

Intrinsic Capacity

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Abstract—Every channel can be expressed as a convex combination of deterministic channels with each deterministic channel corresponding to one particular intrinsic state. Such convex combinations are, in general, not unique, each giving rise to a specific intrinsic-state distribution. In this paper, we study the maximum and minimum capacities of a channel when the realization of its intrinsic state is causally available at the encoder and/or the decoder. Several conclusive results are obtained for binary-input channels and binary-output channels. By-products of our investigation include a generalization of the Birkhoff-von Neumann theorem and a condition on the uselessness of causal state information at the encoder.

Index Terms—Birkhoff-von Neumann theorem, channel capacity, deterministic channel, state information.

I. INTRODUCTION

A DISCRETE channel is commonly viewed as a black box with the input-output relation characterized by a stochastic matrix. In practice, it is often possible to obtain some additional information (known as the state information) by probing the channel. The knowledge of the state information might be useful in increasing the channel capacity. Note that, given each state, the channel can again be viewed as a black box and can potentially be further probed. One may continue this process until the black box is fully opened, i.e., the channel becomes deterministic given the acquired state information. This line of thought suggests that every channel has its own intrinsic state, which fully captures the randomness of the channel, and any state information acquired via channel probing is a degenerate version of this intrinsic state. As such, the intrinsic capacity, defined as the capacity of a channel when its intrinsic state is revealed, determines the ultimate capacity gain one can hope for by probing the channel.

It turns out that the intrinsic capacity of a channel is not necessarily uniquely defined. Consider a binary

symmetric channel with crossover probability 0.5: $W = (W_{x,y})_{x \in \{0,1\}, y \in \{0,1\}} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$, where each entry $W_{x,y}$ denotes the conditional probability $W(y | x)$ of output y given input x . The capacity of W is clearly zero. For this channel, we consider the following two models:

$$F(x) = x \oplus N \quad \text{and} \quad G(x) = N,$$

where \oplus denotes the modulo-2 addition and N is uniformly distributed over $\{0, 1\}$. It is easy to verify that they both have the conditional probability distribution W . If the actual model of W is F , then for every realization of N , W becomes a deterministic perfect channel, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that the capacity of W with N available at the encoder and/or the decoder increases to one. On the other hand, if the actual model of W is G , then for every realization of N , W becomes a deterministic useless channel, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, and hence, even with N known at both sides, the capacity of W is still zero. In fact, it will be seen that, for every number $r \in [0, 1]$, one can find a model for W such that the resulting intrinsic capacity is r .

This example indicates that a channel may admit different decompositions into deterministic channels. All these decompositions are mathematically legitimate though the actual way the deterministic channels are mixed to produce the given channel depends on the underlying physical mechanism. In this work we study the minimum and the maximum intrinsic capacities of a channel over all admissible decompositions. They will be referred to as the lower intrinsic capacity and the upper intrinsic capacity. For the aforementioned channel W , its lower and upper intrinsic capacities are 0 and 1, respectively. Since the causal state information may be available at the encoder, the decoder, or both, there are totally three different notions of lower and upper intrinsic capacities of a channel W , denoted by $\underline{\text{IC}}_f(W)$ and $\overline{\text{IC}}_f(W)$, for $f = 10, 01, 11$, where the two bits indicate if the state information is available at the encoder and the decoder, respectively.

The main contributions of this work are:

- 1) We study the structure of the convex polytope $\text{dec}(W)$, which consists of all convex combinations of deterministic channels for channel W , with a particular focus on its vertices. It is shown that $\overline{\text{IC}}_f(W)$ for all $f \in \{10, 01, 11\}$ and $\underline{\text{IC}}_{11}(W)$ are attained at certain vertices of $\text{dec}(W)$ (Theorem 2).
- 2) We prove a generalization of the Birkhoff-von Neumann theorem for a family $\mathcal{W}[a, b]$ of channel matrices with integer-valued column-sum vector constraints a and b from below and above, respectively (Theorem 19). It is shown that $\mathcal{W}[a, b]$ is convex and its vertices are

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exactly all deterministic channels in $\mathcal{W}[a, b]$. Using this fundamental result, we determine the exact values of $\underline{\text{IC}}_{11}(W)$ and $\overline{\text{IC}}_{11}(W)$ when the input or the output is binary. General lower and upper bounds are further provided for the nonbinary cases (Theorems 8 and 9), and in some cases, the exact value of $\overline{\text{IC}}_{11}(W)$ is also determined.

- 3) We obtain the exact values of $\underline{\text{IC}}_{10}(W)$ and $\overline{\text{IC}}_{10}(W)$ when W is a binary-output channel (Theorem 10), and obtain the exact values of $\underline{\text{IC}}_{01}(W)$ and $\overline{\text{IC}}_{01}(W)$ when W is a binary-input channel (Proposition 11). An interesting phenomenon observed is that $\underline{\text{IC}}_{10}(W) = C(W)$ for binary-output W , where $C(W)$ denotes the capacity of W . In other words, every binary-output channel can be generated through a certain mechanism such that the capacity remains the same if the source of randomness is causally revealed to the encoder. We further prove that the causal state information at the encoder is useless for a broad class of channels (Theorem 24). Finally, by providing some counterexamples, we show that the results such as $\underline{\text{IC}}_{10}(W) = C(W)$ and $\overline{\text{IC}}_{01}(W) = \overline{\text{IC}}_{11}(W)$ are specific to binary-input or binary-output channels, and do not hold in general (Example 46 and Proposition 48).

The rest of this paper is organized as follows. Section II lists some common notations used throughout this paper. Section III provides the definitions of various notions of (lower/upper) intrinsic capacity and a summary of the main results of this paper. The proofs and some other relevant findings are presented in Section IV and the appendices.

II. NOTATIONS

Although most notations will be defined at their first occurrences, some common ones are listed here for easy reference.

$\llbracket x, y \rrbracket$	The set of integers in the interval $[x, y]$.
B^A	The set of all maps $f: A \rightarrow B$, or equivalently, the set of all indexed families $x = (x_i \in B)_{i \in A}$ (a generalized form of sequences). If $A = \llbracket 1, n \rrbracket$, then B^A degenerates to the Cartesian product B^n . In this paper, a vector (for example, in \mathbf{R}^n) will be regarded as a row vector, and an all- c vector is usually denoted by \mathbf{c} .
$x \wedge y$	The minimum of x and y .
$x \vee y$	The maximum of x and y .
$\text{supp}(x)$	The support set $\{i \in I : x_i \neq 0\}$ of $x = (x_i)_{i \in I}$.
$\text{wt}(x)$	The weight $ \text{supp}(x) $ of $x = (x_i)_{i \in I}$.
$\lfloor x \rfloor$	The largest integer $\leq x$. If the argument is a sequence $x = (x_i)_{i \in I}$, then $\lfloor x \rfloor := (\lfloor x_i \rfloor)_{i \in I}$. The same convention also applies to other functions such as $ x $, $\lceil x \rceil$, $(x)_+$, and $(x)_-$.
$\lceil x \rceil$	The smallest integer $\geq x$.
$(x)_+$	$x \vee 0$.
$(x)_-$	$x \wedge 0$.
$\log x$	$\log_2 x$.

$D(P \parallel Q)$ The relative entropy $\sum_{a \in A} P(a) \log(P(a)/Q(a))$ of P with respect to Q , where P and Q are probability distributions over finite set A .

III. DEFINITIONS AND MAIN RESULTS

Let \mathcal{X} and \mathcal{Y} be two finite sets. A channel $W: \mathcal{X} \rightarrow \mathcal{Y}$ is a stochastic matrix with each entry $W_{x,y}$, or conventionally, $W(y | x)$ denoting the probability of output $y \in \mathcal{Y}$ given input $x \in \mathcal{X}$. A deterministic channel $D: \mathcal{X} \rightarrow \mathcal{Y}$ is a special channel whose stochastic matrix is a zero-one matrix, as such it uniquely identifies a map of \mathcal{X} into \mathcal{Y} . In the sequel, deterministic channels and maps will be regarded as equivalent objects and denoted using the same notation.

It is clear that the set of all channels forms a convex polytope in $\mathbf{R}^{\mathcal{X} \times \mathcal{Y}}$. We denote this polytope by $\mathcal{W}_{\mathcal{X}, \mathcal{Y}}$, or succinctly, \mathcal{W} . The deterministic channels are exactly the vertices of \mathcal{W} , and every channel can be expressed as a convex combination of them. This simple observation suggests that, for any channel, one can define a random state variable (referred to as the intrinsic state) given which the channel becomes deterministic. We are interested in characterizing the capacity of a channel when its intrinsic state is available at the encoder and/or the decoder. Such capacity results are of fundamental importance since they delineate the potential gain that can be achieved by probing the channel.

For a given channel, there are often multiple ways to write it as a convex combination of deterministic channels; as a consequence, the distribution of its intrinsic state is in general not uniquely defined. Let $\mathcal{D}_{\mathcal{X}, \mathcal{Y}}$ (or simply \mathcal{D}) denote the set of all deterministic channels $\mathcal{X} \rightarrow \mathcal{Y}$. Then the set of all possible convex decompositions of a channel W is given by

$$\text{dec}(W) := \{\lambda \in \mathcal{P}_{\mathcal{D}} : W = \sum_{D \in \mathcal{D}} \lambda_D D\},$$

where $\mathcal{P}_{\mathcal{D}}$ is the set of all probability distributions over \mathcal{D} and can be regarded as the set $\mathcal{W}_{\{\emptyset\}, \mathcal{D}}$ of matrices or vectors. For each intrinsic-state distribution $\lambda \in \mathcal{P}_{\mathcal{D}}$, we define the resulting capacities when the intrinsic state is causally available at the encoder, the decoder, or both, by

$$\begin{aligned} C_{10}(\lambda) &:= \max_{\mu \in \mathcal{P}_{\mathcal{X}^{\mathcal{D}}}} J_{10}(\lambda, \mu) \\ &= \max_{\mu \in \mathcal{P}_{\mathcal{X}^{\mathcal{D}}}} I\left(\mu, \left(\sum_{D \in \mathcal{D}} \lambda_D D_{u(D), y}\right)_{u \in \mathcal{X}^{\mathcal{D}}, y \in \mathcal{Y}}\right), \\ C_{01}(\lambda) &:= \max_{\mu \in \mathcal{P}_{\mathcal{X}}} J_{01}(\lambda, \mu) = \max_{\mu \in \mathcal{P}_{\mathcal{X}}} \sum_{D \in \mathcal{D}} \lambda_D I(\mu, D), \\ C_{11}(\lambda) &:= \max_{\kappa \in \mathcal{W}_{\mathcal{D}, \mathcal{X}}} \sum_{D \in \mathcal{D}} \lambda_D I(\kappa_{D, *}, D) \\ &= \sum_{D \in \mathcal{D}} \lambda_D \log \text{rank}(D), \end{aligned}$$

respectively (Remark 1), where

$$I(\mu, W) := \sum_x \mu_x D(W_{x, *}, \mu W)$$

is the mutual information given input distribution μ and channel W , and the flag $f \in \{10, 01, 11\}$ indicates the availability of the intrinsic state at the encoder and the decoder. For example, 10 means that the intrinsic state is available at the encoder but not at the decoder. For completeness, we also define the capacity with no encoder and decoder side information:

$$\begin{aligned} C(W) = C_{00}(\lambda) &:= \max_{\mu \in \mathcal{P}_{\mathcal{X}}} I\left(\mu, \sum_{D \in \mathcal{D}} \lambda_D D\right) \\ &= \max_{\mu \in \mathcal{P}_{\mathcal{X}}} I(\mu, W). \end{aligned}$$

Remark 1: Every intrinsic-state distribution $\lambda \in \mathcal{P}_{\mathcal{D}}$ identifies a channel $W(y | x, s) = s_{x,y}$ with state distribution $p_S(s) = \lambda_s$, where $s \in \mathcal{D}$. So the definitions of $C_{01}(\lambda)$ and $C_{11}(\lambda)$ are easy consequences of [2, eqs. (7.2) and (7.3)], respectively. The definition of $C_{10}(\lambda)$ follows from [2, Th. 7.2 and Remark 7.6]. In this case, we obtain by the so-called Shannon strategy an equivalent channel

$$W'(y | u, s) = s_{u(s),y}$$

with state distribution $p_S(s) = \lambda_s$, where $s \in \mathcal{D}$ and the input $u \in \mathcal{X}^{\mathcal{D}}$ is a map from \mathcal{D} to \mathcal{X} , which provides an input strategy for all the cases of s . After averaged with respect to p_S , this channel is finally converted into a channel with no state:

$$W''(y | u) = \sum_{s \in \mathcal{D}} \lambda_s s_{u(s),y}.$$

Reference [2, Th. 7.2] shows that its capacity is exactly the capacity of the original one with state information causally available at the encoder. Since it is customary to denote by D a deterministic channel, we use D instead of s in the definition of $C_f(\lambda)$ for $f \in \{10, 01, 11\}$.

Then, given a channel W , we can define its intrinsic-capacity set by

$$\text{IC}_f(W) := \{C_f(\lambda) : \lambda \in \text{dec}(W)\}.$$

Furthermore, we define the lower intrinsic capacity and the upper intrinsic capacity of W for $f \in \{10, 01, 11\}$ by

$$\underline{\text{IC}}_f(W) := \inf \text{IC}_f(W)$$

and

$$\overline{\text{IC}}_f(W) := \sup \text{IC}_f(W),$$

respectively. The set $\text{IC}_f(W)$ turns out to be a closed interval, so its infimum and supremum are the minimum and the maximum, respectively. More facts about $\text{dec}(W)$ and $\text{IC}_f(W)$ are given by the next theorem as well as the results in Appendices A and B.

Theorem 2: The set $\text{dec}(W)$ is a bounded, closed convex polytope. For each $f \in \{10, 01, 11\}$, $\text{IC}_f(W)$ is a closed interval and $\overline{\text{IC}}_f(W)$ can be attained at some vertex of $\text{dec}(W)$. Furthermore, $\underline{\text{IC}}_{11}(W)$ can also be attained at some vertex of $\text{dec}(W)$.

Proof: By definition, it is clear that $\text{dec}(W)$ is a bounded, closed convex polytope, so that $\text{IC}_f(W)$ is a closed interval (Proposition 35). It is also easy to see that $C_f(\lambda)$ attains

its maximum $\overline{\text{IC}}_f(W)$ at some vertex of $\text{dec}(W)$ and that $C_{11}(\lambda)$ attains its minimum $\underline{\text{IC}}_{11}(W)$ at some vertex of $\text{dec}(W)$ (Proposition 35 and [3, Proposition 3.4.1]). \square

The definitions of $\overline{\text{IC}}_f(W)$ and $\underline{\text{IC}}_f(W)$ may look somewhat artificial. The next two remarks explain their significance.

Remark 3: A channel W with any form of state information, say S , can be decomposed into the following form:

$$W = \sum_{s \in \mathcal{S}} p_S(s) W'_s.$$

For every $s \in \mathcal{S}$, we choose a distribution $\lambda_s \in \text{dec}(W'_s)$. Note that p_S and λ_s , $s \in \mathcal{S}$, induce an intrinsic state S' with distribution

$$p_{S'} := \sum_{s \in \mathcal{S}} p_S(s) \lambda_s \in \text{dec}(W).$$

Moreover, S can be represented as a function of (S', U) for some random variable U independent of S' (the functional representation lemma [2, p. 626], [4, Lemma 1]); therefore, S' is at least as informative as S . This shows that $\overline{\text{IC}}_f(W)$ is the maximum of the capacities of W with any form of state information whose availability at the encoder and the decoder is specified by f .

Remark 4: $\underline{\text{IC}}_f(W)$ is exactly the capacity of the compound channel $(S)_{p_S \in \text{dec}(W)}$ with the availability of S at the encoder and the decoder specified by f , where S is \mathcal{D} -valued and p_S is selected arbitrarily from $\text{dec}(W)$. For example, for $f = 10$, it follows from the minimax theorem [5] and Proposition 34 that

$$\begin{aligned} \underline{\text{IC}}_{10}(W) &= \min_{\lambda \in \text{dec}(W)} C_{10}(\lambda) \\ &= \min_{\lambda \in \text{dec}(W)} \max_{\mu \in \mathcal{P}_{\mathcal{X}^{\mathcal{D}}}} J_{10}(\lambda, \mu) \\ &= \max_{\mu \in \mathcal{P}_{\mathcal{X}^{\mathcal{D}}}} \min_{\lambda \in \text{dec}(W)} J_{10}(\lambda, \mu), \end{aligned}$$

which is the capacity of the compound channel $(S)_{p_S \in \text{dec}(W)}$ with state information S causally available at the encoder [2, Ths. 7.1 and 7.2 and Remark 7.6 as well as a remark before Example 7.2]. Note that the state of this compound channel is p_S , which remains the same throughout the transmission block, and that once p_S is chosen, we are faced with a memoryless channel S with causal state information S available at the encoder. Other cases of f can be proved similarly using the minimax theorem, Proposition 34, and [2, eqs. (7.2) and (7.3)].

The main results of this paper are given as follows. With no loss of generality, we assume from now on that the channel W is from $[[1, m]]$ to $[[1, n]]$, where $m, n \geq 2$.

Since $C_{11}(\lambda)$ can be rewritten as

$$C_{11}(\lambda) = \sum_{r=1}^{m \wedge n} \log r \sum_{D \in \mathcal{D}: \text{rank}(D)=r} \lambda_D,$$

the total probability of all D with the same rank is an important quantity for analyzing $\underline{\text{IC}}_{11}(W)$ and $\overline{\text{IC}}_{11}(W)$. This quantity also turns out to be important for other kinds of lower and upper intrinsic capacities (Theorem 10 and Proposition 11). So let us begin with the ‘‘rank probability’’.

Definition 5: The rank probability function $\Gamma_\lambda(r)$ induced by $\lambda \in \text{dec}(W)$ is defined by

$$\Gamma_\lambda(r) := \lambda\{D \in \mathcal{D} : \text{rank}(D) = r\} = \sum_{D \in \mathcal{D} : \text{rank}(D) = r} \lambda_D.$$

Definition 6: The lower and the upper rank- r probabilities of W are defined by

$$\underline{\Gamma}_W(r) := \min_{\lambda \in \text{dec}(W)} \Gamma_\lambda(r) \quad \text{and} \quad \overline{\Gamma}_W(r) := \max_{\lambda \in \text{dec}(W)} \Gamma_\lambda(r),$$

respectively.

Note that $\text{dec}(W)$ is compact and the finite summation in the definition of $\Gamma_\lambda(r)$ is continuous with respect to λ , so the minimum and the maximum operations in the definitions of $\underline{\Gamma}_W(r)$ and $\overline{\Gamma}_W(r)$ are justified. In the sequel, we are mainly interested in the cases $r = 1$ and $r = m \wedge n$. Most of our results will be expressed in terms of these quantities. Bounds for $\overline{\Gamma}_W(m \wedge n)$ will be given in Proposition 20. Exact values of $\underline{\Gamma}_W(1)$ and $\overline{\Gamma}_W(1)$ are determined in the next proposition.

Proposition 7:

$$\underline{\Gamma}_W(1) = (g - m + 1)_+,$$

$$\overline{\Gamma}_W(1) = \sum_{j=1}^n \alpha_j,$$

where

$$g = \max_{1 \leq j \leq n} (\mathbf{1}W)_j,$$

$$\alpha = \left(\min_{1 \leq i \leq m} W_{i,j} \right)_{j \in \llbracket 1, n \rrbracket}.$$

Theorem 8 (Bounds for $\underline{\text{IC}}_{11}(W)$):

$$\underline{\text{IC}}_{11}(W) \leq \begin{cases} (1 - \overline{\Gamma}_W(1)) \log \gamma, & \text{if } \overline{\Gamma}_W(1) < 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\underline{\text{IC}}_{11}(W) \geq 1 - \overline{\Gamma}_W(1),$$

where

$$\gamma = (m + \text{wt}(a) - a\mathbf{1}^\top) \wedge n, \quad a = \lfloor \mathbf{1}W' \rfloor,$$

$$W' = \frac{W - \sum_{j=1}^n \alpha_j U_j}{1 - \overline{\Gamma}_W(1)},$$

$\mathbf{1}$ denotes the m - or n -dimensional all-one row vectors so that $a = (\lfloor \sum_{i=1}^m W'_{i,j} \rfloor)_{j=1}^n$, and U_j is the deterministic channel with its j -th column being all one.

If $m = 2$ or $n = 2$, then $\underline{\text{IC}}_{11}(W) = 1 - \overline{\Gamma}_W(1)$.

Theorem 9 (Bounds for $\overline{\text{IC}}_{11}(W)$): If $\underline{\Gamma}_W(1) > 0$ or $m = 2$ or $n = 2$, then

$$\overline{\text{IC}}_{11}(W) = 1 - \underline{\Gamma}_W(1);$$

otherwise,

$$\log \gamma' \leq \overline{\text{IC}}_{11}(W) \leq \log(o - 1) + \overline{\Gamma}_W(o) \log \frac{o}{o - 1},$$

where

$$\gamma' = \text{wt}(a) + \left(m - \sum_{j \in \text{supp}(a)} b_j \right)_+, \quad o = m \wedge n,$$

$$a = \lfloor \mathbf{1}W \rfloor, \quad b = \lceil \mathbf{1}W \rceil.$$

TABLE I

THE CASES WHERE $\underline{\text{IC}}_{11}(W)$ OR $\overline{\text{IC}}_{11}(W)$ ARE DETERMINED

Case	$\underline{\text{IC}}_{11}(W)$	$\overline{\text{IC}}_{11}(W)$
$m = 2$	$1 - \underline{\Gamma}_W(1)$	$1 - \underline{\Gamma}_W(1)$
$n = 2$	$1 - \overline{\Gamma}_W(1)$	$1 - \overline{\Gamma}_W(1)$
$m \leq n$ and $\mathbf{1}W \leq \mathbf{1}$?	$\log m$
$m \geq n$ and $\mathbf{1}W \geq \mathbf{1}$?	$\log n$
$\underline{\Gamma}_W(1) > 0$?	$1 - \underline{\Gamma}_W(1)$
$\overline{\Gamma}_W(1) = 1$	0	?

If $m \leq n$ and $\mathbf{1}W \leq \mathbf{1}$ (i.e., $\sum_{i=1}^m W_{i,j} \leq 1$ for all $1 \leq j \leq n$), then $\overline{\text{IC}}_{11}(W) = \log m$.

If $m \geq n$ and $\mathbf{1}W \geq \mathbf{1}$ (i.e., $\sum_{i=1}^m W_{i,j} \geq 1$ for all $1 \leq j \leq n$), then $\overline{\text{IC}}_{11}(W) = \log n$.

The upper bound for $\overline{\text{IC}}_{11}(W)$ is expressed in terms of $\overline{\Gamma}_W(m \wedge n)$ whose exact value is only known in some special cases. To have an explicit bound, we need the upper bound for $\overline{\Gamma}_W(m \wedge n)$ given by Proposition 20.

It is clear that the bounds given in Theorems 8 and 9 are not tight in general, and may be very loose in some special cases. Possible ways to improve these bounds will be discussed in Remark 23. Table I lists all the cases where $\underline{\text{IC}}_{11}(W)$ or $\overline{\text{IC}}_{11}(W)$ are determined.

Theorem 10 ($\underline{\text{IC}}_{10}(W)$ and $\overline{\text{IC}}_{10}(W)$ for Binary-Output Channels): If $n = 2$, then

$$\underline{\text{IC}}_{10}(W) = C(W)$$

and

$$\overline{\text{IC}}_{10}(W) = C \left(\begin{pmatrix} 1 & 0 \\ \underline{\Gamma}_W(1) & 1 - \underline{\Gamma}_W(1) \end{pmatrix} \right).$$

Proposition 11 ($\underline{\text{IC}}_{01}(W)$ and $\overline{\text{IC}}_{01}(W)$ for Binary-Input Channels): If $m = 2$, then for every $\lambda \in \text{dec}(W)$, $C_{01}(\lambda) = C_{11}(\lambda)$, so that $\underline{\text{IC}}_{01}(W) = 1 - \overline{\Gamma}_W(1)$ and $\overline{\text{IC}}_{01}(W) = 1 - \underline{\Gamma}_W(1)$.

The above results enable us to obtain explicit characterizations of all lower and upper intrinsic capacities for binary-input binary-output channels. The relevant expressions are collected in the following example.

Example 12: Suppose

$$W = \begin{pmatrix} 1 - \epsilon_1 & \epsilon_1 \\ \epsilon_2 & 1 - \epsilon_2 \end{pmatrix}.$$

From Theorem 8 and Proposition 11, it follows that

$$\begin{aligned} \underline{\text{IC}}_{11}(W) &= \underline{\text{IC}}_{01}(W) \\ &= 1 - \overline{\Gamma}_W(1) \\ &= \begin{cases} \epsilon_1 + \epsilon_2 - 1, & \epsilon_1 + \epsilon_2 \geq 1 \\ 1 - \epsilon_1 - \epsilon_2, & \epsilon_1 + \epsilon_2 \leq 1 \end{cases} \\ &= |1 - \epsilon_1 - \epsilon_2|. \end{aligned}$$

Theorem 10 shows that

$$\underline{\text{IC}}_{10}(W) = C(W)$$

$$\stackrel{(a)}{=} \begin{cases} \log \left(2^{\frac{\epsilon_2 h(\epsilon_1) - (1 - \epsilon_1) h(\epsilon_2)}{1 - \epsilon_1 - \epsilon_2}} + 2^{\frac{\epsilon_1 h(\epsilon_2) - (1 - \epsilon_2) h(\epsilon_1)}{1 - \epsilon_1 - \epsilon_2}} \right), \\ \text{if } \epsilon_1 + \epsilon_2 \neq 1, \\ 0, \text{ otherwise,} \end{cases}$$

where (a) uses the method in [6, eq. (4.5.11)], and $h(\epsilon) := -\epsilon \log \epsilon - (1 - \epsilon) \log(1 - \epsilon)$ is the binary entropy function. Theorem 9 and Proposition 11 give

$$\begin{aligned} \overline{\text{IC}}_{11}(W) &= \overline{\text{IC}}_{01}(W) \\ &= 1 - \underline{\Gamma}_W(1) \\ &= \begin{cases} 1 + \epsilon_1 - \epsilon_2, & \epsilon_1 \leq \epsilon_2 \\ 1 - \epsilon_1 + \epsilon_2, & \epsilon_1 \geq \epsilon_2 \end{cases} \\ &= 1 - |\epsilon_1 - \epsilon_2|. \end{aligned}$$

Theorem 10 together with the above result gives

$$\begin{aligned} \overline{\text{IC}}_{10}(W) &= C \left(\begin{pmatrix} 1 & 0 \\ |\epsilon_1 - \epsilon_2| & 1 - |\epsilon_1 - \epsilon_2| \end{pmatrix} \right) \\ &= \begin{cases} 1, & \epsilon_1 = \epsilon_2, \\ \log \left[1 + (1 - |\epsilon_1 - \epsilon_2|) |\epsilon_1 - \epsilon_2|^{\frac{|\epsilon_1 - \epsilon_2|}{1 - |\epsilon_1 - \epsilon_2|}} \right], & |\epsilon_1 - \epsilon_2| \in (0, 1), \\ 0, & |\epsilon_1 - \epsilon_2| = 1. \end{cases} \end{aligned}$$

If W is a binary symmetric channel with crossover probability ϵ (i.e., $\epsilon_1 = \epsilon_2 = \epsilon$), then

$$\begin{aligned} \underline{\text{IC}}_{11}(W) &= \underline{\text{IC}}_{01}(W) = |1 - 2\epsilon|, \\ \underline{\text{IC}}_{10}(W) &= C(W) = 1 - h(\epsilon), \\ \overline{\text{IC}}_{11}(W) &= \overline{\text{IC}}_{01}(W) = \overline{\text{IC}}_{10}(W) = 1. \end{aligned}$$

If W is a Z-channel with crossover probability θ (i.e., $\epsilon_1 = 0$ and $\epsilon_2 = \theta$), then

$$\begin{aligned} \underline{\text{IC}}_{11}(W) &= \underline{\text{IC}}_{01}(W) = \overline{\text{IC}}_{11}(W) = \overline{\text{IC}}_{01}(W) = 1 - \theta, \\ \underline{\text{IC}}_{10}(W) &= \overline{\text{IC}}_{10}(W) = C(W) \\ &= \begin{cases} 1, & \theta = 0, \\ \log \left(1 + (1 - \theta) \theta^{\frac{\theta}{1 - \theta}} \right), & \theta \in (0, 1), \\ 0, & \theta = 1. \end{cases} \end{aligned}$$

The case of Z-channel is special, because in this case W admits a unique convex decomposition into deterministic channels:

$$\theta \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + (1 - \theta) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The lower and the upper intrinsic capacities of these two special channels are plotted in Figs. 1 and 2.

IV. PROOFS OF MAIN RESULTS

Theorem 2 shows that $\overline{\text{IC}}_f(W)$ for all $f \in \{10, 01, 11\}$ and $\underline{\text{IC}}_{11}(W)$ are attained at certain vertices of $\text{dec}(W)$. As such, it is of great importance to study the structure of $\text{dec}(W)$. A series of results on the vertices of $\text{dec}(W)$ is provided in Appendix A. Although these results shed useful light on the structure of $\text{dec}(W)$, the characterizations are still too coarse for our purpose. It will be seen that additional insights can be gained by taking the objective functions into consideration.

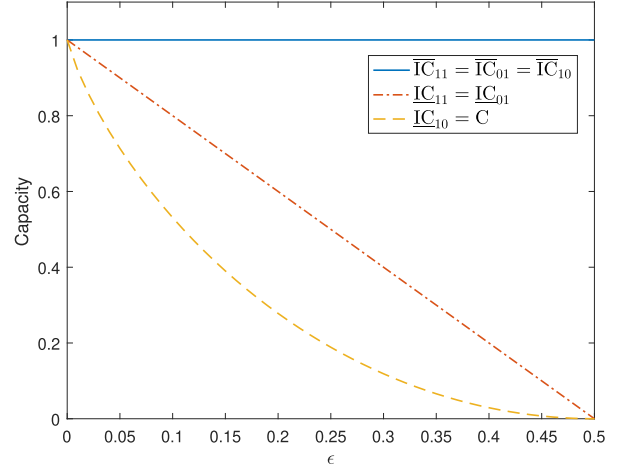


Fig. 1. The lower and the upper intrinsic capacities of a binary symmetric channel with crossover probability ϵ .

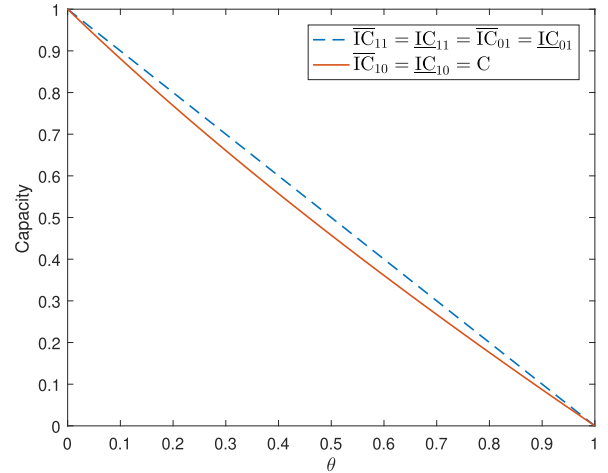


Fig. 2. The lower and the upper intrinsic capacities of a Z-channel with crossover probability θ .

A. $\underline{\text{IC}}_{11}(W)$ and $\overline{\text{IC}}_{11}(W)$

In this subsection, we will prove Theorems 8 and 9. To this end, we will establish a characterization of λ that achieves $\underline{\text{IC}}_{11}(W)$ or $\overline{\text{IC}}_{11}(W)$ (Proposition 13) and a generalization of the Birkhoff-von Neumann theorem (Theorem 19). The exact values of $\underline{\Gamma}_W(1)$ and $\overline{\Gamma}_W(1)$ (Proposition 7), upper bounds for $\overline{\Gamma}_W(m \wedge n)$ (Proposition 20), and the relation between $\underline{\Gamma}_W(1)$ and $\overline{\text{IC}}_{11}(W)$ (Proposition 21) are then investigated. Based on these results, Theorems 8 and 9 are proved. Various ways to improve the bounds in Theorems 8 and 9 are discussed in Remark 23.

We first provide a complete characterization of λ that achieves $\underline{\text{IC}}_{11}(W)$ or $\overline{\text{IC}}_{11}(W)$.

Proposition 13: Let

$$\begin{aligned} \mathfrak{A}_+ &= \left\{ \text{supp}((\alpha)_+): \alpha \in \mathbf{R}^{\mathcal{D}}, \right. \\ &\quad \left. \sum_{D \in \mathcal{D}} \alpha_D D = 0, \sum_{D \in \mathcal{D}} \alpha_D \log \text{rank}(D) > 0 \right\} \end{aligned}$$

and

$$\mathfrak{U}_- = \left\{ \text{supp}((\alpha)_-): \alpha \in \mathbf{R}^{\mathcal{D}}, \right. \\ \left. \sum_{D \in \mathcal{D}} \alpha_D D = 0, \sum_{D \in \mathcal{D}} \alpha_D \log \text{rank}(D) > 0 \right\}.$$

For $\lambda \in \text{dec}(W)$, $C_{11}(\lambda) = \underline{\text{IC}}_{11}(W)$ iff there is no $U \in \mathfrak{U}_+$ such that $U \subseteq \text{supp}(\lambda)$; $C_{11}(\lambda) = \overline{\text{IC}}_{11}(W)$ iff there is no $U \in \mathfrak{U}_-$ such that $U \subseteq \text{supp}(\lambda)$.

Proof: It suffices to prove the first part, because the second part can be proved in the same vein.

(Sufficiency) If there exists some $\beta \in \text{dec}(W)$ such that $C_{11}(\beta) < C_{11}(\lambda)$, then

$$\sum_{D \in \mathcal{D}} (\lambda_D - \beta_D) D = 0$$

and

$$\sum_{D \in \mathcal{D}} (\lambda_D - \beta_D) \log \text{rank}(D) > 0,$$

so that $U = \text{supp}((\lambda - \beta)_+) \in \mathfrak{U}_+$ and $U \subseteq \text{supp}(\lambda)$, a contradiction.

(Necessity) For $U \in \mathfrak{U}_+$, if $U \subseteq \text{supp}(\lambda)$, then there is a vector $\alpha \in \mathbf{R}^{\mathcal{D}}$ such that $\text{supp}((\alpha)_+) \subseteq \text{supp}(\lambda)$, $\sum_D \alpha_D D = 0$, and $\sum_D \alpha_D \log \text{rank}(D) > 0$. Let $\beta = \lambda - t\alpha$. For sufficiently small $t > 0$, it can be verified that $\beta \in \text{dec}(W)$ and $C_{11}(\beta) < C_{11}(\lambda) = \underline{\text{IC}}_{11}(W)$, which is absurd. \square

Definition 14: A subset $S \subseteq \mathcal{D}$ is said to be IC_{11} -minimized, or succinctly, IC-minimized (resp., IC-maximized) if there is a $\lambda \in \mathcal{P}_{\mathcal{D}}$ such that $\text{supp}(\lambda) = S$ and $C_{11}(\lambda) = \underline{\text{IC}}_{11}(W)$ (resp., $C_{11}(\lambda) = \overline{\text{IC}}_{11}(W)$), where $W = \sum_{D \in \mathcal{D}} \lambda_D D$.

The following result is a simple consequence of Proposition 13.

Proposition 15: If $S \subseteq \mathcal{D}$ is IC-minimized (resp., IC-maximized), then any $\lambda \in \mathcal{P}_{\mathcal{D}}$ supported on S achieves $\underline{\text{IC}}_{11}(W)$ (resp., $\overline{\text{IC}}_{11}(W)$), where $W = \sum_{D \in \mathcal{D}} \lambda_D D$. As a consequence, any nonempty subset of S is also IC-minimized (resp., IC-maximized).

By Proposition 15, it is important to identify patterns of sets that are not IC-minimized or IC-maximized. Some simple patterns that are not IC-minimized or IC-maximized are given as follows and their proofs are relegated to Appendix C.

Proposition 16: If $m \leq n$, then any deterministic perfect channels P_1, \dots, P_ℓ such that at least one column of $P_1 + \dots + P_\ell$ has a weight greater than one are not IC-minimized.

Proposition 17: If $m \geq n$, then any deterministic perfect channels P_1, \dots, P_ℓ such that at least one column of $P_1 + \dots + P_\ell$ has no entry equal to ℓ are not IC-minimized.

Proposition 18: For $D \in \mathcal{D}$, if $\text{wt}(D_{*,j}) \leq m - 2$, then $\{D, U_j\}$ is not IC-maximized.

The next result is a generalization of the Birkhoff-von Neumann theorem, which plays a crucial role in proving Theorems 8 and 9. Our proof hinges on an extension of the ideas in [7] and [8].

Theorem 19: Let a and b be two n -dimensional integer-valued vectors such that $a \leq b$, namely, $a_j \leq b_j$ for

$1 \leq j \leq n$. Let

$$\mathcal{W}[a, b] := \{W \in \mathcal{W}: a \leq \mathbf{1}W \leq b\}$$

and $\mathcal{D}[a, b] := \mathcal{W}[a, b] \cap \mathcal{D}$, where $\mathbf{1}$ denotes the m -dimensional all-one row vector. If $\mathcal{W}[a, b]$ is not empty, then $\mathcal{W}[a, b]$ is convex and the vertices of $\mathcal{W}[a, b]$ are exactly the matrices in $\mathcal{D}[a, b]$.

When $m = n$ and $a = b = \mathbf{1}$, Theorem 19 reduces to the classical Birkhoff-von Neumann theorem for doubly stochastic matrices.

Proof: It is clear that $\mathcal{W}[a, b]$, if nonempty, is a convex set. We will show that any matrix $W \in \mathcal{W}[a, b]$ with non-integer entries cannot be a vertex of $\mathcal{W}[a, b]$. There are two cases:

Case (a): There is a non-integer entry in a non-boundary column.

Case (b): All non-integer entries are in the boundary columns.

Here, a column is called a boundary column if its sum is either a_j or b_j , where j is the index of the column.

In either case, we can pick a non-integer entry, say the (i_0, j_0) entry, which in Case (a) must be a non-integer entry in a non-boundary column. By the following argument, we will find a chain or loop of non-integer entries of the matrix, which will be used to prove that the matrix is not extremal.

Because the (i_0, j_0) entry is not an integer, there exists at least another entry in the same row that is also not an integer, say the (i_0, j_1) entry. If the j_1 -th column is not on the boundary, then we are done. If however the j_1 -th column is on the boundary, then there exists at least another non-integer entry in the same column, say (i_1, j_1) . In general, after t steps, we have visited $t + 1$ columns, with the chain

$$(i_0, j_0), (i_0, j_1), (i_1, j_1), \dots, (i_{t-1}, j_t), (i_t, j_t).$$

Except for the j_0 -th column, every column has exactly one inbound entry (i_{s-1}, j_s) and one outbound entry (i_s, j_s) , where $1 \leq s \leq t$. Now in the $(t + 1)$ -th step, by the same argument, we find the (i_t, j_{t+1}) entry in the j_{t+1} -th column. If this column has already been visited, then $j_{t+1} = j_s$ for some $0 \leq s \leq t - 1$ and we are done. If this column is new but not on the boundary, we are also done. If however this new column is on the boundary, then we can further find an outbound entry in this column, say (i_{t+1}, j_{t+1}) , and proceed to the $(t + 2)$ -th step. Because there are finite columns, we will always end up with a chain

$$(i_0, j_0), (i_0, j_1), (i_1, j_1), \dots, (i_{k-1}, j_{k-1}), (i_{k-1}, j_k)$$

which only happens in Case (a), or a loop

$$(i_\ell, j_\ell), (i_\ell, j_{\ell+1}), (i_{\ell+1}, j_{\ell+1}), \dots, (i_{k-1}, j_k), \\ (i_k, j_k) = (i_\ell, j_\ell)$$

for some $0 \leq \ell < k - 1$.

Then we can construct a matrix N by setting all outbound entries (in the chain or the loop) $N_{i_s, j_s} = 1$, all inbound entries $N_{i_{s-1}, j_s} = -1$, and all other entries to be zero. It is clear that

$$\mathbf{1}N = \mathbf{e}_{j_0} - \mathbf{e}_{j_k}, \quad N\mathbf{1}^\top = 0$$

in the former case and

$$\mathbf{1}N = 0, \quad N\mathbf{1}^\top = 0$$

in the latter case, where $\mathbf{e}_k = (1\{j = k\})_{j \in \llbracket 1, n \rrbracket}$.

Let $U = W + \epsilon N$ and $V = W - \epsilon N$. It is clear that $U, V \in \mathcal{W}[a, b]$ for sufficiently small $\epsilon > 0$. It is also clear that $W = \frac{1}{2}U + \frac{1}{2}V$ and $U \neq V$, that is, W is not a vertex of $\mathcal{W}[a, b]$.

Therefore, we have $\mathcal{V} \subseteq \mathcal{D}[a, b]$, where \mathcal{V} denotes the set of all vertices of $\mathcal{W}[a, b]$. It remains to show that $\mathcal{D}[a, b] \subseteq \mathcal{V}$. For any $W \in \mathcal{D}[a, b]$, if $W = \alpha U + (1 - \alpha)V$ with $U, V \in \mathcal{W}[a, b]$ and $\alpha \in (0, 1)$, then for every $1 \leq i \leq m$,

$$\mathbf{e}_i W = \alpha \mathbf{e}_i U + (1 - \alpha) \mathbf{e}_i V,$$

which however implies that $\mathbf{e}_i U = \mathbf{e}_i V$ for every $1 \leq i \leq m$, or $U = V$. \square

Equipped with Theorem 19, we proceed to study the lower and the upper rank probabilities (see Definitions 5 and 6).

Proof of Proposition 7: By Theorem 19, W can be expressed as a convex combination of deterministic channels of rank ≥ 2 if $g \leq m - 1$, in which case, $\underline{\Gamma}_W(1) = 0$. Otherwise, let ℓ be the index of the column with the sum $g > m - 1$. Consider the convex combination

$$W = tU_\ell + (1 - t)W'.$$

It is clear that W' cannot be a convex combination of deterministic channels of rank ≥ 2 unless the sum of its ℓ -th column is $\leq m - 1$. To this end, we set $t = g - m + 1$, which is the minimum value required, and we have

$$(\mathbf{1}W')_\ell = \frac{(\mathbf{1}W)_\ell - tm}{1 - t} = m - 1$$

and

$$(\mathbf{1}W')_j = \frac{(\mathbf{1}W)_j}{1 - t} \leq 1$$

for $j \neq \ell$, so that $\underline{\Gamma}_W(1) = g - m + 1$.

If W has the following convex decomposition

$$W = \left(1 - \sum_{j=1}^n s_j\right) W' + \sum_{j=1}^n s_j U_j,$$

then W' is a valid stochastic matrix iff $s_j \leq \alpha_j$ for all j . Therefore, $\overline{\Gamma}_W(1) = \sum_{j=1}^n \alpha_j$. \square

Proposition 20: If $m \leq n$, then

$$\overline{\Gamma}_W(m) \leq 1 - \beta,$$

where

$$\beta = 0 \vee \max_{1 \leq j \leq n} \beta'_j \quad (1)$$

and

$$\beta'_j = \begin{cases} \frac{(\mathbf{1}W)_j - 1}{\text{wt}(W_{*,j}) - 1}, & \text{if } \text{wt}(W_{*,j}) > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $\beta = 0$, then $\overline{\Gamma}_W(m) = 1$.

If $m \geq n$, then

$$\overline{\Gamma}_W(n) \leq h,$$

where

$$h = 1 \wedge \min_{1 \leq j \leq n} (\mathbf{1}W)_j. \quad (2)$$

If $h = 1$, then $\overline{\Gamma}_W(n) = 1$.

Proof: If $m \leq n$, then the sum of every column of a deterministic channel of rank m is at most 1, and for every $1 \leq j \leq n$, W admits a convex decomposition into deterministic channels with the j -th column sum at most $\text{wt}(W_{*,j})$. Thus for every $\lambda \in \text{dec}(W)$ and every j ,

$$\begin{aligned} (\mathbf{1}W)_j &\leq (1 \wedge \text{wt}(W_{*,j})) \Gamma_\lambda(m) + \text{wt}(W_{*,j})(1 - \Gamma_\lambda(m)) \\ &= \text{wt}(W_{*,j}) - (\text{wt}(W_{*,j}) - 1)_+ \Gamma_\lambda(m), \end{aligned}$$

so that

$$\Gamma_\lambda(m) \leq 1 - \frac{(\mathbf{1}W)_j - 1}{\text{wt}(W_{*,j}) - 1}$$

for $\text{wt}(W_{*,j}) > 1$ and hence $\overline{\Gamma}_W(m) \leq 1 - \beta$. If $\beta = 0$, which implies that $(\mathbf{1}W)_j \leq 1$ for all $1 \leq j \leq n$, then $\overline{\Gamma}_W(m) = 1$ (Theorem 19).

If $m \geq n$, then the sum of every column of a deterministic channel of rank n is at least 1, so that, for every $\lambda \in \text{dec}(W)$ and every $1 \leq j \leq n$,

$$(\mathbf{1}W)_j \geq \Gamma_\lambda(n),$$

and hence $\overline{\Gamma}_W(n) \leq h$. If $h = 1$, which implies $(\mathbf{1}W)_j \geq 1$ for all $1 \leq j \leq n$, then $\overline{\Gamma}_W(n) = 1$ (Theorem 19). \square

The next result establishes the relation between $\underline{\Gamma}_W(1)$ and $\overline{\text{IC}}_{11}(W)$.

Proposition 21: If $\lambda \in \text{dec}(W)$ achieves $\overline{\text{IC}}_{11}(W)$, then $\Gamma_\lambda(1) = \underline{\Gamma}_W(1)$. In particular, if $\underline{\Gamma}_W(1) > 0$, then $\lambda_{U_\ell} = \underline{\Gamma}_W(1)$ and $\Gamma_\lambda(2) = 1 - \underline{\Gamma}_W(1)$, where $\ell = \arg \max_{1 \leq j \leq n} (\mathbf{1}W)_j$.

Proof: If λ is zero on all deterministic useless channels, then $\Gamma_\lambda(1) = \underline{\Gamma}_W(1) = 0$.

If $\lambda_{U_j} > 0$ for some j , then λ must be zero on all deterministic channels whose j -th column weight is less than $m - 1$ (Propositions 15 and 18). Therefore, we must have $\lambda_{U_j} = \Gamma_\lambda(1) = \underline{\Gamma}_W(1)$ (Proposition 7) and $\Gamma_\lambda(2) = 1 - \underline{\Gamma}_W(1)$. \square

Is there a similar relation between $\overline{\Gamma}_W(1)$ and $\underline{\text{IC}}_{11}(W)$? The following example shows that the λ attaining $\underline{\text{IC}}_{11}(W)$ does not necessarily satisfy $\Gamma_\lambda(1) = \overline{\Gamma}_W(1)$.

Example 22: Let

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

If we decompose W as follows:

$$W = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

then the second channel does not have a composition containing deterministic channels of rank less than 3, so that the associated intrinsic capacity is greater than or equal to $2 \log 3/3 \approx 1.057$. On the other hand, it can be shown via linear programming that $\underline{\text{IC}}_{11}(W) = 1$ with the decomposition

$$W = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In this case, the optimal λ satisfies $\Gamma_\lambda(1) = \Gamma_\lambda(3) = \Gamma_\lambda(4) = 0$ and $\Gamma_\lambda(2) = 1$.

We are now ready to prove Theorems 8 and 9.

Proof of Theorem 8: To establish a good upper bound for $\underline{\text{IC}}_{11}(W)$, we need to find an “unfavorable” convex decomposition of W . To this end, we can first extract from W a collection of useless channels with the total probability $\bar{\Gamma}_W(1)$ (Proposition 7), that is,

$$W = \sum_{j=1}^n \alpha_j U_j + (1 - \bar{\Gamma}_W(1)) W'.$$

If $\bar{\Gamma}_W(1) = 1$, then $\underline{\text{IC}}_{11}(W) = 0$; otherwise,

$$\underline{\text{IC}}_{11}(W) \leq (1 - \bar{\Gamma}_W(1)) \underline{\text{IC}}_{11}(W').$$

It is clear that $W' \in \mathcal{W}[a, \mathbf{m}]$, where \mathbf{m} denotes the all- m row vector. The best deterministic channels in $\mathcal{W}[a, \mathbf{m}]$ are those with the number of nonzero columns maximized. The rank of those matrices is

$$\gamma = \left(\text{wt}(a) + m - \sum_{j=1}^n a_j \right) \wedge n,$$

so $\underline{\text{IC}}_{11}(W') \leq \log \gamma$ (Theorem 19).

Let λ be a vertex of $\text{dec}(W)$ that attains $\underline{\text{IC}}_{11}(W)$. Then

$$\begin{aligned} \underline{\text{IC}}_{11}(W) &= \sum_{D \in \mathcal{D}} \lambda_D \log \text{rank}(D) \\ &\geq 1 - \Gamma_\lambda(1) \geq 1 - \bar{\Gamma}_W(1). \end{aligned}$$

Finally, the special case of $m = 2$ or $n = 2$ can be easily verified. \square

Proof of Theorem 9: Let λ be a vertex of $\text{dec}(W)$ that attains $\bar{\text{IC}}_{11}(W)$.

If $\underline{\Gamma}_W(1) > 0$ or $m = 2$ or $n = 2$, then $\Gamma_\lambda(r) = 0$ for all $r > 2$ (Proposition 21), so that $\bar{\text{IC}}_{11}(W) = 1 - \underline{\Gamma}_W(1)$ (Proposition 7). The remaining case is then $\underline{\Gamma}_W(1) = 0$.

To establish a good lower bound for $\bar{\text{IC}}_{11}(W)$, we need to find a “favorable” convex decomposition of W . It is clear that $W \in \mathcal{W}[a, b]$, so $\bar{\text{IC}}_{11}(W)$ is bounded below by the capacity of the worst deterministic channels in $\mathcal{W}[a, b]$ (Theorem 19), which are obviously those with the number of nonzero columns minimized. The rank of those matrices is

$$\gamma' = \text{wt}(a) + \left(m - \sum_{j \in \text{supp}(a)} b_j \right)_+,$$

so $\bar{\text{IC}}_{11}(W) \geq \log \gamma'$.

On the other hand,

$$\begin{aligned} \bar{\text{IC}}_{11}(W) &= \sum_{D \in \mathcal{D}} \lambda_D \log \text{rank}(D) \\ &\leq (1 - \Gamma_\lambda(o)) \log(o - 1) + \Gamma_\lambda(o) \log o \\ &= \log(o - 1) + \Gamma_\lambda(o) \log \frac{o}{o - 1} \\ &\leq \log(o - 1) + \bar{\Gamma}_W(o) \log \frac{o}{o - 1}, \end{aligned}$$

where $o = m \wedge n$. The remaining part of the proof is straightforward. \square

Remark 23: The bounds given by Theorems 8 and 9 can be improved in various ways. In Theorem 8, if $\gamma = m \wedge n$, then the upper bound for $\bar{\Gamma}_W(m \wedge n)$ in Proposition 20 can be used to improve the upper bound for $\underline{\text{IC}}_{11}(W)$; if $\gamma = m = n$, the upper bound for $\underline{\text{IC}}_{11}(W)$ can be improved by Proposition 16 (see Example 43). The lower bound for $\underline{\text{IC}}_{11}(W)$ can also be improved to $(1 - \bar{\Gamma}_W(1)) \vee C(W)$ because $C(W) \leq \underline{\text{IC}}_{11}(W)$. However, all these improvements are somewhat ad hoc. The fundamental problem to be solved is how we can choose λ in order to approach or achieve the lower or the upper intrinsic capacities. Example 22 shows that naive greedy strategies do not work for this problem in general.

B. $\underline{\text{IC}}_{10}(W)$ and $\bar{\text{IC}}_{10}(W)$

Although it is difficult to compute $\underline{\text{IC}}_{10}(W)$ and $\bar{\text{IC}}_{10}(W)$ in general, their exact values can be determined in the binary-output case, as is shown by Theorem 10. In this subsection, we will prove Theorem 10. Moreover, we will study the so-called “universally useless (causal) state information” phenomenon and the identity $\underline{\text{IC}}_{10}(W) = C(W)$ in Theorem 10 turns out to be a special case of this phenomenon (Theorem 24, Definition 26, and Theorem 27). The proofs of Theorems 10 and 24 rely heavily on the sufficient and necessary condition of a capacity-achieving input probability distribution as well as the corresponding geometric interpretation. Readers not familiar with this result are referred to [6, Th. 4.5.1] and Appendix D.

Proof of Theorem 10: Since $n = 2$, we only need to choose two maps from all the $m^{|\mathcal{D}|} = m^{2^m}$ maps of \mathcal{D} into $\llbracket 1, m \rrbracket$ for constructing the capacity-achieving distributions. We denote these two maps by u and v . The optimal strategy for choosing u, v is to maximize $W'_{u,1}$ and minimize $W'_{v,1}$, where $W'_{u,y} = \sum_{D \in \mathcal{D}} \lambda_D D_{u(D),y}$. There are only two classes of deterministic channels in \mathcal{D} , rank 1 and rank 2. For D of rank 1, it does not matter how to choose the values of $u(D)$ and $v(D)$. For D of rank 2, however, we choose $u(D) = i_1$ such that $D_{i_1,1} = 1$ and choose $v(D) = i_2$ such that $D_{i_2,1} = 0$. Then we have

$$W'_{u,*} = (1 - \lambda_{U_2}, \lambda_{U_2})$$

and

$$W'_{v,*} = (\lambda_{U_1}, 1 - \lambda_{U_1}).$$

By Proposition 7, the maximum of $\Gamma_\lambda(1) = \lambda_{U_1} + \lambda_{U_2}$ is $\alpha_1 + \alpha_2$ with each α_j being the maximum of feasible values

of λ_{U_j} , so that

$$\underline{\text{IC}}_{10}(W) = C \left(\begin{pmatrix} 1 - \alpha_2 & \alpha_2 \\ \alpha_1 & 1 - \alpha_1 \end{pmatrix} \right).$$

Observing that these two rows are exactly those of W , we further have $\underline{\text{IC}}_{10}(W) = C(W)$. Again by Proposition 7, the minimum $\underline{\Gamma}_W(1)$ of $\Gamma_\lambda(1)$ is $(g - m + 1)_+$. With no loss of generality, we suppose $g = (1W)_1$. Then the minima of feasible values of λ_{U_1} and λ_{U_2} are $(g - m + 1)_+$ and 0, respectively, so that

$$\overline{\text{IC}}_{10}(W) = C \left(\begin{pmatrix} 1 & 0 \\ \underline{\Gamma}_W(1) & 1 - \underline{\Gamma}_W(1) \end{pmatrix} \right). \quad \square$$

The fact that $\underline{\text{IC}}_{10}(W) = C(W)$ for binary-output channels is quite intriguing (although it is not true in general when the output is non-binary (Example 46)). It implies that every binary-output channel can be simulated in a certain way that the capacity cannot be increased even when the encoder has causal access to the source of randomness, i.e., the intrinsic state. The following result shows that, in fact for a fairly broad class of channels, the causal state information at the encoder is useless as far as the capacity is concerned.

Theorem 24: Let $W = W'W''$, where W' is a channel with binary output and W'' is a channel with binary input and $W''_{1,*} \neq W''_{2,*}$. Suppose

$$W' = \sum_{s \in \mathcal{S}} p_S(s) K^{(s)},$$

where \mathcal{S} denotes the channel state and p_S is its distribution. The capacity of W cannot be increased by the causal state information S at the encoder iff all $K^{(s)}$ with $p_S(s) > 0$ are (i_1, i_2) -ended for some fixed i_1 and i_2 , where a binary output channel K is said to be (i_1, i_2) -ended if $K_{i_1,1} = \min_i K_{i,1}$ and $K_{i_2,1} = \max_i K_{i,1}$. In other words, all row vectors of K are contained in the line segment from endpoint $K_{i_1,*}$ to endpoint $K_{i_2,*}$.

Proof: (Sufficiency) By [2, Th. 7.2 and Remark 7.6], we consider the channel $V: \llbracket 1, m \rrbracket^{\mathcal{S}} \rightarrow \llbracket 1, n \rrbracket$ given by $V = V'W''$ and

$$V_{u,*} = \sum_{s \in \mathcal{S}} p_S(s) K_{u(s),*}^{(s)}.$$

Because every channel $K^{(s)}$ is (i_1, i_2) -ended, it is easy to show that V' is also (i_1, i_2) -ended, where i_1 and i_2 are regarded as two constant maps from \mathcal{S} to $\llbracket 1, m \rrbracket$. Then every row vector of V is contained in the line segment between $V'_{i_1,*}W''$ and $V'_{i_2,*}W''$, which implies that V has a capacity-achieving input probability distribution supported on $\{i_1, i_2\}$ (Proposition 44), and consequently the capacity of W cannot be increased by the causal state information at the encoder.

(Necessity) If the capacity of W cannot be increased by its causal state information at the encoder, then a capacity-achieving input probability distribution of V must have a support, say $\{i_1, i_2\}$, so that for every map $u: \mathcal{S} \rightarrow \llbracket 1, m \rrbracket$, the vector

$$V_{u,*} = V'_{u,*}W'' = \left(\sum_{s \in \mathcal{S}} p_S(s) K_{u(s),*}^{(s)} \right) W''$$

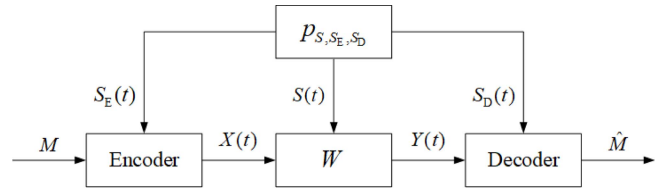


Fig. 3. A generalized channel model.

is contained in the line segment between $V_{i_1,*}$ and $V_{i_2,*}$ (Proposition 45), where i_1 and i_2 are understood as two constant maps from \mathcal{S} to $\llbracket 1, m \rrbracket$. With no loss of generality, we assume $V'_{i_1,1} \leq V'_{i_2,1}$. For any $t \in \mathcal{S}$ and any $i_0 \in \llbracket 1, m \rrbracket$, we can take $u(t) = i_0$ and $u(s) = i_1$ for $s \neq t$, then we get $V'_{u,1} \geq V'_{i_1,1}$, so that $K_{i_0,1}^{(t)} \geq K_{i_1,1}^{(t)}$. Similarly, we have $K_{i_0,1}^{(t)} \leq K_{i_2,1}^{(t)}$. Therefore, every $K^{(s)}$ is (i_1, i_2) -ended. \square

It can be shown via a perturbation and continuity argument that the uselessness of the causal state information at the encoder is not restricted to the channels covered by Theorem 24. However, we have not been able to identify a simple explicit condition under which the sufficiency part of Theorem 24 can be extended. For example, consider a seemingly natural condition postulated by the following conjecture.

Conjecture 25: Let W be a channel from $\llbracket 1, 2 \rrbracket$ to $\llbracket 1, n \rrbracket$. Suppose

$$W = \sum_{s \in \mathcal{S}} p_S(s) K^{(s)},$$

where \mathcal{S} denotes the state of channel. If for every $1 \leq j \leq n$, $K_{1,j}^{(s)}$ and $K_{2,j}^{(s)}$ have an order (either \leq or \geq) independent of s , then the capacity of W cannot be increased by the causal state information available at the encoder.

This conjecture is obviously true for $n = 2$. Numerical results indicate that it also holds in many cases when $n > 2$. However it turns out to be false in general as shown by Example 47.

Theorem 24 imposes no restriction on the distribution of the channel state. This universal property motivates us to introduce the following definition.

Definition 26: The state information S of a channel $W(y | x, s)$ is said to be *universally useless at the encoder* if for any p_S , the capacity of W with S causally available at the encoder is equal to the capacity of $W'(y | x) := \sum_s p_S(s) W(y | x, s)$.

This definition is not void in view of Theorem 24 (in fact, according to our numerical results, many channels not covered by Theorem 24 also satisfy this definition). Now consider the channel model shown in Fig. 3, where the channel state S is distributed according to p_S , and (noisy) state observations S_E and S_D generated by S through $p_{S_E, S_D | S}$ are causally available at the encoder and the decoder, respectively. Let $C(W, S_E, S_D, p_S)$ denote the capacity of this channel model.

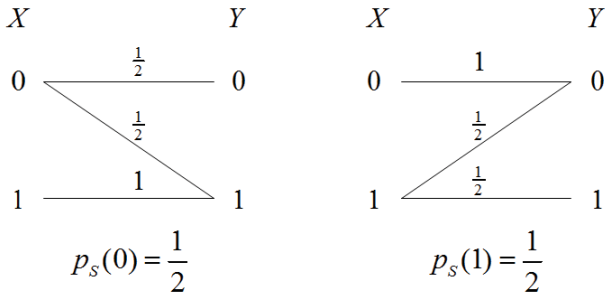


Fig. 4. Illustration of W and p_S given by (3) and (4), respectively.

It is instructive to study the following example (see also Fig. 4) where

$$W(y | x, s) = \begin{cases} \frac{1}{2}, & (x, y, s) = (0, 0, 0), (0, 1, 0), (1, 0, 1) \\ & \text{or } (1, 1, 1), \\ 0, & (x, y, s) = (1, 0, 0) \text{ or } (0, 1, 1), \\ 1, & (x, y, s) = (1, 1, 0) \text{ or } (0, 0, 1), \end{cases} \quad (3)$$

$$p_S(0) = p_S(1) = \frac{1}{2}. \quad (4)$$

The matrix form of the channel W with state S is

$$W^{(0)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad W^{(1)} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

It is clear that both $W^{(0)}$ and $W^{(1)}$ are (1, 0)-ended (supposing the row indices are input symbols 0 and 1), so the state information S of W is universally useless at the encoder (Theorem 24 and Definition 26). For this example, we assume that $p_{S_E|S}$ is a binary symmetric channel with crossover probability $p \in [0, \frac{1}{2}]$, and $p_{S_D|S}$ is a binary symmetric channel with crossover probability $q = 0.25$; furthermore, we assume that $p_{S_E|S}$ is physically degraded with respect to $p_{S_D|S}$ when $p \geq q = 0.25$, and the other way around when $p \leq q = 0.25$. To gain a better understanding, we plot $C(W, S_E, S_D, p_S)$ against p for $p \in [0, \frac{1}{2}]$ in Fig. 5. It turns out that, somewhat counterintuitively, $C(W, S_E, S_D, p_S)$ is maximized when the encoder side information coincides with the decoder side information (i.e., $p = 0.25$) rather than when the encoder has access to the perfect state information S (i.e., $p = 0$). As shown by the following theorem, this is in fact a general phenomenon for any channel whose state information is universally useless at the encoder.

Theorem 27: If the state information of W is universally useless at the encoder, then $C(W, S_E, S_D, p_S)$ is maximized when $S_E = S_D$ almost surely (assuming p_S, p_{S_D} is fixed but $p_{S_E|S, S_D}$ can be arbitrary).

Proof: It is clear that among all possible forms of encoder side information S_E , $C(W, S_E, S_D, p_S)$ is maximized when $S_E = (S, S_D)$ (since any other form of S_E can be viewed as its degenerate version), i.e.,

$$C(W, S_E, S_D, p_S) \leq C(W, (S, S_D), S_D, p_S).$$

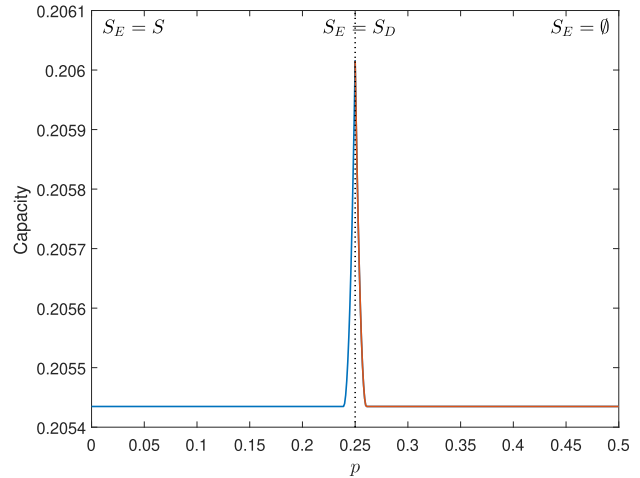


Fig. 5. Plot of $C(W, S_E, S_D, p_S)$ against p for $p \in [0, 0.5]$, where W and p_S are given by (3) and (4), respectively.

Note that

$$\begin{aligned} C(W, (S, S_D), S_D, p_S) &= \sum_{s_D} p_{S_D}(s_D) C(W, S, \emptyset, p_{S|S_D=s_D}) \\ &\stackrel{(a)}{=} \sum_{s_D} p_{S_D}(s_D) C(W, \emptyset, \emptyset, p_{S|S_D=s_D}) \\ &= C(W, S_D, S_D, p_S), \end{aligned}$$

where (a) follows from the universal-uselessness property of the state information of W , and the constant \emptyset means no information. This completes the proof. \square

Roughly speaking, Theorem 27 implies that, for the class of channels satisfying Definition 26, what the encoder really needs to know is not the state information, but the decoder's knowledge of the state information; in other words, for such channels, it is important to maintain consensus between the encoder and the decoder. It is also worth noting that Theorem 27 reduces to Definition 26 when there is no decoder side information.

Another surprising phenomenon revealed by Fig. 5 is that, as p moves away from 0.25, the capacity not only decreases but actually drops to the value corresponding to the no encoder side information case once p passes certain thresholds. Again, such a phenomenon is not confined to that specific example. An investigation of this phenomenon in the context where the encoder side information is a degenerate version of the decoder side information can be found in [9].

Similar to Theorem 10, we can also determine the exact values of $\underline{IC}_{01}(W)$ and $\overline{IC}_{01}(W)$ when the input is binary. In this case, we have $C_{01}(\lambda) = C_{11}(\lambda)$ for all $\lambda \in \text{dec}(W)$, so that $\underline{IC}_{01}(W) = \underline{IC}_{11}(W)$ and $\overline{IC}_{01}(W) = \overline{IC}_{11}(W)$ (see Proposition 11 and Appendix F). The general case of $\underline{IC}_{01}(W)$ and $\overline{IC}_{01}(W)$ is however quite difficult. Currently, we only know that $\overline{IC}_{01}(W) = \overline{IC}_{11}(W)$ does not hold in general (Proposition 48).

V. CONCLUSION

We have studied the lower and the upper intrinsic capacities of a channel W , denoted by $\underline{IC}_f(W)$ and $\overline{IC}_f(W)$, for three

different scenarios ($f = 10, 01, 11$) in terms of the availability of the causal state information at the encoder and/or the decoder. Their values are determined in almost all cases when the input or the output are binary, with only two exceptions (which are the binary-input nonbinary-output channels for $f = 10$ and the nonbinary-input binary-output channels for $f = 01$). A deeper understanding of the relevant optimization problems (especially the structure of $\text{dec}(W)$) is needed for further progress.

The lower and the upper intrinsic capacities are inherent properties of a channel with clear operational meanings. In particular, they characterize the potential capacity gains that can be achieved with a direct access to the generator of channel randomness by the encoder and/or the decoder. More generally, the notion of intrinsic capacity provides a useful perspective for studying the values of encoder and decoder side information. For example, our analysis of $\underline{\text{IC}}_{10}(W)$ reveals that for a broad class of channels, the capacity is not necessarily maximized when the encoder has access to the perfect state information. We believe that this surprising finding is just the tip of the iceberg, and this line of research can be fruitfully pursued to uncover many previously unknown phenomena.

APPENDIX A THE STRUCTURE OF $\text{dec}(W)$

In light of Theorem 2, we proceed to study the structure of $\text{dec}(W)$ with a focus on its vertices. Our approach is analogous to [7].

Proposition 28: Let

$$\mathfrak{S} = \left\{ \text{supp}(\alpha) : \alpha \in \mathbf{R}^{\mathcal{D}}, \sum_{D \in \mathcal{D}} \alpha_D D = 0 \right\}$$

or

$$\left\{ \text{supp}(\alpha) : \alpha \in \mathbf{R}^{\mathcal{D}}, \alpha I = 0 \right\},$$

where

$$\begin{aligned} I &:= (I_{D,(i,j)})_{D \in \mathcal{D}, (i,j) \in \llbracket 1,m \rrbracket \times \llbracket 1,n \rrbracket} \\ &= (D_{i,j})_{D \in \mathcal{D}, (i,j) \in \llbracket 1,m \rrbracket \times \llbracket 1,n \rrbracket} \end{aligned} \quad (5)$$

is called the incidence matrix. A probability distribution $\lambda \in \text{dec}(W)$ is a vertex iff for $S \in \mathfrak{S}$, $S \subseteq \text{supp}(\lambda)$ implies $S = \emptyset$, or in other words, iff $\text{rank}(I_{S,*}) = |S|$.

Proof: Note that for every $i \in \llbracket 1, m \rrbracket$,

$$\sum_{D \in \mathcal{D}} \alpha_D = \sum_{j=1}^n \sum_{D \in \mathcal{D}} \alpha_D D_{i,j}. \quad (6)$$

(Sufficiency) If $\lambda = t\beta + (1-t)\gamma$ for some $\beta, \gamma \in \text{dec}(W)$ and some $0 < t < 1$, then $\beta - \gamma = (\lambda - \gamma)/t$ and $\text{supp}(\gamma) \subseteq \text{supp}(\lambda)$, so that $\text{supp}(\beta - \gamma) \in \mathfrak{S}$ and $\text{supp}(\beta - \gamma) \subseteq \text{supp}(\lambda)$, hence $\text{supp}(\beta - \gamma) = \emptyset$, and therefore $\lambda = \beta = \gamma$ is a vertex.

(Necessity) For every nonempty $S \in \mathfrak{S}$, there is a vector $\alpha \in \mathbf{R}^{\mathcal{D}}$ such that $\text{supp}(\alpha) = S$ and $\sum_D \alpha_D D = 0$. Let $\beta = \lambda + t\alpha$ and $\gamma = \lambda - t\alpha$ with $t \neq 0$, so that $\lambda = (\beta + \gamma)/2$ with $\beta \neq \gamma$. Since λ is a vertex, β and γ must not be elements of $\text{dec}(W)$ for all $t \neq 0$, or equivalently, $S \not\subseteq \text{supp}(\lambda)$. \square

Below are several easy consequences of Proposition 28.

Proposition 29: Let

$$\mathfrak{T} = \{\text{supp}(\alpha) : \alpha \in \text{dec}(W)\}.$$

A probability distribution $\lambda \in \text{dec}(W)$ is a vertex iff $\text{supp}(\lambda)$ is minimal in \mathfrak{T} , where a minimal pattern in \mathfrak{T} is a set $T \subseteq \mathcal{D}$ such that $T = \text{supp}(\alpha)$ for some $\alpha \in \text{dec}(W)$ and for every $\beta \in \text{dec}(W)$, $\text{supp}(\beta) \subseteq T$ implies $\beta = \alpha$.

Proposition 30: If $\lambda \in \text{dec}(W)$ is a vertex, then

$$\text{wt}(\lambda) \leq \text{wt}(W) - m + 1.$$

Sketch of Proof: Because of (6), the equations $\alpha I = 0$ have at most $m(n-1) + 1$ linearly independent equations. This number can be further reduced to $\text{wt}(W) - m + 1$ by utilizing the information of W , because all the variables α_D with $D_{i,j} = 1$ must be zero if $\sum_{D \in \mathcal{D}} \lambda_D D_{i,j} = W_{i,j} = 0$. The remaining part of the proof is then straightforward. \square

Proposition 30 provides an upper bound for the support size of a vertex in $\text{dec}(W)$. On the other hand, the following result provides a lower bound for the support size of points in $\text{dec}(W)$, including all the vertices of $\text{dec}(W)$.

Proposition 31: For any $\lambda \in \text{dec}(W)$,

$$\text{wt}(\lambda) \geq \lceil \log_2 s \rceil \vee \text{wt}(W_{1,*}) \vee \cdots \vee \text{wt}(W_{m,*}),$$

where $s = |\{W_{i,j}\}_{i \in \llbracket 1,m \rrbracket, j \in \llbracket 1,n \rrbracket}|$.

Proof: By the definition of $\text{dec}(W)$, we have

$$W_{i,j} = \sum_{D \in \mathcal{D}} \lambda_D D_{i,j}.$$

Since $D_{i,j}$ is either 0 or 1, the right-hand side can yield at most $2^{\text{wt}(\lambda)}$ different values, so that

$$2^{\text{wt}(\lambda)} \geq s = |\{W_{i,j}\}_{i \in \llbracket 1,m \rrbracket, j \in \llbracket 1,n \rrbracket}|$$

or $\text{wt}(\lambda) \geq \lceil \log_2 s \rceil$.

On the other hand, every equation

$$\sum_{D \in \mathcal{D}} \lambda_D D_{i,j} = W_{i,j} > 0$$

must have at least one positive λ_D for some

$$D \in \mathfrak{D}_{i,j} = \{D \in \mathcal{D} : D_{i,j} = 1\}.$$

Since for every i , the sets $\mathfrak{D}_{i,1}, \mathfrak{D}_{i,2}, \dots, \mathfrak{D}_{i,n}$ are mutually disjoint, we conclude that $\text{wt}(\lambda) \geq \text{wt}(W_{i,*})$. \square

Algorithm 32: Let f be an arbitrary one-to-one map of $\llbracket 1, n^m \rrbracket$ onto \mathcal{D} . the following algorithm with W and f as arguments can yield a vertex of $\text{dec}(W)$.

```

function VERTEX( $W, f$ )
 $\lambda \leftarrow 0, K \leftarrow W, i \leftarrow 1$ 
while  $K \neq 0$  and  $1 \leq i \leq n^m$  do
     $D \leftarrow f(i)$ 
     $\lambda_D \leftarrow \min_{1 \leq r \leq m} K_{r,D(r)}$ 
     $K \leftarrow K - \lambda_D D$ 
     $i \leftarrow i + 1$ 
end while
return  $\lambda$ 
end function
    
```

Sketch of Proof: Let λ be the vertex output by the algorithm. Let $S = \text{supp}(\lambda)$. Then by checking Algorithm 32, it is easy

to verify that for every $D \in S$, there exists an $i \in \llbracket 1, m \rrbracket$ such that $I_{D,(i,D(i))} = 1$ and $I_{D',(i,D(i))} = 0$ for all $D' \in S$ with $f^{-1}(D') > f^{-1}(D)$, so that $\text{rank}(I_{S,*}) = |S|$. \square

Remark 33: We can replace the map f in Algorithm 32 with some one-to-one map $f': \llbracket 1, \ell \rrbracket \rightarrow \mathcal{D}$, where $1 \leq \ell < n^m$. Then we have a modified algorithm returning a pair (λ', K) such that

$$W = K + \sum_{D \in \mathcal{D}} \lambda'_D D.$$

Suppose the nontrivial case $K \neq 0$, so that $\alpha = \sum_{D \in \mathcal{D}} \lambda'_D < 1$. Let $W' = K/(1 - \alpha)$. If we have another algorithm to find a vertex of $\text{dec}(W')$, say λ'' , then it is easy to show that $\lambda = \lambda' + (1 - \alpha)\lambda''$ is a vertex of $\text{dec}(W)$.

APPENDIX B PROPERTIES OF J_f AND C_f

This section provides some basic results on the analytic properties of J_f and C_f defined in Section III. For any $p, p' \in \mathcal{P}_A$,

$$d(p, p') := \frac{1}{2} \|p - p'\|_1 = \frac{1}{2} \sum_{a \in A} |p_a - p'_a|$$

is called the *statistical distance* on \mathcal{P}_A . Given the product space $(A, d_A) \times (B, d_B)$, we define its product metric by

$$d_\vee((p, q), (p', q')) := d_A(p, p') \vee d_B(q, q'),$$

which induces the usual product topology. Thus for any channels $W, W' \in \mathcal{W}_{A,B}$, we have the *channel distance*

$$\begin{aligned} d(W, W') &:= d_\vee((W_{a,*})_{a \in A}, (W'_{a,*})_{a \in A}) \\ &= \max_{a \in A} d(W_{a,*}, W'_{a,*}). \end{aligned}$$

Proposition 34: (a) $J_{10}(\lambda, \mu)$ is uniformly continuous, and it is convex in λ for fixed μ and is concave in μ for fixed λ .

(b) $J_{01}(\lambda, \mu)$ is uniformly continuous, and it is linear in λ for fixed μ and is concave in μ for fixed λ .

Proof: (a) The function $J_{10}(\lambda, \mu)$ can be rewritten as $I(\mu, g(\lambda))$ where

$$g(\lambda) = ((\lambda V(u))_y)_{u \in \mathcal{X}^{\mathcal{D}}, y \in \mathcal{Y}}$$

with $V(u) = (D_{u(D),y})_{D \in \mathcal{D}, y \in \mathcal{Y}}$. By Proposition 37, for $\lambda, \lambda' \in \mathcal{P}_{\mathcal{D}}$,

$$d(g(\lambda), g(\lambda')) = \max_{u \in \mathcal{X}^{\mathcal{D}}} d(\lambda V(u), \lambda' V(u)) \leq d(\lambda, \lambda'),$$

so that g is uniformly continuous, and hence $J_{10}(\lambda, \mu)$ is uniformly continuous (Proposition 39). It is also clear that g is linear, so that $J_{10}(\lambda, \mu)$ is convex for fixed μ and is concave for fixed λ ([10, Th. 2.7.4]).

(b) The function $J_{01}(\lambda, \mu)$ can be written as $\lambda(g(\mu))^\top$ where $g(\mu) = (I(\mu, D))_{D \in \mathcal{D}}$. By Propositions 36 and 37, $I(\mu, D)$ is uniformly continuous on $\mathcal{P}_{\mathcal{X}}$ and is bounded by

$\log(|\mathcal{X}| \wedge |\mathcal{Y}|)$. Then for $\lambda, \lambda' \in \mathcal{P}_{\mathcal{D}}$ and $\mu, \mu' \in \mathcal{P}_{\mathcal{X}}$, we have

$$\begin{aligned} &|\lambda(g(\mu))^\top - \lambda'(g(\mu'))^\top| \\ &= |\lambda(g(\mu))^\top - \lambda'(g(\mu))^\top + \lambda'(g(\mu))^\top - \lambda'(g(\mu'))^\top| \\ &\leq |\lambda(g(\mu))^\top - \lambda'(g(\mu))^\top| + |\lambda'(g(\mu))^\top - \lambda'(g(\mu'))^\top| \\ &\leq |\lambda - \lambda'| (g(\mu))^\top + \lambda' |g(\mu) - g(\mu')|^\top \\ &\leq \log(|\mathcal{X}| \wedge |\mathcal{Y}|) \|\lambda - \lambda'\|_1 + \|g(\mu) - g(\mu')\|_1, \end{aligned}$$

which implies that J_{01} is uniformly continuous. The remaining part is straightforward ([10, Th. 2.7.4]). \square

Proposition 35: For $f \in \{10, 01, 11\}$, $C_f(\lambda)$ is uniformly continuous and convex (and in fact linear for $f = 11$).

Sketch of Proof: Use Theorem 34 and Proposition 40 for $f = 10$ or 01 . The case of $f = 11$ is trivial because $C_{11}(\lambda)$ is a linear function of λ . \square

Proposition 36 [11, Th. 2]: For $\mu, \mu' \in \mathcal{P}_A$ and $W, W' \in \mathcal{W}_{A,B}$,

$$|I(\mu, W) - I(\mu', W')| \leq 3\delta \log(|A||B| - 1) + 3h(\delta),$$

where $\delta = d(\text{diag}(\mu)W, \text{diag}(\mu')W')$.

Proposition 37 (cf. [12, Lemma 3]): For $\mu, \mu' \in \mathcal{P}_A$ and $W \in \mathcal{W}_{A,B}$,

$$d(\text{diag}(\mu)W, \text{diag}(\mu')W) = d(\mu, \mu')$$

and

$$d(\mu W, \mu' W) \leq d(\mu, \mu').$$

Proposition 38 (cf. [12, Lemma 3]): For $\mu, \mu' \in \mathcal{P}_A$ and $W, W' \in \mathcal{W}_{A,B}$,

$$\begin{aligned} d(\text{diag}(\mu)W, \text{diag}(\mu')W') &\leq d(\mu, \mu') + d(W, W') \\ &\leq 2d_\vee((\mu, W), (\mu', W')), \end{aligned}$$

so that $I(\mu, W)$ is uniformly continuous on $(\mathcal{P}_A \times \mathcal{W}_{A,B}, d_\vee)$.

Sketch of Proof: Use the triangle inequality and Propositions 36, 37 and 41. \square

Proposition 39: Let g be a map from $\mathcal{P}_{\mathcal{C}}$ to $\mathcal{W}_{A,B}$. If g is uniformly continuous, then $I(\mu, g(\lambda))$ is uniformly continuous on $(\mathcal{P}_A \times \mathcal{P}_{\mathcal{C}}, d_\vee)$, where $\mu \in \mathcal{P}_A$ and $\lambda \in \mathcal{P}_{\mathcal{C}}$.

Sketch of Proof: Use Propositions 36 and 38 and the observation that $I(\mu, g(\lambda))$ is a composition of uniformly continuous mappings. \square

Proposition 40: If $g: A \times B \rightarrow \mathbf{R}$ is uniformly continuous on $(A \times B, d_\vee)$, then $f(x) := \sup_{b \in B} g(x, b)$ is uniformly continuous.

Proof: Since g is uniformly continuous, for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $a, a' \in A$ and any $b \in B$, $d_\vee((a, b), (a', b)) < \delta$ implies $|g(a, b) - g(a', b)| < \epsilon$. In other words, for any $b \in B$, $d_A(a, a') < \delta$ implies $|g(a, b) - g(a', b)| < \epsilon$. Then

$$\sup_{b \in B} g(a, b) - \sup_{b \in B} g(a', b) \leq \sup_{b \in B} (g(a, b) - g(a', b)) < \epsilon$$

and similarly, $\sup_{b \in B} g(a', b) - \sup_{b \in B} g(a, b) < \epsilon$, so that $f(x)$ is uniformly continuous. \square

Proposition 41: For $\mu \in \mathcal{P}_A$ and $W, W' \in \mathcal{W}_{A,B}$,

$$d(\text{diag}(\mu)W, \text{diag}(\mu)W') \leq d(W, W').$$

Proof:

$$\begin{aligned} & d(\text{diag}(\mu)W, \text{diag}(\mu)W') \\ &= \frac{1}{2} \sum_{a,b} |\mu_a W_{a,b} - \mu_a W'_{a,b}| \\ &= \frac{1}{2} \sum_a \mu_a \sum_b |W_{a,b} - W'_{a,b}| \\ &= \sum_a \mu_a d(W_{a,*}, W'_{a,*}) \leq d(W, W'). \quad \square \end{aligned}$$

APPENDIX C

PROOFS AND EXAMPLES OF SECTION IV-A

Proof of Proposition 16: Let $W = (P_1 + \dots + P_\ell)/\ell$ and j be the column such that $\text{wt}(W_{*,j}) > 1$. It is clear that $W = xD + (1-x)W'$ for some $x \in (0, 1)$ and some $D \in \mathcal{D}$ such that $\text{wt}(D_{*,j}) > 1$, so that $\underline{\text{IC}}_{11}(W) < \log m$, and hence P_1, \dots, P_ℓ are not IC-minimized. \square

Proof of Proposition 17: Let $W = (P_1 + \dots + P_\ell)/\ell$ and j be the column of which all entries are less than 1. It is clear that $W = xD + (1-x)W'$ for some $x \in (0, 1)$ and some $D \in \mathcal{D}$ such that $\text{wt}(D_{*,j}) = 0$, so that $\underline{\text{IC}}_{11}(W) < \log n$, and hence P_1, \dots, P_ℓ are not IC-minimized. \square

Proof of Proposition 18: With no loss of generality, we assume that $D_{m,j} = 0$. It is then clear that

$$\frac{1}{2}D + \frac{1}{2}U_j = \frac{1}{2}D' + \frac{1}{2}D'',$$

where

$$D'(i) = \begin{cases} j, & \text{if } i = m, \\ D(i), & \text{otherwise,} \end{cases}$$

and

$$D''(i) = \begin{cases} D(m), & \text{if } i = m, \\ j, & \text{otherwise.} \end{cases}$$

It is clear that $\text{rank}(D') \geq (\text{rank}(D)-1)\vee 2$ and $\text{rank}(D'') = 2$, so that

$$\begin{aligned} \frac{1}{2} \log \text{rank}(D) + \frac{1}{2} \log \text{rank}(U_j) &< \frac{1}{2} \log \text{rank}(D') \\ &+ \frac{1}{2} \log \text{rank}(D''), \end{aligned}$$

and therefore $\{D, U_j\}$ is not IC-maximized. \square

Example 42: If

$$W = \begin{pmatrix} 1/n & 1/n & \dots & 1/n \\ 1/n & 1/n & \dots & 1/n \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/n & \dots & 1/n \end{pmatrix},$$

which is the probability transition matrix seen in the well-known random binning scheme, then $\underline{\text{IC}}_{11}(W) = 0$ and $\overline{\text{IC}}_{11}(W) = \log(m \wedge n)$ (Theorems 8 and 9).

Example 43:

$$W = \begin{pmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.1 & 0.5 \end{pmatrix}$$

It can be computed using linear programming that $\underline{\text{IC}}_{11}(W) = 0.4$ and $\overline{\text{IC}}_{11}(W) = 0.2 + 0.8 \log 3 \approx 1.4680$. The decompositions of W for $\underline{\text{IC}}_{11}(W)$ and $\overline{\text{IC}}_{11}(W)$ are

$$\begin{aligned} W &= 0.2 \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 0.1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &+ 0.3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + 0.1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &+ 0.1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 0.1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &+ 0.1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} W &= 0.1 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 0.1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &+ 0.1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 0.3 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ &+ 0.1 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 0.3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

respectively. Using Theorems 8 and 9 and Proposition 20, we have

$$0.4 \leq \underline{\text{IC}}_{11}(W) \leq 0.4 \log 3 \approx 0.6340$$

and

$$1 \leq \overline{\text{IC}}_{11}(W) \leq 1 + \overline{\Gamma}_W(3) \log \frac{3}{2} \leq 1 + 0.9 \log \frac{3}{2} \approx 1.5265.$$

From Proposition 16, it follows that the optimal decomposition λ' for $\underline{\text{IC}}_{11}(W')$ can have at most one perfect channel, so that $\Gamma_{\lambda'}(3) \leq 0.25$, where

$$W' = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix}$$

is computed by the formula in Theorem 8. Then we have an improved bound: $\underline{\text{IC}}_{11}(W) \leq 0.4 \underline{\text{IC}}_{11}(W') = 0.3 + 0.1 \log 3 \approx 0.4585$.

APPENDIX D
CAPACITY-ACHIEVING INPUT
PROBABILITY DISTRIBUTIONS

Let W be a channel in $\mathcal{W}_{\mathcal{X}, \mathcal{Y}}$. According to [6, Th. 4.5.1], an input probability distribution μ maximizes the mutual information $I(\mu, W)$ iff

$$D(W_{x,*} \| \tau) = C \quad \text{for } x \in \text{supp}(\mu)$$

and

$$D(W_{x,*} \| \tau) \leq C \quad \text{for } x \notin \text{supp}(\mu),$$

where $\tau = \mu W$. Based on this sufficient and necessary condition, we have the following results concerning the support of capacity-achieving input probability distributions. In the sequel, we denote by $\text{conv}(V)$ the convex hull of all vectors in V .

Proposition 44: Let $A \subseteq \mathcal{X}$. If all row vectors of W are contained in $\text{conv}(\{W_{x,*}\}_{x \in A})$, then there exists a capacity-achieving probability distribution μ such that $\text{supp}(\mu) \subseteq A$.

Proof: Let ν be a capacity-achieving probability distribution of the sub-matrix $W_{A,*}$. Extending ν with zero values, we obtain a probability distribution μ over \mathcal{X} . It is clear that

$$D(W_{x,*} \parallel \tau) = C \quad \text{for } x \in \text{supp}(\mu)$$

and

$$D(W_{x,*} \parallel \tau) \leq C \quad \text{for } x \in A \setminus \text{supp}(\mu),$$

where $\tau = \mu W = \nu W_{A,*}$. It remains to show that

$$D(W_{x,*} \parallel \tau) \leq C \quad \text{for } x \notin A,$$

which is obvious, because

$$\begin{aligned} D(W_{x,*} \parallel \tau) &= D\left(\sum_{a \in A} \alpha_a W_{a,*} \parallel \tau\right) \\ &\leq \sum_{a \in A} \alpha_a D(W_{a,*} \parallel \tau) \leq C \end{aligned}$$

for some nonnegative coefficients $(\alpha_a)_{a \in A}$ with $\sum_{a \in A} \alpha_a = 1$. \square

Proposition 45: Let μ be a capacity-achieving probability distribution of W and let $A = \text{supp}(\mu)$. For any $a \in A$ and any $b \notin A$, $W_{a,*} \notin \text{conv}(\{W_{x,*}\}_{x \in A \cup \{b\}} \setminus \{W_{a,*}\})$.

Proof: It is clear that $D(W_{x,*} \parallel \tau) = C$ for all $x \in A$, where $\tau = \mu W$. We first show that $W_{a,*} \notin \text{conv}(\{W_{x,*}\}_{x \in A} \setminus \{W_{a,*}\})$, which corresponds to the case $W_{b,*} = W_{a,*}$. If it is false, then

$$W_{a,*} = \sum_{x \in A'} \alpha_x W_{x,*},$$

where $A' = \{x \in A : W_{x,*} \neq W_{a,*}\}$, $\alpha_x \geq 0$, and $\sum_{x \in A'} \alpha_x = 1$. It is clear that $\alpha_x < 1$ for all $x \in A'$, so that

$$D(W_{a,*} \parallel \tau) < \sum_{x \in A'} \alpha_x D(W_{x,*} \parallel \tau) = C,$$

a contradiction. Now suppose that

$$W_{a,*} \in \text{conv}(\{W_{x,*}\}_{x \in A \cup \{b\}} \setminus \{W_{a,*}\})$$

for some $b \notin A$ with $W_{b,*} \neq W_{a,*}$. Let $A'' = A' \cup \{b\}$. Then

$$W_{a,*} = \sum_{x \in A''} \alpha_x W_{x,*},$$

where $\alpha_x \geq 0$ and $\sum_{x \in A''} \alpha_x = 1$. It is clear that $0 < \alpha_b < 1$, and therefore

$$\begin{aligned} C &= D(W_{a,*} \parallel \tau) < \alpha_b D(W_{b,*} \parallel \tau) + \sum_{x \in A'' \setminus \{b\}} \alpha_x D(W_{x,*} \parallel \tau) \\ &= \alpha_b D(W_{b,*} \parallel \tau) + (1 - \alpha_b)C, \end{aligned}$$

so that $D(W_{b,*} \parallel \tau) > C$, which is absurd. \square

APPENDIX E COUNTEREXAMPLES FOR SECTION IV-B

Example 46: $\underline{\text{IC}}_{10}(W) > C(W)$ for

$$W = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.6 & 0.35 & 0.05 \end{pmatrix}.$$

Proof: Let $S = \{D \in \mathcal{D} : D(1) \in \{1, 2\}, D(2) = 3\}$. It is then clear that, for every $\lambda \in \text{dec}(W)$,

$$\sum_{D \in S} \lambda_D = 0.05.$$

If we define the map $u : \mathcal{D} \rightarrow \llbracket 1, 2 \rrbracket$ by

$$u(D) = \begin{cases} 1, & \text{if } D \in S, \\ 2, & \text{otherwise,} \end{cases}$$

then the row vector $v = (\sum_{D \in S} \lambda_D D_{u(D), y})_{y \in \llbracket 1, 3 \rrbracket}$ is always on the line segment L with endpoints $(0.65, 0.35, 0)$ and $(0.6, 0.4, 0)$.

By numerical computation, we know that

$$D(W_{1,*} \parallel c) = D(W_{2,*} \parallel c) \approx 0.03541501,$$

where

$$c := \mu W \approx (0.71339243, 0.26495568, 0.02165189)$$

and

$$\mu \approx (0.56696216, 0.43303784)$$

is the capacity-achieving input distribution of W . Furthermore, it can be verified that all points x of L satisfy

$$D(x \parallel c) > 0.0369.$$

This implies that μ , if extended to $\llbracket 1, 2 \rrbracket^{\mathcal{D}}$, cannot be a capacity-achieving distribution ([6, Theorem 4.5.1]). In other words, for every $\lambda \in \text{dec}(W)$, the intrinsic capacity $C_{10}(\lambda) > C(W)$, so that $\underline{\text{IC}}_{10}(W) > C(W)$. \square

Example 47: Let state alphabet $\mathcal{S} = \llbracket 1, 2 \rrbracket$ and let

$$\begin{aligned} W &= \sum_{s \in \mathcal{S}} p_S(s) K^{(s)} = \frac{17}{18} K^{(1)} + \frac{1}{18} K^{(2)} \\ &= \begin{pmatrix} \delta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0.05 & 0.1 & 0.85 \\ 0 & 0.05 & 0.95 \end{pmatrix}, \end{aligned}$$

where

$$K^{(1)} = \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0.1 & 0.9 \\ 0 & 0.05 & 0.95 \end{pmatrix}$$

and

$$K^{(2)} = \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0.9 & 0.1 & 0 \\ 0 & 0.05 & 0.95 \end{pmatrix}.$$

It is easy to show that $\mu = (0.603123, 0.396877)$ is the capacity-achieving input distribution for W , so that the output distribution is

$$\tau := \mu_1 \delta + \mu_2 \gamma \approx (0.01984385, 0.06984385, 0.9103123)$$

and $D(\delta \parallel \tau) = D(\gamma \parallel \tau) \approx 0.0238286$. However, for the channel $V : \llbracket 1, 2 \rrbracket^{\mathcal{S}} \rightarrow \llbracket 1, 3 \rrbracket$ given by

$$V_{u,*} = \sum_{s \in \mathcal{S}} p_S(s) K_{u(s),*}^{(s)} \quad ([2, \text{Remark 7.6}],$$

if we choose the map $u(s) = s$, then the corresponding row vector

$$\begin{aligned} \zeta &:= V_{u,*} = p_1 K_{1,*}^{(1)} + p_2 K_{2,*}^{(2)} \\ &= \frac{17}{18}\alpha + \frac{1}{18}\gamma \approx (0, 0.09722222, 0.90277778) \end{aligned}$$

and $D(\zeta \parallel \tau) \approx 0.0246518 > D(\gamma \parallel \tau)$. This implies that μ , if extended to $\llbracket 1, 2 \rrbracket^S$, cannot be a capacity-achieving distribution for V ([6, Th. 4.5.1]). In other words, the capacity of W can be increased by the causal state information S at the encoder.

APPENDIX F $\overline{\mathbf{IC}}_{01}(W)$ AND $\overline{\mathbf{IC}}_{11}(W)$

Proof of Proposition 11: Because $m = 2$, the binary uniform distribution is capacity-achieving for every deterministic channel, rank 1 or rank 2. Thus we have $C_{01}(\lambda) = C_{11}(\lambda)$ for every $\lambda \in \text{dec}(W)$. The remaining part is an easy consequence of Theorems 8 and 9. \square

Proposition 48: Let W be a channel $\llbracket 1, 3 \rrbracket \rightarrow \llbracket 1, 2 \rrbracket$. If all probabilities $W_{i,j}$ are distinct and the sum of each column of W is greater than or equal to 1, then $\overline{\mathbf{IC}}_{01}(W) < \overline{\mathbf{IC}}_{11}(W)$.

Proof: By Proposition 20, $\overline{\Gamma}_W(2) = 1$, so that W can be expressed as a convex combination of perfect channels and hence $\overline{\mathbf{IC}}_{11}(W) = 1$.

Let

$$S = \{\lambda \in \text{dec}(W) : \Gamma_\lambda(2) = 1\}.$$

If $\overline{\mathbf{IC}}_{01}(W) = 1$, then there exists a $\lambda \in S$ such that the capacity-achieving input distribution, denoted μ , is capacity-achieving for every perfect channel $D \in \text{supp}(\lambda)$. Thus at least one entry of μ must be $1/2$. With no loss of generality, we assume $\mu_1 = 1/2$.

If μ_2 and μ_3 are both positive, then μ is capacity-achieving only for perfect channels

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

By Proposition 31, every $\lambda \in \text{dec}(W)$ satisfies $\text{supp}(\lambda) \geq \lceil \log_2 6 \rceil = 3$, which implies that μ is not capacity-achieving for $\lambda \in S$.

If $\mu_2 = 0$, then μ is capacity-achieving for perfect channels

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

However, any convex combination of these four matrices can only yield a channel matrix with at most four distinct probability values, and hence μ is not capacity-achieving for $\lambda \in S$.

In all cases, we have shown that μ is not capacity-achieving, which contradicts the assumption $\overline{\mathbf{IC}}_{01}(W) = 1$. Therefore, we have $\overline{\mathbf{IC}}_{01}(W) < 1 = \overline{\mathbf{IC}}_{11}(W)$. \square

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