Chapter 6 - Solved Problems

Solved Problem 6.1. *Contributed by - James Welsh, University of Newcastle, Australia.*
Find suitable values for the PID parameters using the Z-N tuning strategy for the nominal plant G_o.
where

$$G_o(s) = \frac{e^{-s}}{s+1} \tag{1}$$

Solutions to Solved Problem 6.1

Solved Problem 6.2. *A plant has a nominal given by*

$$G_o(s) = \frac{1}{(s-1)^2} \tag{2}$$

Prove that this system cannot be stabilized with a PI controller

Solutions to Solved Problem 6.2

Solved Problem 6.3. *Show, using Root Locus analysis that the plant in Problem 6.2 can be stabilized using a PID controller.*

Solutions to Solved Problem 6.3

Solved Problem 6.4. *Consider a plant with nominal model given by*

$$G_o(s) = \frac{1}{s+2} \tag{3}$$

Compute the parameters of a PI controller so that the natural modes of the closed loop response decay at least as fast as e^{-5t}.

Solutions to Solved Problem 6.4

Solved Problem 6.5. *Assume that the Ziegler-Nichols ultimate gain method is used to tune a PID controller for a plant with model*

$$G_o(s) = 2\frac{e^{-s}}{(2s+1)^2} \tag{4}$$

Determine the parameters of the PID controller.

Solutions to Solved Problem 6.5

Solved Problem 6.6. *Assume that the theory presented in section §6.5 of the book is used to tune a PI controller for a process with transfer function*

$$G(s) = \frac{24(s+1.5)e^{-0.2s}}{(s+2)^2(s+6)} \tag{5}$$

6.6.1 *Find the approximate model which underlies the tuning strategy. Compare its step response with that of the true process.*

6.6.2 *Compute the parameters of a PI controller as per Table 6.2 in the book.*

6.6.3 *Compare the performance of the PI controller acting on the model with that of PI controller acting on the true process.*

Solutions to Solved Problem 6.6

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Chapter 6 - Solutions to Solved Problems

Solution 6.1. *We first determine the critical frequency.*
We set $s = j\omega$.

$$G_o(j\omega) = \frac{e^{-j\omega}}{j\omega + 1} \tag{6}$$

$$= \frac{\cos\omega - j\sin\omega}{j\omega + 1} \tag{7}$$

$$= \frac{(\cos\omega - j\sin\omega)(-j\omega + 1)}{\omega^2 + 1} \tag{8}$$

$$= \frac{1}{\omega^2 + 1}(-j\omega\cos\omega - \omega\sin\omega + \cos\omega - j\sin\omega) \tag{9}$$

$$= \frac{1}{\omega^2 + 1}(\cos\omega - \omega\sin\omega - j(\omega\cos\omega + \sin\omega)) \tag{10}$$

at the critical frequency, the imaginary term = 0.

$$\omega\cos\omega + \sin\omega = 0 \tag{11}$$

$$\omega = -\tan\omega \tag{12}$$

which when solved, for the smallest +ve value of ω, yields

$$\omega_c = 2.0288 \quad \text{rad/sec} \tag{13}$$

hence $P_c = 3.097$ seconds.
Now the critical gain occurs when

$$K_c G_o(j\omega_c) = -1 \tag{14}$$

$$\therefore \quad K_c = \frac{-1}{G_o(j\omega_c)} \tag{15}$$

$$= 2.2619 \tag{16}$$

Using Table 6.1 in the book we find the PID parameters to be

$$K_p = 1.3571 \tag{17}$$

$$T_r = 1.548 \tag{18}$$

$$T_d = 0.3871 \tag{19}$$

We then choose

$$\tau_D = 0.1T_d \tag{20}$$

$$= 0.03871 \tag{21}$$

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Solution 6.2. *We consider the general form of a PI controller as*

$$C(s) = \frac{as + b}{s} \tag{22}$$

Thus, the closed loop characteristic polynomial, $A_{cl}(s)$, is given by

$$A_{cl}(s) = \text{numerator of } \{1 + G_o(s)C(s)\} = s^3 - 2s^2 + (1+a)s + b \tag{23}$$

Hence, there are no values for a and b such that $A_{cl}(s)$ is strictly Hurwitz. This originates from the fact that one of the coefficients of the polynomial $A_{cl}(s)$ is negative no matter what values we choose for a and b.

Solution 6.3. *Since Root Locus analysis is to be used, the PID controller transfer function is first expressed in pole-zero form, i.e.,*

$$C(s) = K\frac{(s - c_1)(s - c_2)}{s(s - p_2)} \tag{24}$$

Then, the open loop transfer function $G_o(s)C(s)$ has relative degree equal to 2, with zeros at c_1 and c_2 and poles at $p_1 = 0$, p_2, $p_3 = 1$ and $p_4 = 1$. Thus implies that the Root Locus has two asymptotes with slopes $\pm\frac{\pi}{2}$ (for $K > 0$) and the asymptotes intersect at $(\sigma, 0)$, where

$$\sigma = \frac{\sum_{i=1}^{4} p_i - \sum_{i=1}^{2} c_i}{2} = \frac{2 + p_2 - c_1 - c_2}{2} \tag{25}$$

To ensure stability for a sufficiently large value of K, we choose the controller zeros and the controller pole so that $\sigma < 0$, this requires that

$$c_1 + c_2 > 2 + p_2 \tag{26}$$

Say we choose $p_2 = -12$, then (26) is satisfied if we choose, for instance $c_1 = -3$ and $c_2 = -4$. The Root Locus for $K > 0$ is shown in Figure 1.

The above results yield a PID controller transfer function

$$C(s) = K\frac{(s + 3)(s + 4)}{s(s + 12)} \tag{27}$$

*Using the MATLAB command **rlocfind** we can compute the critical value, K_c, such that the closed loop is stable for all $K > K_c$. This is done using the MATLAB code*

```
≫ Go=tf(1,[1 -2 1]);Co=tf([1 7 12],[1 12 0]);GoCo=Go*Co;
≫ rlocus(GoCo);
≫ rlocfind(GoCo);
```

The last command generates a cross-hair that we move until it coincides with the imaginary axis crossing of the root locus. A mouse click delivers $K_c \approx 135$

*Alternatively one can enter the MATLAB environment **rltool** with the command*

```
≫ rltool(Go,Co);
```
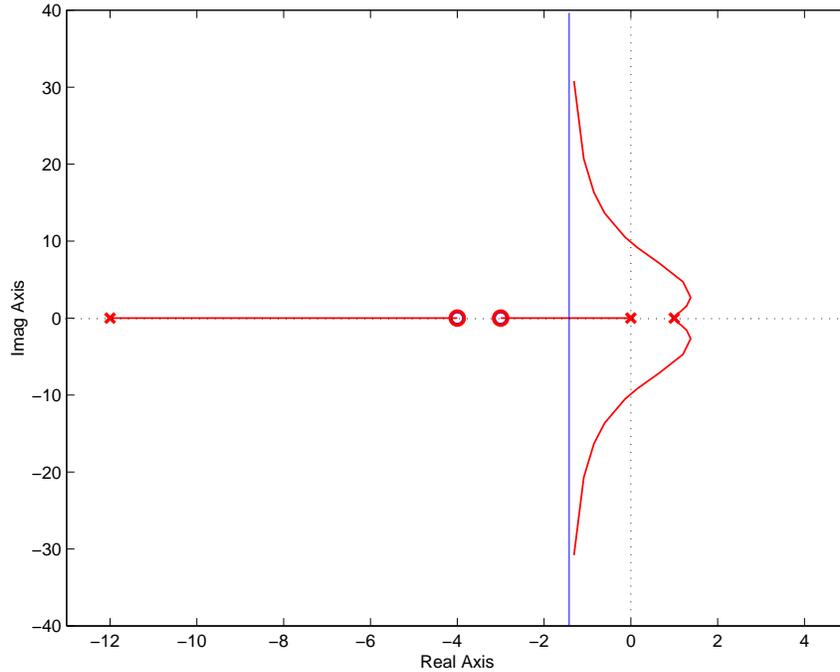
Figure 1: Root locus

Solution 6.4. *A PI controller has transfer function given by*

$$C(s) = \frac{as+b}{s}; \quad where \ a = K_p; \ \ b = \frac{K_p}{T_r} \tag{28}$$

The the closed loop characteristic polynomial, $A_{cl}(s)$, is given by

$$A_{cl}(s) = numerator \ of \ \ \{1 + G_o(s)C(s)\} = s^2 + (2+a)s + b \tag{29}$$

We choose the controller to obtain a pair of complex conjugate poles. To achieve a closed loop transient as fast as e^{-5t}, those poles must have real parts equal to -5. This requires $a = 8$ and $b = 49$. Hence

$$C(s) = \frac{8s+49}{s}; \tag{30}$$

The closed loop response to a unit step reference is shown in Figure 2.

Solution 6.5. *We have to find the critical frequency, ω_c and the critical gain, K_c. This can be done using, for instance, the Bode diagrams shown in Figure 3.*
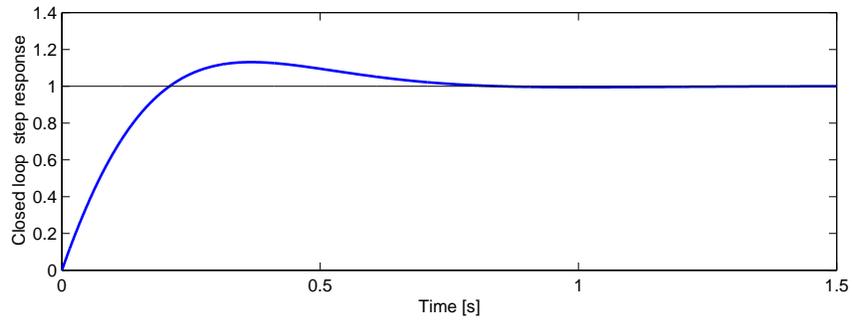
Figure 2: Closed loop unit step response
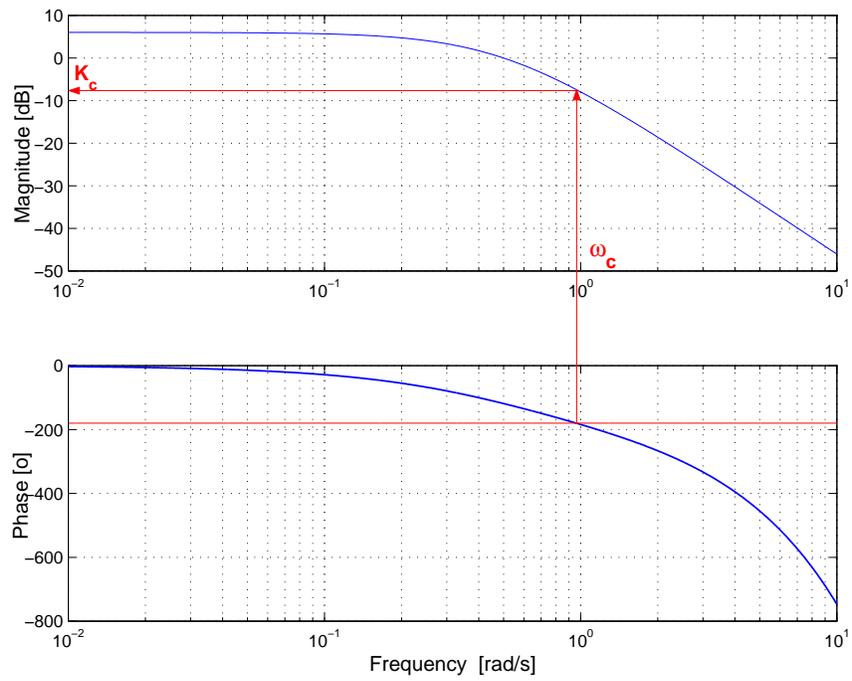
Figure 3: Bode diagrams

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From the diagrams in Figure 3 we obtain $\omega_c = 0.96\,[rad/s]$ and $K_c = -6.5\,[dB] = 0.4732$. The oscillation period is then $P_c = 2\pi/\omega_c = 6.55$. Finally, from Table 6.1 in the book, the parameters of the PID controller are

$$K_p = 0.6K_c = 0.2839; \quad T_r = 0.5P_c = 3.27; \quad T_d = 0.125P_c = 0.819 \tag{31}$$

Solution 6.6.

6.6.1 *The methods based on using process step response implicitly use a (nominal) model having the general form*

$$G_o(s) = \frac{K_o e^{-\tau_o s}}{\nu_o s + 1} \tag{32}$$

The model parameters K_o, τ_o and ν_o are obtained from the process step response. For the system in equation (5), the step response is as shown in Figure 4.
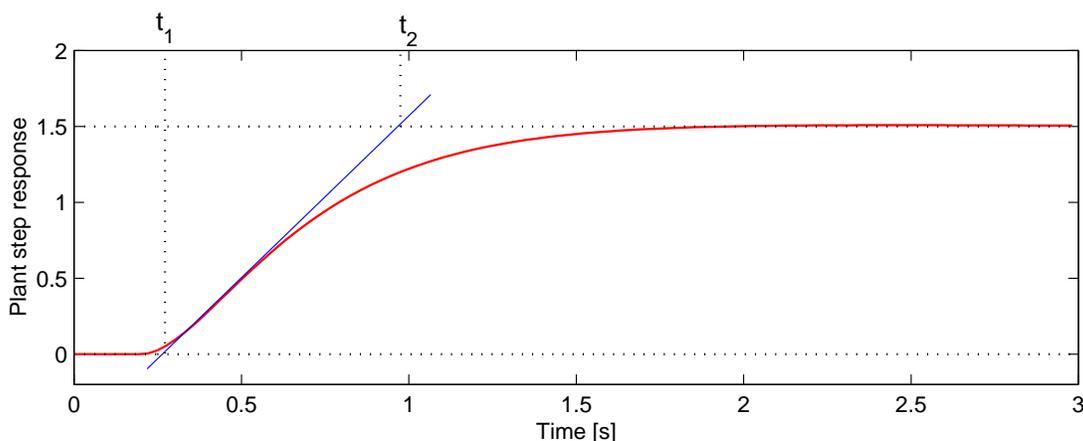
Figure 4: Plant response to an input $u(t) = \mu(t)$

*From Figure 4 we have that $K_o = 1.5$, $\nu_o = t_2 - t_1 = 0.703$ and $\tau_o = t_1 = 0.272$ (use the MATLAB command **ginput**. Therefore the approximate model is*

$$G_o(s) = \frac{1.5e^{-0.272s}}{0.703s + 1} \tag{33}$$

We next apply a unit step input to both, $G(s)$ and $G_o(s)$. The results are shown in Figure 5.

6.6.2 *The controller parameters are then computed using Table 6.2 in the book employing the values obtained above for ν_o, K_o and $\tau_o = t_1$. This yields a controller given by*

$$C(s) = K_p\left(1 + \frac{1}{T_r s}\right) = 1.55\left(1 + \frac{1}{0.81s}\right) \tag{34}$$

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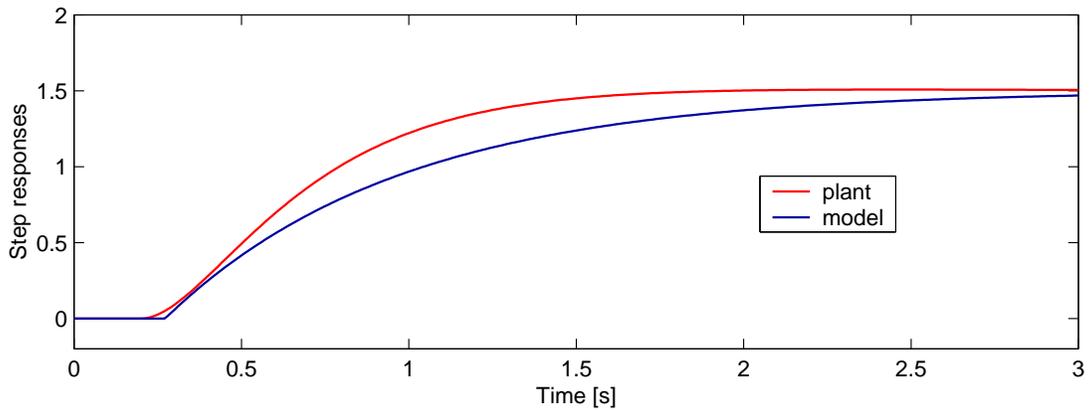
Figure 5: Comparison of the step response of the process and its model

6.6.3 *We next compare the performance of the two control loops: one, where the controller is used to control the original process, and the other, where the controller is used to control a system having the nominal model* (33). *This is done using the SIMULINK schematic shown in Figure 6* [1]. *In the upper loop the original process is considered, while in the lower loop we use the model* (33). *Both loops are driven by the same reference. For the simulation we use a unit step reference at $t = 1$ [s].*
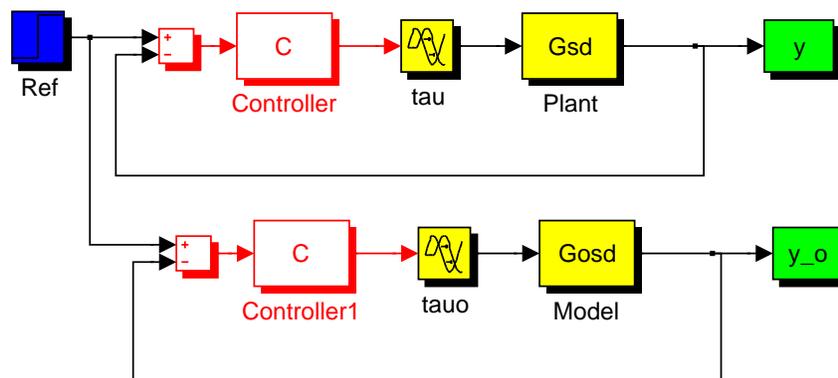
Figure 6: SIMULINK schematic

The simulation is run and the results are shown in Figure 7.

Figure 7 shows that better performance is achieved when the controller is acting on the nominal model (33). *This can be explained by the fact that the approximate model turns out to have a larger rise time than the real process, as shown in Figure 5.*

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[1] The transfer function blocks C, Gsd and Gosd correspond to LTI Simulink blocks (Control System toolbox).
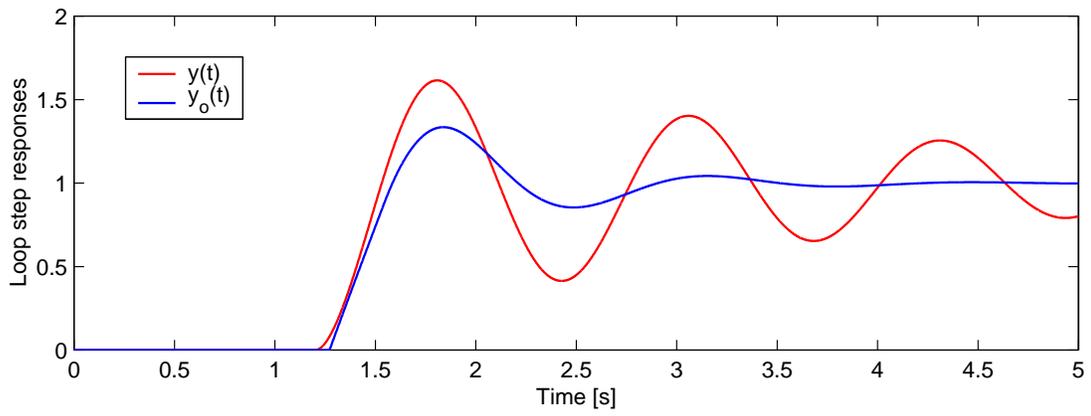
Figure 7: Closed loop step responses