} are cut-off frequencies on either side of the pass band or reject band (for low pass filters, $w_{c1} = 0$).

Figure 4.8: *Frequency response of a bandpass filter*



Fourier Transform

Definition of the Fourier Transform

$$\mathcal{F}\left[f(t)\right] = F(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t) dt$$

$$\mathcal{F}^{-1}\left[F(j\omega)\right] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} F(j\omega) d\omega$$

Table 4.3: *Fourier transform table*

$f(t) \quad \forall t \in \mathbb{R}$	$\mathcal{F}[f(t)]$
1	$2\pi\delta(\omega)$
$\delta_D(t)$	1
$\mu(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\mu(t) - \mu(t - t_o)$	$\frac{1 - e^{-j\omega t_o}}{j\omega}$
$e^{\alpha t}\mu(t) \quad \Re\{\alpha\} < 0$	$\frac{1}{j\omega - \alpha}$
$te^{\alpha t}\mu(t) \quad \Re\{\alpha\} < 0$	$\frac{1}{(j\omega - \alpha)^2}$
$e^{-\alpha\|t\|} \quad \alpha \in \mathbb{R}^+$	$\frac{2\alpha}{\omega^2 + \alpha^2}$
$\cos(\omega_o t)$	$\pi\left(\delta(\omega - \omega_o) + \delta(\omega - \omega_o)\right)$
$\sin(\omega_o t)$	$j\pi\left(\delta(\omega + \omega_o) - \delta(\omega - \omega_o)\right)$
$\cos(\omega_o t)\mu(t)$	$\pi\left(\delta(\omega - \omega_o) + \delta(\omega - \omega_o)\right) + \frac{j\omega}{-\omega^2 + \omega_o^2}$
$\sin(\omega_o t)\mu(t)$	$j\pi\left(\delta(\omega + \omega_o) - \delta(\omega - \omega_o)\right) + \frac{\omega_o}{-\omega^2 + \omega_o^2}$
$e^{-\alpha t}\cos(\omega_o t)\mu(t) \quad \alpha \in \mathbb{R}^+$	$\frac{j\omega + \alpha}{(j\omega + \alpha)^2 + \omega_o^2}$
$e^{-\alpha t}\sin(\omega_o t)\mu(t) \quad \alpha \in \mathbb{R}^+$	$\frac{\omega_o}{(j\omega + \alpha)^2 + \omega_o^2}$

Table 4.4: *Fourier transforms properties. Note that $F_i(jw) = F[f_i(t)]$ and $Y(jw) = F[y(t)]$.*

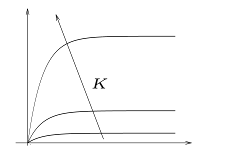
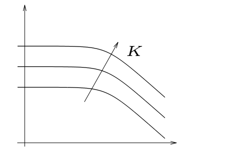
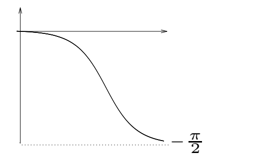
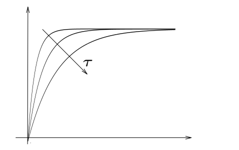
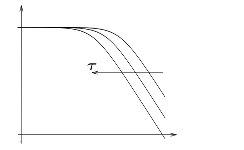
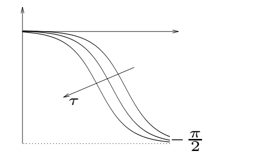
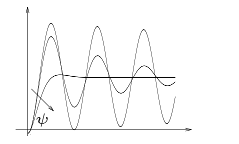
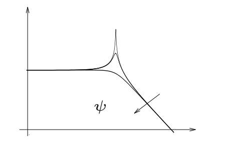
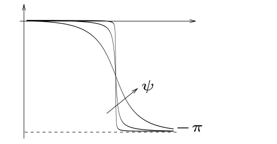
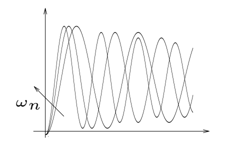
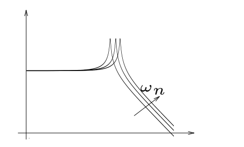
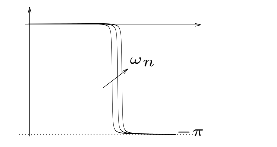
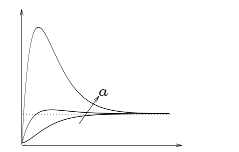
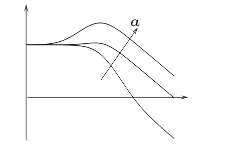
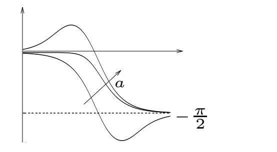
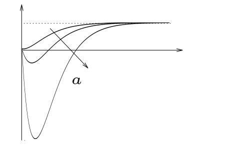
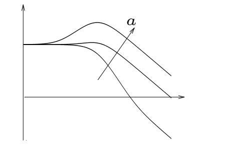
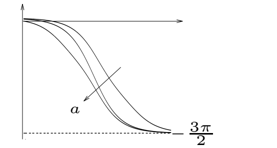
$f(t)$	$\mathcal{F}[f(t)]$	Description
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(j\omega)$	Linearity
$\frac{dy(t)}{dt}$	$j\omega Y(j\omega)$	Derivative law
$\frac{d^k y(t)}{dt^k}$	$(j\omega)^k Y(j\omega)$	High order derivative
$\int_{-\infty}^{t} y(\tau)d\tau$	$\frac{1}{j\omega}Y(j\omega) + \pi Y(0)\delta(\omega)$	Integral law
$y(t-\tau)$	$e^{-j\omega\tau}Y(j\omega)$	Delay
$y(at)$	$\frac{1}{\|a\|}Y\left(j\frac{\omega}{a}\right)$	Time scaling
$y(-t)$	$Y(-j\omega)$	Time reversal
$\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau$	$F_1(j\omega)F_2(j\omega)$	Convolution
$y(t)\cos(\omega_o t)$	$\frac{1}{2}\left\{Y(j\omega - j\omega_o) + Y(j\omega + j\omega_o)\right\}$	Modulation (cosine)
$y(t)\sin(\omega_o t)$	$\frac{1}{j2}\left\{Y(j\omega - j\omega_o) - Y(j\omega + j\omega_o)\right\}$	Modulation (sine)
$F(t)$	$2\pi f(-j\omega)$	Symmetry
$f_1(t)f_2(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(j\zeta)F_2(j\omega - j\zeta)d\zeta$	Time domain product
$e^{at}f_1(t)$	$F_1(j\omega - a)$	Frequency shift

A useful result: *Parseval's Theorem*

Theorem 4.1: *Let F(jw) and G(jw) denote the Fourier transform of f(t) and g(t) respectively. Then*

$$\int_{-\infty}^{\infty} f(t)g(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)G(-j\omega)\, d\omega$$

Table 4.5: *System models and influence of parameter variations*

System	Parameter	Step response	Bode (gain)	Bode(phase)
$\frac{K}{\tau s+1}$	K			
	τ			
$\frac{\omega_n^2}{s^2+2\psi\omega_n s+\omega_n^2}$	ψ			
	ω_n			
$\frac{as+1}{(s+1)^2}$	a			
$\frac{-as+1}{(s+1)^2}$	a			

Modeling Errors for Linear Systems

If a linear model is used to approximate a linear system, then modeling errors due to errors in parameters and/or complexity can be expressed in transfer function form as

$$Y(s) = G(s)U(s) = (G_o(s) + G_\epsilon(s))U(s) = G_o(s)(1 + G_\Delta(s))U(s)$$

where $G_\varepsilon(s)$ denotes the AME and $G_\Delta(s)$ denotes the MME, introduced in Chapter 3.

AME and MME are two different ways of capturing the same modeling error. The advantage of the MME is that it is a relative quantity, whereas the AME is an absolute quantity.

Example: Time Delays

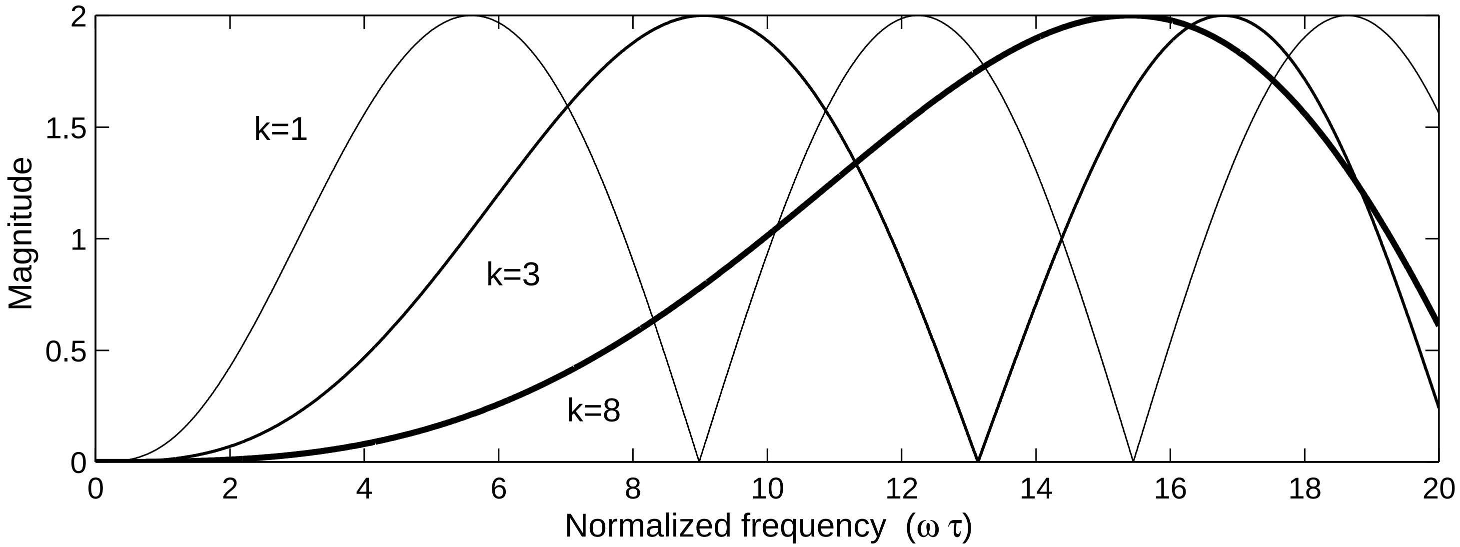
Time delays do not yield rational functions in the Laplace domain. Thus a common strategy is to approximate the delay by a suitable rational expression. One possible approximation is

$$e^{-\tau s} \approx \left(\frac{-\tau s + 2k}{\tau s + 2k} \right)^k \qquad k \in \langle 1, 2, \dots \rangle$$

where *k* determines the accuracy of the approximation.

For this approximation, we can determine the magnitude of the frequency response of the MME as shown below.

Figure 4.9: *MME for all pass rational approximation of time delays*



Missing resonance effect

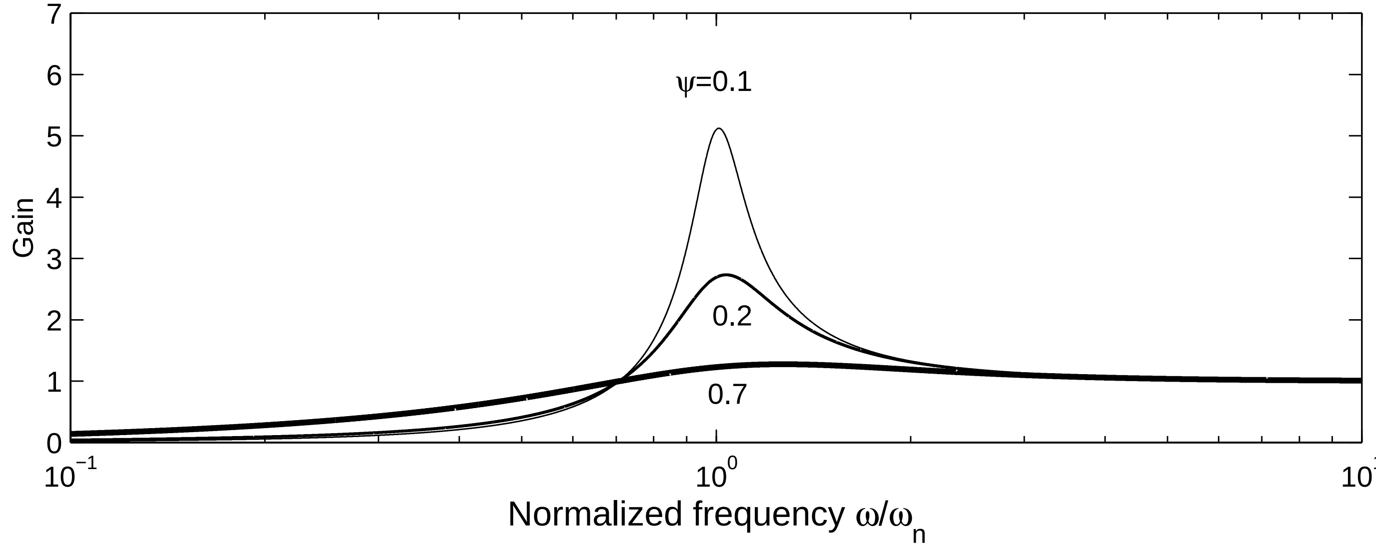
The omission of resonant modes is very common when modeling certain classes of systems, such as robots, arms, antennas and other large flexible structures. This situation may be described by

$$G_\epsilon(s) = \frac{-s(s+2\psi\omega_n)}{s^2+2\psi\omega_n s+\omega_n^2}F(s) \qquad G_\Delta(s) = \frac{-s(s+2\psi\omega_n)}{s^2+2\psi\omega_n s+\omega_n^2}$$

The modeling errors are now given by

$$G(s) = \frac{\omega_n^2}{s^2+2\psi\omega_n s+\omega_n^2}F(s) \qquad G_o(s) = F(s) \qquad 0 < \psi < 1$$

Figure 4.10: *MME frequency response for omitted resonance, for different values of the damping factor φ*



Bounds for Modeling Errors

In control system design it is often desirable to account for model errors in some way. A typical specification might be

$$|G_\Delta(j\omega)| < \epsilon(\omega)$$

where $\varepsilon(w)$ is some given positive function of w.

Summary

- There are two key approaches to linear dynamic models:
  - the, so-called, time domain, and
  - the so-called, frequency domain
- Although these two approaches are largely equivalent, they each have their own particular advantages and it is therefore important to have a good grasp of each.

- In the time domain,
  - systems are modeled by differential equations
  - systems are characterized by the evolution of their variables (output etc.) in time
  - the evolution of variables in time is computed by solving differential equations

- In the frequency domain,
  - modeling exploits the key linear system property that the steady state response to a sinusoid is again a sinusoid of the same frequency; the system only changes amplitude and phase of the input in a fashion uniquely determined by the system at that frequency,
  - systems are modeled by transfer functions, which capture this impact as a function of frequency.

- With respect to the important characteristic of stability, a continuous time system is
  - stable if and only if the real parts of all poles are strictly negative
  - marginally stable if at least one pole is strictly imaginary and no pole has strictly positive real part
  - unstable if the real part of at least one pole is strictly positive
  - non-minimum phase if the real part of at least one zero is strictly positive.

- All models contain modeling errors.
- Modeling errors can be described as an additive (AME) or multiplicative (MME) quantity.
- Modeling errors are necessarily unknown and frequently described by upper bounds.