Lecture 7: Dualities in Multiuser Information Theory

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1 A Fundamental Function:

Definition 1.1. *For any set of random variables* $\{Z_1, Z_2, \cdots, Z_M\}$, *let*

$$\psi(Z_1, \cdots, Z_M) = \sum_{i=1}^{M-1} I(Z_1, \cdots, Z_i; Z_{i+1})$$

if $M \geq 2$ *and let* $\psi(Z_1, \cdots, Z_M) = 0$ *if* $M = 1$.

The following consistency lemma shows that $\psi(Z_1, \cdots, Z_M)$ is well-defined.

Lemma 1.1. *Let* π *be a permutation on* $\mathcal{I}_M$. *The sum*

$$\sum_{i=1}^{M-1} I(Z_{\pi(1)}, \cdots, Z_{\pi(i)}; Z_{\pi(i+1)})$$

is invariant under π.

Proof. Since $I(Z_{\pi(1)}, \cdots, Z_{\pi(i)}; Z_{\pi(i+1)}) = H(Z_{\pi(1)}, \cdots, Z_{\pi(i)}) + H(Z_{\pi(i+1)}) - H(Z_{\pi(1)}, \cdots, Z_{\pi(i+1)})$, it follows that

$$\sum_{i=1}^{M-1} I(Z_{\pi(1)}, \cdots, Z_{\pi(i)}; Z_{\pi(i+1)}) = \sum_{i=1}^{M} H(Z_i) - H(Z_i, i \in \mathcal{I}_M),$$

which clearly does not dependent on π. $\square$

Remark: We choose $\sum_{i=1}^{M-1} I(Z_1, \cdots, Z_i; Z_{i+1})$ instead of $\sum_{i=1}^{M} H(Z_i) - H(Z_i, i \in \mathcal{I}_M)$ as the definition of $\psi(Z_1, \cdots, Z_M)$ because its connection with successive coding schemes is more transparent.

Theorem 1.1. *Function* $\psi(\cdot)$ *has the following properties:*

1. $\psi(Z_i, i \in \mathcal{S}) \leq \psi(Z_i, i \in \mathcal{T})$ *if* $\mathcal{S} \subseteq \mathcal{T}$.
2. $\psi(Z_i, i \in \mathcal{S}) + \psi(Z_i, i \in \mathcal{T}) \leq \psi(Z_i, i \in \mathcal{S} \cap \mathcal{T}) + \psi(Z_i, i \in \mathcal{S} \cup \mathcal{T})$.

Proof. Property 1 is obvious. So we only need to verify Property 2. For any $\mathcal{S}, \mathcal{T} \subseteq \mathcal{I}_M$, suppose $\mathcal{S} = \{i_1, \cdots, i_j, i_{j+1}, \cdots, i_k\}$, $\mathcal{T} = \{i_{j+1}, \cdots, i_k, i_{k+1}, \cdots, i_l\}$, and $\mathcal{S} \cap \mathcal{T} = \{i_{j+1}, \cdots, i_k\}$. We have

$$\begin{aligned}
& \psi(Z_i, i \in \mathcal{S}) + \psi(Z_i, i \in \mathcal{T}) \\
= & \sum_{m=1}^{k-1} I(Z_{i_1}, \cdots, Z_{i_m}; Z_{i_{m+1}}) + \sum_{m=j+1}^{l-1} I(Z_{i_{j+1}}, \cdots, Z_{i_m}; Z_{i_{m+1}}) \\
= & \sum_{m=1}^{k-1} I(Z_{i_1}, \cdots, Z_{i_m}; Z_{i_{m+1}}) + \psi(X, Z_i, i \in \mathcal{S} \cap \mathcal{T}) + \sum_{m=k}^{l-1} I(Z_{i_{j+1}}, \cdots, Z_{i_m}; Z_{i_{m+1}}) \\
\leq & \sum_{m=1}^{k-1} I(Z_{i_1}, \cdots, Z_{i_m}; Z_{i_{m+1}}) + \psi(Z_i, i \in \mathcal{S} \cap \mathcal{T}) + \sum_{m=k}^{l-1} I(Z_{i_1}, \cdots, Z_{i_m}; Z_{i_{m+1}}) \\
= & \psi(Z_i, i \in \mathcal{S} \cap \mathcal{T}) + \psi(Z_i, i \in \mathcal{S} \cup \mathcal{T}),
\end{aligned}$$

which completes the proof. $\square$

2 Polymatroid and Contra-polymatroid

2.1 Multiaccess Rate Region

Let

$$\mathcal{R}_{MC}(X_i, i \in \mathcal{I}_L) = \left\{ (R_1, \cdots, R_L) : \sum_{i \in \mathcal{A}} R_i \leq I(X_{\mathcal{A}}; Y | X_{\mathcal{A}^c}), \ \forall \mathcal{A} \subseteq \mathcal{I}_L \right\}.$$

Since

$$\begin{aligned}
I(X_{\mathcal{A}}; Y | X_{\mathcal{A}^c}) &= I(X_{\mathcal{I}_L}; Y) - I(X_{\mathcal{A}^c}; Y) \\
&= I(X_{\mathcal{I}_L}; Y) - \psi(Y, X_i, i \in \mathcal{A}^c),
\end{aligned}$$

it follows that $\mathcal{R}_{MC}(X_i, i \in \mathcal{I}_L)$ is a polymatroid.

Dominant Face:

$$\mathcal{D}_{MC}(X_i, i \in \mathcal{I}_L) = \left\{ (R_1, \cdots, R_L) \in \mathcal{R}_{MC}(X_i, i \in \mathcal{I}_L) : \sum_{i=1}^{L} R_i = I(X_{\mathcal{I}_L}; Y) \right\}.$$

Vertices: If π is a permutation on $\mathcal{I}_L$, define the vector $(R_1(\pi), \cdots, R_L(\pi))$ by

$$\begin{aligned}
R_{\pi(i)}(\pi) &= I(X_{\pi(i)}; Y | X_{\pi(i+1)}, \cdots, X_{\pi(L)}) \\
&= I(X_{\pi(i)}; Y, X_{\pi(i+1)}, \cdots, X_{\pi(L)}), \ i = 1, \cdots, L-1, \\
R_{\pi(L)}(\pi) &= I(X_{\pi(L)}; Y).
\end{aligned}$$

Then $(R_1(\pi), \cdots, R_L(\pi))$ is a vertex of $\mathcal{R}_{MC}(X_i, i \in \mathcal{I}_L)$ for every permutation π.

2.2 Berger-Tung Region

Let

$$\mathcal{R}_{BT}(W_i, i \in \mathcal{I}_L) = \left\{ (R_1, \cdots, R_L) : \sum_{i \in \mathcal{A}} R_i \geq I(Y_{\mathcal{A}}; W_{\mathcal{A}} | W_{\mathcal{A}^c}), \ \forall \mathcal{A} \subseteq \mathcal{I}_L \right\}.$$

Since

$$I(Y_{\mathcal{A}};W_{\mathcal{A}}|W_{\mathcal{A}^c}) = \sum_{i\in\mathcal{A}} I(Y_i;W_i) - \psi(W_i, i\in\mathcal{I}_L) + \psi(W_i, i\in\mathcal{A}^c),$$

it follows that $\mathcal{R}_{BT}(W_i, i\in\mathcal{I}_L)$ is a contra-polymatroid.

Dominant Face:

$$\mathcal{D}_{BT}(W_i, i\in\mathcal{I}_L) = \left\{(R_1,\cdots,R_L)\in\mathcal{R}_{BT}(W_i, i\in\mathcal{I}_L) : \sum_{i=1}^{L} R_i = I(Y_{\mathcal{I}_L};W_{\mathcal{I}_L})\right\}.$$

Vertices: If π is a permutation on $\mathcal{I}_L$, define the vector $(R_1(\pi),\cdots,R_L(\pi))$ by

$$\begin{aligned} R_{\pi(i)}(\pi) &= I(Y_{\pi(i)};W_{\pi(i)}|W_{\pi(i+1)},\cdots,W_{\pi(L)}) \\ &= I(Y_{\pi(i)};W_{\pi(i)}) - I(W_{\pi(i)};W_{\pi(i+1)},\cdots,W_{\pi(L)}),\ i=1,\cdots,L-1, \\ R_{\pi(L)}(\pi) &= I(Y_{\pi(L)};W_{\pi(L)}). \end{aligned}$$

Then $(R_1(\pi),\cdots,R_L(\pi))$ is a vertex of $\mathcal{R}_{BT}(W_i, i\in\mathcal{I}_L)$ for every permutation π.

2.3 EGC Region

Let

$$\mathcal{R}_{MD}(U_i, i\in\mathcal{I}_L) = \left\{(R_1,\cdots,R_L) : \sum_{i\in\mathcal{A}} R_i \geq \psi(X, U_i, i\in\mathcal{A}),\ \forall\mathcal{A}\subseteq\mathcal{I}_L\right\}.$$

Clearly, $\mathcal{R}_{MD}(U_i, i\in\mathcal{I}_L)$ is a contra-polymatroid.

Dominant Face:

$$\mathcal{D}_{MD}(U_i, i\in\mathcal{I}_L) = \left\{(R_1,\cdots,R_L)\in\mathcal{R}_{MD}(U_i, i\in\mathcal{I}_L) : \sum_{i=1}^{L} R_i = \psi(X, U_i, i\in\mathcal{I}_L)\right\}.$$

Vertices: If π is a permutation on $\mathcal{I}_L$, define the vector $(R_1(\pi),\cdots,R_L(\pi))$ by

$$\begin{aligned} R_{\pi(1)}(\pi) &= I(X;U_{\pi(1)}), \\ R_{\pi(i)}(\pi) &= I(X, U_{\pi(1)},\cdots,U_{\pi(i-1)};U_{\pi(i)}),\ i=2,\cdots,L. \end{aligned}$$

Then $(R_1(\pi),\cdots,R_L(\pi))$ is a vertex of $\mathcal{R}_{MD}(U_i, i\in\mathcal{I}_L)$ for every permutation π.

2.4 Marton's Region

Let

$$\mathcal{R}_{BC}(V_i, i\in\mathcal{I}_L) = \left\{(R_1,\cdots,R_L)\in\mathcal{R}_+^L : \sum_{i\in\mathcal{A}} R_i \leq \sum_{i\in\mathcal{A}} I(V_i;Y_i) - \psi(V_i, i\in\mathcal{A}),\ \forall\mathcal{A}\subseteq\mathcal{I}_L\right\}.$$

$\mathcal{R}_{BC}(V_i, i\in\mathcal{I}_L)$ is a polymatroid if $I(V_i;Y_i) - I(V_i;V_{\{i\}^c}) \geq 0$ for all i.

Dominant Face:

$$\mathcal{D}_{BC}(V_i, i\in\mathcal{I}_L) = \left\{(R_1,\cdots,R_L)\in\mathcal{R}_{MD}(U_i, i\in\mathcal{I}_L) : \sum_{i=1}^{L} R_i = \sum_{i\in\mathcal{I}_L} I(V_i;Y_i) - \psi(V_i, i\in\mathcal{I}_L)\right\}.$$

Vertices: If π is a permutation on $\mathcal{I}_L$, define the vector $(R_1(\pi), \cdots, R_L(\pi))$ by

$$\begin{aligned} R_{\pi(1)}(\pi) &= I(V_{\pi(1)}; Y_{\pi(1)}), \\ R_{\pi(i)}(\pi) &= I(V_{\pi(i)}; Y_{\pi(i)}) - I(V_{\pi(1)}, \cdots, V_{\pi(i-1)}; V_{\pi(i)}),\ i = 2, \cdots, L. \end{aligned}$$

Then $(R_1(\pi), \cdots, R_L(\pi))$ is a vertex of $\mathcal{R}_{BC}(U_i, i \in \mathcal{I}_L)$ for every permutation π.

3 Sufficient Statistics and Gram-Schmidt Orthogonalization

3.1 Gaussian Multiaccess Channel

Channel Model: $Y = X_1 + \cdots + X_L$.

$$\begin{aligned} R_{\pi(L)}(\pi) &= I(X_{\pi(L)}; Y) = I(X_{\pi(L)}; \mathbb{E}(X_{\pi(L)}|Y)) \\ R_{\pi(i)}(\pi) &= I(X_{\pi(i)}; Y, X_{\pi(i+1)}, \cdots, X_{\pi(L)}) \\ &= I(X_{\pi(i)}; \mathbb{E}(X_{\pi(i)}|Y, X_{\pi(i+1)}, \cdots, X_{\pi(L)})) \\ &= I(X_{\pi(i)}; \mathbb{E}(X_{\pi(i)}|Y - X_{\pi(i+1)} - \cdots - X_{\pi(L)})),\ i = L-1, \cdots, 1. \end{aligned}$$

3.2 Gaussian CEO Problem

$$\begin{aligned} R_{\pi(L)}(\pi) &= I(Y_{\pi(L)}; W_{\pi(L)}), \\ R_{\pi(i)}(\pi) &= I(Y_{\pi(i)}; W_{\pi(i)}|W_{\pi(i+1)}, \cdots, W_{\pi(L)}) \\ &= I(Y_{\pi(i)}; W_{\pi(i)}) - I(W_{\pi(i)}; W_{\pi(i+1)}, \cdots, W_{\pi(L)}) \\ &= I(Y_{\pi(i)}; W_{\pi(i)}) - I(W_{\pi(i)}; \mathbb{E}(W_{\pi(i)}|W_{\pi(i+1)}, \cdots, W_{\pi(L)})),\ i = L-1, \cdots, 1. \end{aligned}$$

3.3 Gaussian Multiple Description Problem

$$\begin{aligned} R_{\pi(1)}(\pi) &= I(X; U_{\pi(1)}), \\ R_{\pi(i)}(\pi) &= I(X, U_{\pi(1)}, \cdots, U_{\pi(i-1)}; U_{\pi(i)}) \\ &= I(\mathbb{E}(U_{\pi(i)}|X, U_{\pi(1)}, \cdots, U_{\pi(i-1)}); U_{\pi(i)}),\ i = 2, \cdots, L. \end{aligned}$$

3.4 Gaussian Broadcast Channel

$$\begin{aligned} R_{\pi(1)}(\pi) &= I(V_{\pi(1)}; Y_{\pi(1)}), \\ R_{\pi(i)}(\pi) &= I(V_{\pi(i)}; Y_{\pi(i)}) - I(V_{\pi(1)}, \cdots, V_{\pi(i-1)}; V_{\pi(i)}) \\ &= I(V_{\pi(i)}; Y_{\pi(i)}) - I(\mathbb{E}(V_{\pi(i)}|V_{\pi(1)}, \cdots, V_{\pi(i-1)}); V_{\pi(i)}),\ i = 2, \cdots, L. \end{aligned}$$

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