ELEC ENG 4CL4:

Control System Design

Notes for Lecture #8

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Chapter 5

Analysis of SISO Control Loops

Topics to be covered

For a given controller and plant connected in feedback we ask and answer the following questions:

- Is the loop stable?
- What are the sensitivities to various disturbances?
- What is the impact of linear modeling errors?
- How do small nonlinearities impact on the loop?

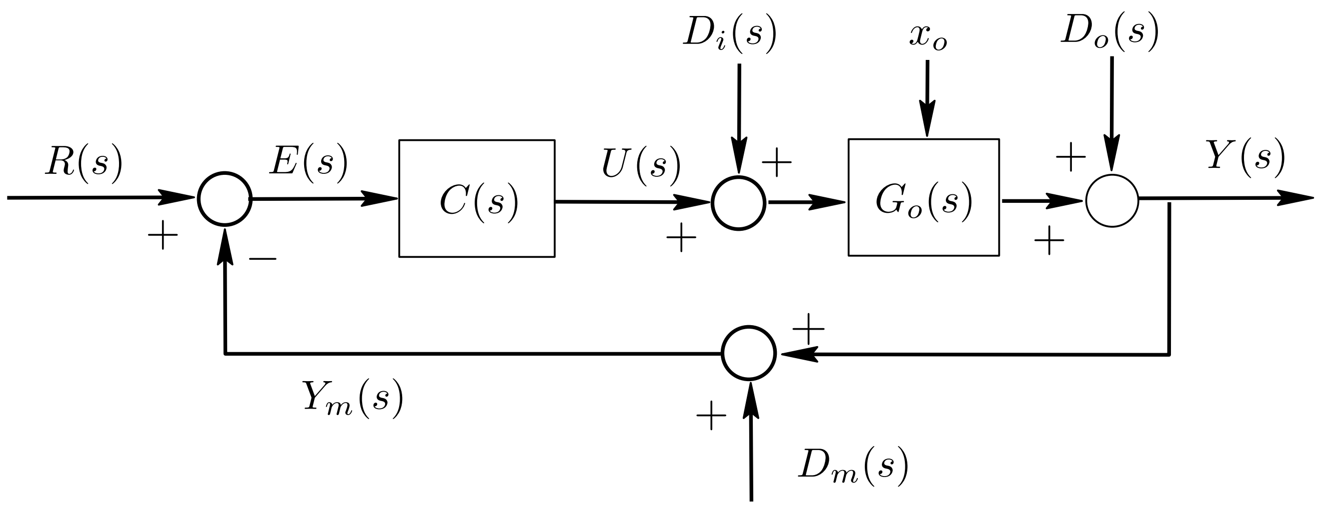
We also introduce several analysis tools; specifically

- Root locus
- Nyquist stability analysis

Feedback Structures

We will see that feedback can have many desirable properties such as the capacity to reduce the effect of disturbances, to decrease sensitivity to model errors or to stabilize an unstable system. We will also see, however, that ill-applied feedback can make a previously stable system unstable, add oscillatory behaviour into a previously smooth response or result in high sensitivity to measurement noise.

Figure 5.1: *Simple feedback control system*



In the loop shown in Figure 5.1 we use transfer functions and Laplace transforms to describe the relationships between signals in the loop. In particular, $C(s)$ and $G_0(s)$ denote the transfer functions of the controller and the nominal plant model respectively, which can be represented in fractional form as:

$$C(s) = \frac{P(s)}{L(s)}$$

$$G_o(s) = \frac{B_o(s)}{A_o(s)}$$

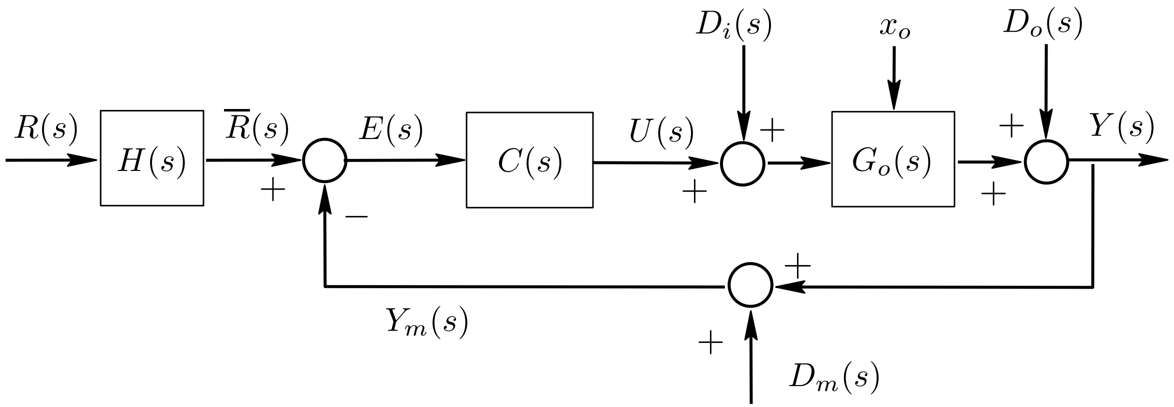
Laplace Transforms of System Input and Output

$$U(s) = \frac{C(s)}{1 + G_o(s)C(s)} \left(R(s) - D_m(s) - D_o(s) - G_o(s)D_i(s) - \frac{f(s, x_o)}{A(s)} \right)$$

and

$$Y(s) = \frac{1}{1 + G_o(s)C(s)} \left[G_o(s)C(s)(R(s) - D_m(s)) + D_o(s) + G_o(s)D_i(s) + \frac{f(s, x_o)}{A(s)} \right]$$

Figure 5.2: *Two degree of freedom closed loop*



$$Y(s) = \frac{G_o(s)C(s)H(s)}{1+G_o(s)C(s)}R(s) + \frac{1}{1+G_o(s)C(s)}\left(D_o(s) + \frac{f(s,x_o)}{A_o(s)}\right) + \frac{G_o(s)}{1+G_o(s)C(s)}D_i(s) - \frac{G_o(s)C(s)}{1+G_o(s)C(s)}D_m(s)$$

Nominal Sensitivity Functions

$$T_o(s) \triangleq \frac{G_o(s)C(s)}{1 + G_o(s)C(s)} = \frac{B_o(s)P(s)}{A_o(s)L(s) + B_o(s)P(s)}$$

$$S_o(s) \triangleq \frac{1}{1 + G_o(s)C(s)} = \frac{A_o(s)L(s)}{A_o(s)L(s) + B_o(s)P(s)}$$

$$S_{io}(s) \triangleq \frac{G_o(s)}{1 + G_o(s)C(s)} = \frac{B_o(s)L(s)}{A_o(s)L(s) + B_o(s)P(s)}$$

$$S_{uo}(s) \triangleq \frac{C(s)}{1 + G_o(s)C(s)} = \frac{A_o(s)P(s)}{A_o(s)L(s) + B_o(s)P(s)}$$

These functions are given specific names as follows:

$T_0(s)$	:	Nominal complementary sensitivity
$S_0(t)$	:	Nominal sensitivity
$S_{i0}(s)$	:	Nominal input disturbance sensitivity
$S_{u0}(s)$	:	Nominal control sensitivity

Relationship between sensitivities

$$S_o(s) + T_o(s) = 1$$

$$S_{io}(s) = S_o(s)G_o(s) = \frac{T_o(s)}{C(s)}$$

$$S_{uo}(s) = S_o(s)C(s) = \frac{T_o(s)}{G_o(s)}$$

Internal Stability

Definition 5.1 (Internal stability). We say that the nominal loop is internally stable if and only if all eight transfer functions in the equation below are stable.

$$\begin{bmatrix} Y_o(s) \\ U_o(s) \end{bmatrix} = \frac{\begin{bmatrix} G_o(s)C(s) & G_o(s) & 1 & -G_o(s)C(s) \\ C(s) & -G_o(s)C(s) & -C(s) & -C(s) \end{bmatrix}}{1 + G_o(s)C(s)} \begin{bmatrix} H(s)R(s) \\ D_i(s) \\ D_o(s) \\ D_m(s) \end{bmatrix}$$

Link to Characteristic Equation

Lemma 5.1 (Nominal internal stability)

Consider the nominal closed loop depicted in Figure 5.2. Then the nominal closed loop is internally stable if and only if the roots of the nominal closed loop characteristic equation

$$A_o(s)L(s) + B_o(s)P(s) = 0$$

all lie in the open left half plane. We call $A_0L + B_0P$ *the nominal closed-loop characteristic polynomial.*

Stability and Polynomial Analysis

Consider a polynomial of the following form:

$$p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

The problem to be studied deals with the question of whether that polynomial has any root with nonnegative real part. Obviously, this equation can be answered by *computing the n roots* of *p*(*s*). However, in many applications it is of special interest to study the interplay between the location of the roots and certain polynomial coefficients.

Some Polynomial Properties of Special Interest

Property 1: The coefficient $a_{n\text{-}1}$ satisfies

$$a_{n-1} = -\sum_{i=1}^{n} \lambda_i$$

Property 2: The coefficient a_0 satisfies

$$a_0 = (-1)^n \prod_{i=1}^{n} \lambda_i$$

Property 3: If all roots of p(s) have negative real parts, it is necessary that $a_i > 0$, $i \in \{0, 1, \ldots, n\text{-}1\}$.

Property 4: If any of the polynomial coefficients is nonpositive (negative or zero), then, one or more of the roots have nonnegative real parts.

Routh's Algorithm

$$p(s) = \sum_{i=0}^{n} a_i s^i$$

The Routh's algorithm is based on the following numerical array:

$$\begin{array}{llllll}
\mathbf{s^n} & \gamma_{0,1} & \gamma_{0,2} & \gamma_{0,3} & \gamma_{0,4} & \cdots \\
\mathbf{s^{n-1}} & \gamma_{1,1} & \gamma_{1,2} & \gamma_{1,3} & \gamma_{1,4} & \cdots \\
\mathbf{s^{n-2}} & \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,3} & \gamma_{2,4} & \cdots \\
\mathbf{s^{n-3}} & \gamma_{3,1} & \gamma_{3,2} & \gamma_{3,3} & \gamma_{3,4} & \cdots \\
\mathbf{s^{n-4}} & \gamma_{4,1} & \gamma_{4,2} & \gamma_{4,,3} & \gamma_{4,4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\mathbf{s^2} & \gamma_{n-2,1} & \gamma_{n-2,2} & & & \\
\mathbf{s^1} & \gamma_{n-1,1} & & & & \\
\mathbf{s^0} & \gamma_{n,1} & & & &
\end{array}$$

Table 5.1: *Routh's array*

Where

$$\gamma_{0,i} = a_{n+2-2i}; \quad i = 1, 2, \ldots, m_0 \qquad \text{and} \quad \gamma_{1,i} = a_{n+1-2i}; \quad i = 1, 2, \ldots, m_1$$

with $m_0 = (n+2)/2$ and $m_1 = m_0 - 1$ for n even and $m_1 = m_0$ for n odd. Note that the elements $\gamma_{0,i}$ and $\gamma_{1,i}$ are the coefficients of the polynomials arranged in alternated form. Furthermore

$$\gamma_{k,j} = \frac{\gamma_{k-1,1}\,\gamma_{k-2,j+1} - \gamma_{k-2,1}\,\gamma_{k-1,j+1}}{\gamma_{k-1,1}}; \qquad k = 2, \ldots, n \qquad j = 1, 2, \ldots, m_j$$

Result:

Consider a polynomial $p(s)$ given by (5.5.8) and its associated array as in Table 5.1. Then the number of roots with real part greater than zero is equal to the number of sign changes in the first column of the array.