Discreteness of Sum-Capacity-Achieving Distributions for Discrete-Time Poisson Multiple Access Channels with Peak Constraints

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***Abstract*—This letter introduces the discrete-time Poisson (DTP) multiple access channel (MAC) which is applicable to a number of optical wireless links, most notably intersatellite laser communications. It is shown that the sum-capacity-achieving distributions of the DTP MAC under peak amplitude constraints are discrete with a finite number of mass points.**

***Index Terms*—Discrete-time Poisson channel, multiple access channel, sum-capacity-achieving.**

I. Introduction

FOR a single-user discrete-time Poisson (DTP) channel, the intensity of the input signal is fixed in each time slot but may vary across different time intervals while the receiver is modelled as a photon counter [1], [2], [3]. This model accurately represents shot noise in the detection process as well as the limited signalling bandwidth present in all practical links. Such channels exist in low intensity, direct detection channels such as intersatellite laser links which usually operate over thousands kilometers with limited transmitter power [4]. In [5] Shamai showed that the capacity-achieving distributions for the single-user DTP channel under peak and average amplitude constraints consist of a finite number of mass points.

In this paper, we propose a two-user DTP multiple access channel (MAC) model and prove the discreteness of sum-capacity-achieving distributions. Our proof relies heavily on the methods developed in [5], which is in turn based on Smith's seminal work [6], [7]. One example of where such a DTP MAC channel may arise is in scenarios where a single geostationary satellite (GEO) aggregates earth observation data from two low-Earth orbiting (LEO) satellites for downlink [8], [9], [10]. In addition, our DTP MAC model should be contrasted with its continuous-time counterpart studied in [11], [12] where the signalling bandwidth is not constrained. Although not directly applicable, it is also instructive to compare our work with related work on the Gaussian MAC [13].

The rest of this letter is organized as follows. The two-user DTP MAC model is introduced in Section II. Section III presents the essence of the main result of this work with the technical details relegated to appendices. Section IV contains some concluding remarks.

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II. Channel Model

For a two-user DTP MAC, users output independent optical intensities, X and W [photons/second] in time slots of length ΔT. At the receiver, the photon arrival rate is $X + W + \lambda$ [photons/second] where λ models the combined impact of background radiation and dark current. Without loss of generality, set $\Delta T = 1$ for the balance of this work. Thus, in each time slot, given the channel inputs $X = x$ and $W = w$, the overall channel output Y is Poisson distributed with mean $x + w + \lambda$, i.e.,

$$P_{Y|X,W}(y|x,w) = \frac{(x+w+\lambda)^y}{y!}e^{-(x+w+\lambda)},$$

where $x, w \in \mathbb{R}^+,\ y \in \mathbb{Z}^+$ and $\lambda > 0$.

A peak amplitude constraint $A_i,\ i = 1, 2$, is imposed on the input signal from each user due to power and device limitations. The sum-capacity C is defined as the maximum mutual information over all independent input distribution pairs satisfying peak amplitude constraints

$$C \triangleq \max_{X\perp W, F_X\in\mathcal{F}_1, F_W\in\mathcal{F}_2} I(X,W;Y) \tag{1}$$

where F_X and F_W are the distributions of X and W, respectively, and $\mathcal{F}_i$ denotes the set of distributions over $[0, A_i]$, $i = 1, 2$.

III. Main Result

Our main result is summarized in the following theorem.

Theorem 1. *For a two-user DTP MAC under peak amplitude constraints, the sum-capacity-achieving input distributions are discrete with a finite number of mass points.*

Proof: This theorem is a direct consequence of the following result. ■

Theorem 2. *Let X and W be two independent random variables. For any fixed $F_W \in \mathcal{F}_2$, the optimal solution to the following maximization*

$$\max_{F_X\in\mathcal{F}_1} I(X,W;Y) \tag{2}$$

is unique and is discrete with a finite number of mass points.

A. Proof of Theorem 2

As in [7], define

$$i(x,F_X) = -\sum_{y=0}^{\infty}\int_0^{A_2}\frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!}\gamma_y dF_W + \int_0^{A_2}(x+w+\lambda)\log\frac{x+w+\lambda}{e}dF_W,$$

where

$$\gamma_y \triangleq \log \int_0^{A_1} \int_0^{A_2} e^{-(x+w+\lambda)}(x+w+\lambda)^y dF_W dF_X.$$

We have

$$I(X,W;Y) = \int_0^{A_1} i(x,F_X) dF_X.$$

In the remaining discussion, assume that F_W is fixed. As a consequence, one can define a mapping $\chi : \mathcal{F}_1 \to \mathbb{R}^+$ such that $\chi(F_X) = I(X,W;Y)$ for $F_X \in \mathcal{F}_1$. Note that $\mathcal{F}_1$ is a convex and compact space in the Lévy metric d_L [7, p. 21]. (See [14] for the definition of Lévy metric.)

The proofs of the following two technical results are relegated to Appendices A and B, respectively.

Proposition 3 (KKT conditions)**.** *Let F_X^* be the optimal solution to the maximization problem (2). Then*

$$\begin{aligned} i(x,F_X^*) &\le \chi(F_X^*), \quad x \in [0,A_1] \\ i(x,F_X^*) &= \chi(F_X^*), \quad x \in \psi_X^* \end{aligned} \tag{3}$$

where ψ_X^* is the set of points of increase of F_X^*.

Remark: By Proposition 6 in Appendix A, the optimal solution to (2) is unique.

Proposition 4 (Analyticity of $i(x,F_X)$)**.** *Given $F_X \in \mathcal{F}_1$, $i(x,F_X)$ (as a function of x) is analytic on $\mathcal{R}_\lambda \triangleq \{x \in \mathbb{C} : Re(x) + \lambda > 0\}$ for $\lambda > 0$.*

Given these results, the proof of Theorem 2 follows. Suppose ψ_x^* contains an infinite number of mass points in the interval $[0,A_1]$, then it follows from the Bolzano-Weierstrass theorem that ψ_x^* has a limit point in the interval $[0,A_1] \subset R_\lambda$. Therefore, by Proposition 4 and the identity theorem, equation (3), i.e.,

$$\begin{aligned} &\sum_{y=0}^{\infty} \int_0^A \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} \gamma_y dF_W \\ &\quad = \int_0^A (x+w+\lambda) \log \frac{x+w+\lambda}{e} dF_W - \chi(F_X^*), \end{aligned} \tag{4}$$

must hold for all $x \in \mathcal{R}_\lambda$. We shall show that this leads to a contradiction. Note that

$$\begin{aligned} \gamma_y &= \log \int_0^{A_1} \int_0^{A_2} e^{-(x+w+\lambda)}(x+w+\lambda)^y dF_W dF_X \\ &\le \log \int_0^{A_1} \int_0^{A_2} (A_1+A_2+\lambda)^y dF_W dF_X \\ &= y \log(A_1+A_2+\lambda). \end{aligned}$$

For the LHS of (4), we have

$$\begin{aligned} &\sum_{y=0}^{\infty} \int_0^{A_2} \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} \gamma_y dF_W \\ &\le \sum_{y=0}^{\infty} \int_0^{A_2} \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} y \\ &\qquad \times \log(A_1+A_2+\lambda) dF_W \\ &= \int_0^{A_2} e^{-(x+w+\lambda)} \log(A_1+A_2+\lambda) \\ &\qquad \times \sum_{y=0}^{\infty} \frac{(x+w+\lambda)^{y+1}}{y!} dF_W \\ &= (x + E[W] + \lambda) \log(A_1+A_2+\lambda), \end{aligned}$$

where swapping of sum and integral in second to last step is justified by Fubini's theorem. As a consequence, the LHS of (4) cannot grow faster than $x \log(A_1+A_2+\lambda)$. On the other hand, the RHS of (4) grows as $x \log x$, which leads to a contradiction. Therefore, ψ_x^* must be finite.

IV. Conclusion

This letter introduces the two-user DTP MAC model. It is shown that the sum-capacity-achieving distributions under peak amplitude constraints are discrete with a finite number of mass points. This result can be extended to the general multiuser case in a straightforward manner.

These results have interesting ramifications for system design of space laser communication systems where multiple LEO satellites require downlink simultaneously to a ground-station via a GEO spacecraft.

Appendix A
The KKT Conditions

In the view of [7, Corollary 1] (see also [5], [6]), for the purpose of proving Proposition 3, it suffices to show that χ is a strictly concave, continuous, and weakly differentiable mapping from $\mathcal{F}_1$ to $\mathbb{R}^+$.

Lemma 5. *$P_Y(y;F_X)$ is bounded and continuous in F_X.*

Let $\{F_n\}_{n\ge1}$ be a sequence in $\mathcal{F}_1$ such that F_n converges to F in the Lévy metric for some $F \in \mathcal{F}_1$. Note that

$$\begin{aligned} &\lim_{n\to\infty} P_Y(y;F_n) \\ &\quad = \lim_{n\to\infty} \int_0^{A_1} \int_0^{A_2} \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} dF_W dF_n \\ &\quad \overset{(a)}{=} \int_0^{A_1} \int_0^{A_2} \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} dF_W dF \\ &\quad = P_Y(y;F), \end{aligned}$$

where (a) is due to the Helly-Bray theorem [14]. Now it remains to prove that

$$\int_0^{A_2} \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} dF_W$$

is bounded and continuous in x. Continuity is clear and boundedness follows from

$$\int_0^{A_2} \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} dF_W \le \frac{e^{-\lambda}(A_1+A_2+\lambda)^y}{y!}.$$

Proposition 6. *$\chi(F_X)$ is a strictly concave function of F_X.*

Proof: It is known that $\chi(F_X)$ is a concave function of F_X. To show strict concavity, it suffices to show

$$P_Y(y;F_1) = P_Y(y;F_2) \Rightarrow d_L(F_1,F_2) = 0$$

for $F_1, F_2 \in \mathcal{F}_1$. Let X_1, X_2 be distributed as F_1, F_2, respectively. Define $V_i = X_i + W$, $i = 1,2$. Note that

$$\Phi_{V_i} = \Phi_{X_i}\Phi_W, \quad i = 1,2,$$

where Φ_U denotes the characteristic function of U for any random variable U. By [5, Lemma 2],

$$P_Y(y;F_1)=P_Y(y;F_2)\Rightarrow \Phi_{V_1}=\Phi_{V_2}.$$

Therefore, we have

$$\Phi_W(\Phi_{X_1}-\Phi_{X_2})=0.$$

Since F_W is a distribution with compact support $[0,A_2]$, it follows from Schwartz's Paley-Wiener theorem [15] that Φ_W is an entire function (when extended to the whole complex plane) and consequently its zeros are isolated, which, together with the (uniform) continuity of characteristic functions, implies that

$$\Phi_{X_1}-\Phi_{X_2}=0,$$

i.e., $F_1=F_2$. ■

Proposition 7. *$\chi(F_X)$ is continuous in F_X.*

Proof: Note that

$$\chi(F_X)=H_Y(F_X)-H_{Y|X,W}(F_X),$$

where

$$H_Y(F_X)=\sum_{y=0}^{\infty}-P_Y(y;F_X)\log(P_Y(y;F_X)),$$

and

$$H_{Y|X,W}(F_X)=\int_0^{A_1}\left[\sum_{y=0}^{\infty}\int_0^{A_2}\frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!}\times\log\frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!}dF_W\right]dF_X.$$

We need to prove that both $H_Y(F_X)$ and $H_{Y|X,W}(F_X)$ are continuous in F_X.

1) $H_Y(F_X)$ is continuous in F_X. Let $\{F_n\}_{n\geq 1}$ be a sequence in $\mathcal{F}_1$ such that F_n converges to F in the Lévy metric for some $F\in\mathcal{F}_1$. Note that

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{y=0}^{\infty}-P_Y(y;F_n)\log(P_Y(y;F_n))\\
&\overset{(b)}{=}\sum_{y=0}^{\infty}\lim_{n\to\infty}-P_Y(y;F_n)\log(P_Y(y;F_n))\\
&\overset{(c)}{=}\sum_{y=0}^{\infty}-P_Y(y;F)\log(P_Y(y;F)),
\end{aligned}$$

where (c) is due to Lemma 5. For (b), we need to invoke the dominated convergence theorem [14]. In view of the fact that

$$\frac{e^{-(A_1+A_2+\lambda)}\lambda^y}{y!}\leq P_Y(y;F_n)\leq\frac{e^{-\lambda}(A_1+A_2+\lambda)^y}{y!},$$

we have

$$\Big|-P_Y(y;F_n)\log P_Y(y;F_n)\Big|\leq\frac{e^{-\lambda}(A_1+A_2+\lambda)^y}{y!}\times\Big((A_1+A_2+\lambda)-y\log\lambda+\log y!\Big).$$

Note that

$$\begin{aligned}
&\sum_{y=0}^{\infty}\frac{e^{-\lambda}(A_1+A_2+\lambda)^y}{y!}\times\Big((A_1+A_2+\lambda)-y\log\lambda+\log y!\Big)\\
&\leq(A_1+A_2+\lambda)e^{A_1+A_2}+(A_1+A_2+\lambda)e^{A_1+A_2}\log\lambda+\sum_{y=0}^{\infty}\frac{e^{-\lambda}(A_1+A_2+\lambda)^y}{y!}y\log y\\
&=(A_1+A_2+\lambda)e^{A_1+A_2}+(A_1+A_2+\lambda)e^{A_1+A_2}\log\lambda+\sum_{y=0}^{\infty}\frac{e^{-\lambda}(A_1+A_2+\lambda)^{y+1}}{y!}\log(y+1)\\
&\leq(A_1+A_2+\lambda)e^{A_1+A_2}+(A_1+A_2+\lambda)e^{A_1+A_2}\log\lambda+\sum_{y=0}^{\infty}\frac{e^{-\lambda}(A_1+A_2+\lambda)^{y+1}}{y!}y\\
&=(A_1+A_2+\lambda)e^{A_1+A_2}+(A_1+A_2+\lambda)e^{A_1+A_2}\log\lambda+(A_1+A_2+\lambda)^2e^{A_1+A_2}\\
&<+\infty.
\end{aligned}\tag{5}$$

Therefore, the conditions of the dominated convergence theorem are indeed satisfied.

2) $H_{Y|X,W}(F_X)$ is continuous in F_X. It is obvious that

$$\sum_{y=0}^{\infty}\int_0^{A_2}\frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!}\times\log\frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!}dF_W$$

is continuous in x. Moreover,

$$\begin{aligned}
&\left|\sum_{y=0}^{\infty}\int_0^{A_2}\frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!}\times\log\frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!}dF_W\right|\\
&\leq\sum_{y=0}^{\infty}\int_0^{A_2}\frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!}\times[y\log(x+w+\lambda)+(x+w+\lambda)+\log y!]\,dF_W\\
&\leq\sum_{y=0}^{\infty}\left[\frac{e^{-\lambda}(A_1+A_2+\lambda)^y}{y!}y\log(A_1+A_2+\lambda)+\frac{e^{-\lambda}(A_1+A_2+\lambda)^y}{y!}(A_1+A_2+\lambda)+\frac{e^{-\lambda}(A_1+A_2+\lambda)^y}{y!}\log y!\right]\\
&<+\infty,
\end{aligned}$$

where the last step follows by an argument similar to that for (5). Now one can readily complete the proof by invoking the Helly-Bray theorem. ■

Proposition 8. *$\chi(F_X)$ is weakly differentiable in $\mathcal{F}_1$, i.e., the limit*

$$\lim_{\theta\to 0^+} \frac{\chi((1-\theta)F_X+\theta F)-\chi(F_X)}{\theta}$$

exists for any $F_X, F \in \mathcal{F}_1$.

Proof: The proof follows by an argument similar to that in [7, pp.30–32] and is omitted. ■

Appendix B
Analyticity of $i(x, F_X)$

Consider extending $i(x, F_X)$ for $x \in \mathbb{C}$. The continuity of $i(x, F_X)$ in x over the complex plane can be readily verified by invoking the dominated convergence theorem using an approach similar to that in Proposition 7 part 1. Therefore, in view of Morera's theorem, it suffices to show that

$$\oint_\gamma i(x, F_X)dx = 0 \tag{6}$$

for all closed contours γ in $\mathcal{R}_\lambda$. We shall prove (6) by verifying the following two equations:

$$\oint_\gamma \sum_{y=0}^{\infty} \int_0^{A_2} \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} \times \log \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} dF_W dx = 0, \tag{7}$$

$$\oint_\gamma \sum_{y=0}^{\infty} \int_0^{A_2} \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} \times \log P_Y(y;F_X) dF_W dx = 0. \tag{8}$$

1) Verification of (7). It can be shown by invoking Fubini's theorem that

$$\begin{aligned}\oint_\gamma \sum_{y=0}^{\infty} \int_0^{A_2} &\frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} \times \log \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} dF_W dx \\ &= \sum_{y=0}^{\infty} \int_0^{A_2} \oint_\gamma \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} \times \log \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} dx dF_W. \end{aligned} \tag{9}$$

Since the integrand in (9) is an analytic function of x on $\mathcal{R}_\lambda$, it follows from Cauchy's integral theorem that

$$\oint_\gamma \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} \times \log \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} dx = 0. \tag{10}$$

Substituting (10) into (9) yields (7).

2) Verification of (8). It can be shown by invoking Fubini's theorem that

$$\begin{aligned}\oint_\gamma \sum_{y=0}^{\infty} \int_0^{A_2} &\frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} \times \log P_Y(y;F_X) dF_W dx \\ &= \sum_{y=0}^{\infty} \log P_Y(y;F_X) \times \int_0^{A_2} \oint_\gamma \frac{e^{-(x+w+\lambda)}(x+w+\lambda)^y}{y!} dx dF_W. \end{aligned} \tag{11}$$

Since the integrand in (11) is an analytic function of x on $\mathcal{R}_\lambda$, it follows from from Cauchy's integral theorem that the contour integration in (11) is equal to zero. This completes the verification of (8).

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