Generalized Gaussian Multiterminal Source Coding: The Symmetric Case

Jun Chen, Li Xie, Yameng Chang, Jia Wang, and Yizhong Wang

***Abstract*— Consider a generalized multiterminal source coding system, where $\binom{\ell}{m}$ encoders, each observing a distinct size-m subset of ℓ ($\ell \geq 2$) zero-mean unit-variance exchangeable Gaussian sources with correlation coefficient ρ, compress their observations in such a way that a joint decoder can reconstruct the sources within a prescribed mean squared error distortion based on the compressed data. The optimal rate-distortion performance of this system was previously known only for the two extreme cases $m = \ell$ (the centralized case) and $m = 1$ (the distributed case), and except when $\rho = 0$, the centralized system can achieve strictly lower compression rates than the distributed system under all non-trivial distortion constraints. Somewhat surprisingly, it is established in the present paper that the optimal rate-distortion performance of the afore-described generalized multiterminal source coding system with $m \geq 2$ coincides with that of the centralized system for all distortions when $\rho \leq 0$ and for distortions below an explicit positive threshold (depending on m) when $\rho > 0$. Moreover, when $\rho > 0$, the minimum achievable rate of generalized multiterminal source coding subject to an arbitrary positive distortion constraint d is shown to be within a finite gap (depending on m and d) from its centralized counterpart in the large ℓ limit except for possibly the critical distortion $d = 1 - \rho$.**

***Index Terms*— Gaussian source, mean squared error, multiterminal source coding, rate-distortion, reverse water-filling.**

I. Introduction

MULTITERMINAL source coding deals with the scenarios where (possibly) correlated data collected at different sites are compressed in a distributed manner and then forwarded to a fusion center for joint reconstruction. The fundamental problem here is to characterize the optimal tradeoff between the compression rates and the reconstruction distortions. The lossless version of this problem was largely solved by Slepian and Wolf in their landmark paper [1]. Their result was later partially extended to the lossy case by Wyner and Ziv [2] and by Berger and Tung [3], [4]. Though a complete solution to the general lossy multiterminal source coding problem remains out of reach, significant progress has been made on some special cases of this problem, most notably the quadratic Gaussian case [5]–[11] and the logarithmic loss case [12].

In many applications, the data collected at one site may be partially contained in those collected at another site. For example, in a distributed video surveillance system, the scenes captured by different cameras can potentially overlap with each other. To model such scenarios, a so-called generalized multiterminal source coding problem was introduced in [13]. Specifically, in generalized multiterminal source coding, several encoders, each observing a subset of ℓ jointly distributed sources, compress their observations in such a way that a joint decoder can reconstruct the sources within a prescribed distortion level based on the compressed data. It is shown in [13] that, for Gaussian sources with mean squared error distortion constraints, a generalized multiterminal source coding system can achieve the same rate-distortion performance as that of the centralized point-to-point system in the high-resolution regime if the source-encoder bipartite graph and the probabilistic graphical model of the source distribution satisfy a certain condition.

In this work, we shall continue this line of research by considering a symmetric version of the generalized Gaussian multiterminal source coding problem. Here we have ℓ zero-mean unit-variance exchangeable Gaussian sources with correlation coefficient ρ and $\binom{\ell}{m}$ encoders, each of which has access to a distinct size-m subset of these ℓ sources (see Fig. 1 for an illustration of the special case $(\ell, m) = (3, 2)$); moreover, we impose a normalized mean squared error trace distortion constraint on the joint source reconstruction (or equivalently, identical mean squared error distortion constraints on individual source reconstructions). It is worth mentioning that this seemingly simple symmetric setting is in fact non-trivial. Indeed, the associated rate-distortion function was previously known only for the two extreme cases $m = \ell$ (the centralized case) and $m = 1$ (the distributed case). Furthermore, there are two major benefits to study this symmetric setting. First of all, it enables us to obtain results that are more explicit and conclusive than those for a more generic setting in [13]. More importantly, it is instructive to think of m as a parameter that specifies

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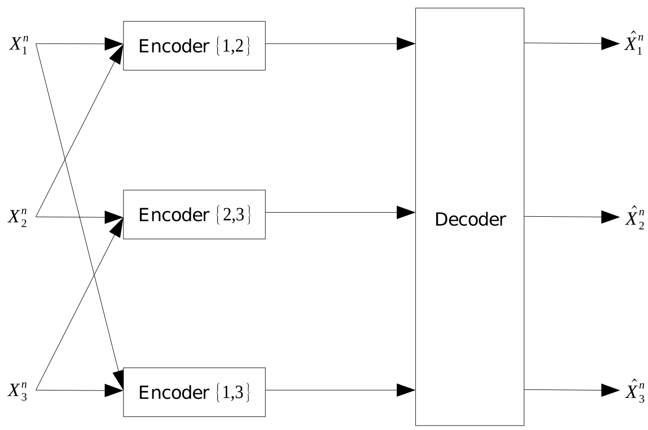

Fig. 1. A generalized multiterminal source coding system with $(\ell, m) = (3, 2)$.

the amount of cooperation among the encoders; as such, one can gain a precise understanding of the value of cooperation in terms of improving compression efficiency by investigating the gradual transition from a distributed system to a centralized system with m varying from 1 to ℓ.

The rest of this paper is organized as follows. Section II contains the problem definition and a review of the relevant results in the literature. We state the main results in Section III. Section IV provides a detailed discussion of the special case $(\ell, m) = (3, 2)$. The proofs of the main results can be found in Sections V, VI, and VII. We present some numerical results in Section VIII. Section IX contains the concluding remarks.

Notation: We use $\mathbb{E}[\cdot]$, $(\cdot)^T$, $\mathrm{tr}(\cdot)$, and $\det(\cdot)$ to denote the expectation operator, the transpose operator, the trace operator, and the determinant operator, respectively. For any random (column) vector Y and random object ω, the distortion covariance matrix incurred by the minimum mean squared error estimator of Y from ω (i.e., $\mathbb{E}[(Y - \mathbb{E}[Y|\omega])((Y - \mathbb{E}[Y|\omega]))^T]$) is denoted by $\mathrm{cov}(Y|\omega)$. We use Y^n as an abbreviation of $(Y(1), \cdots, Y(n))$. The cardinality of a set $\mathcal{S}$ is denoted by $|\mathcal{S}|$. An $\ell \times \ell$ diagonal matrix with the i-th diagonal entry being a_i, $i = 1, \cdots, \ell$, is written as $\mathrm{diag}(a_1, \cdots, a_\ell)$. Throughout this paper, the base of the logarithm function is e.

II. Problem Definition and Known Results

A. Problem Definition

Let $X \triangleq (X_1, \cdots, X_\ell)^T$ be an ℓ-dimensional ($\ell \geq 2$) zero-mean Gaussian random column vector with covariance matrix

$$\Sigma^{(\ell)} = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}.$$

We assume $\rho \in [-\frac{1}{\ell-1}, 1]$ to ensure that $\Sigma^{(\ell)}$ is positive semidefinite and consequently is a valid covariance matrix. Let $X(t) \triangleq (X_1(t), \cdots, X_\ell(t))^T$, $t = 1, 2, \cdots$, be i.i.d. copies of X.

Definition 1: A rate r is said to be achievable by an (ℓ, m) generalized multiterminal source coding system under normalized mean squared error trace distortion constraint d if, for any $\epsilon > 0$, there exist encoding functions $\phi_{\mathcal{S}}^{(n)} : \mathbb{R}^{m \times n} \to \mathcal{C}_{\mathcal{S}}^{(n)}$, $\mathcal{S} \in \mathcal{I}^{(\ell,m)} \triangleq \{\mathcal{S} \subseteq \{1, \cdots, \ell\} : |\mathcal{S}| = m\}$, and a decoding function $\psi^{(n)} : \prod_{\mathcal{S} \in \mathcal{I}^{(\ell,m)}} \mathcal{C}_{\mathcal{S}}^{(n)} \to \mathbb{R}^{\ell \times n}$ such that

$$\frac{1}{n} \sum_{\mathcal{S} \in \mathcal{I}^{(\ell,m)}} \log |\mathcal{C}_{\mathcal{S}}^{(n)}| \leq r + \epsilon,$$

$$\frac{1}{\ell n} \sum_{t=1}^{n} \mathrm{tr}(\mathbb{E}[(X(t) - \hat{X}(t))(X(t) - \hat{X}(t))^T]) \leq d + \epsilon, \quad (1)$$

where

$$\hat{X}^n \triangleq \psi^{(n)}(\phi_{\mathcal{S}}^{(n)}(X_i^n, i \in \mathcal{S}), \mathcal{S} \in \mathcal{I}^{(\ell,m)}).$$

The minimum of such r is denoted by $r^{(\ell,m)}(d)$, which will be referred to as the rate-distortion function of (ℓ, m) generalized multiterminal source coding.

Remark 1: Due to the symmetry of the source distribution, $r^{(\ell,m)}(d)$ remains the same if we replace the normalized mean squared error trace distortion constraint on the joint source reconstruction in (1) with identical mean squared error distortion constraints on individual source reconstructions given below

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{E}[(X_i(t) - \hat{X}_i(t))^2] \leq d + \epsilon, \quad i = 1, \cdots, \ell,$$

where $\hat{X}_i(t)$ is the i-th entry of $\hat{X}(t)$, $i = 1, \cdots, \ell$, $t = 1, \cdots, n$.

Remark 2: It is clear that, for $m = 1, \cdots, \ell$,

$$r^{(\ell,m)}(d) = 0, \quad d \geq 1.$$

Henceforth we shall assume $d \in (0, 1)$.

Remark 3: Note that an encoder that observes X_i^n, $i \in \mathcal{S}$, is at least as powerful as one that observes X_i^n, $i \in \mathcal{S}'$, for some $\mathcal{S}' \subseteq \mathcal{S}$, in the sense that the former can perform any function that the latter can do. Given $1 \leq m' < m \leq \ell$, we can find, for any (ℓ, m') generalized multiterminal source coding system, an (ℓ, m) generalized multiterminal source coding system such that each encoder in the (ℓ, m') system is dominated (in terms of functionality) by an encoder in the (ℓ, m) system. Therefore, we must have $r^{(\ell,m)}(d) \leq r^{(\ell,m')}(d)$ for $m > m'$.

Remark 4: It is easy to prove the following facts.

- For $\rho = -\frac{1}{\ell-1}$ and $m = 1, \cdots, \ell$,

$$r^{(\ell,m)} = \frac{\ell - 1}{2} \log \frac{1}{d}, \quad d \in (0, 1).$$

- For $\rho = 0$ and $m = 1, \cdots, \ell$,

$$r^{(\ell,m)} = \frac{\ell}{2} \log \frac{1}{d}, \quad d \in (0, 1).$$

- For $\rho = 1$ and $m = 1, \cdots, \ell$,

$$r^{(\ell,m)} = \frac{1}{2} \log \frac{1}{d}, \quad d \in (0, 1).$$

Henceforth we shall assume $\rho \in (-\frac{1}{\ell-1}, 0) \cup (0, 1)$.

This work is devoted to the characterization of $r^{(\ell,m)}(d)$. It will be seen that $r^{(\ell,m)}(d)$ (or the best known upper bound if the exact characterization of $r^{(\ell,m)}(d)$ is not available) can be expressed as

$$\frac{1}{2}\log\frac{\det(\Sigma^{(\ell)})}{\det(D^{(\ell,m)})}, \tag{2}$$

where $D^{(\ell,m)}$ is an $\ell\times\ell$ matrix with all its diagonal entries equal to d and all its off-diagonal entries equal to $\theta^{(\ell,m)}$ for some $\theta^{(\ell,m)}$, i.e.,

$$D^{(\ell,m)} \triangleq \begin{pmatrix} d & \theta^{(\ell,m)} & \cdots & \theta^{(\ell,m)} \\ \theta^{(\ell,m)} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \theta^{(\ell,m)} \\ \theta^{(\ell,m)} & \cdots & \theta^{(\ell,m)} & d \end{pmatrix}. \tag{3}$$

Roughly speaking, $D^{(\ell,m)}$ can be interpreted as the distortion covariance matrix induced by the best known (ℓ,m) generalized multiterminal source coding system under normalized mean squared error trace distortion constraint d, and the essential characteristic of such a system is reflected in its associated $\theta^{(\ell,m)}$. The expression in (2) admits an equivalent representation in the eigenspace. Recall that any $\ell\times\ell$ real matrix Π of the form

$$\begin{pmatrix} a & b & \cdots & b \\ b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{pmatrix} \tag{4}$$

can be written as

$$\Pi = O\Lambda O^T,$$

where O is an arbitrary $\ell\times\ell$ real unitary matrix with the last column being $(\frac{1}{\sqrt{\ell}},\cdots,\frac{1}{\sqrt{\ell}})^T$, and $\Lambda \triangleq \mathrm{diag}(\lambda_1,\cdots,\lambda_\ell)$ with

$$\lambda_i \triangleq a-b, \quad i=1,\cdots,\ell-1,$$
$$\lambda_\ell \triangleq a+(\ell-1)b$$

being the eigenvalues of Π. Note that both $\Sigma^{(\ell)}$ and $D^{(\ell,m)}$ are of the form shown in (4). Therefore, their eigenvalues are given by

$$\lambda_i^{(\ell)} \triangleq 1-\rho, \quad i=1,\cdots,\ell-1,$$
$$\lambda_\ell^{(\ell)} \triangleq 1+(\ell-1)\rho,$$

and

$$d_i^{(\ell,m)} \triangleq d-\theta^{(\ell,m)}, \quad i=1,\cdots,\ell-1, \tag{5}$$
$$d_\ell^{(\ell,m)} \triangleq d+(\ell-1)\theta^{(\ell,m)}, \tag{6}$$

respectively; moreover, we can write the expression in (2) equivalently as

$$\sum_{i=1}^{\ell}\frac{1}{2}\log\frac{\lambda_i^{(\ell)}}{d_i^{(\ell,m)}}. \tag{7}$$

The expression in (7) naturally suggests that the best known (ℓ,m) system might be interpreted as performing lossy compression according to a certain form of distortion allocation in the eigenspace. Indeed, for the optimal centralized system (i.e., $m=\ell$), this distortion allocation interpretation (referred to as reverse water-filling) and the associated transform coding scheme are well known. However, in an (ℓ,m) generalized multiterminal source coding system with $m<\ell$, each encoder can only observe a subset of the sources; therefore, in principle it cannot decorrelate the sources simultaneously through a unitary transformation and perform lossy compression in the eigenspace. Nevertheless, since $\Sigma^{(\ell)}$ and $D^{(\ell,m)}$ can be diagonalized by the same unitary matrix, one may still interpret the effect of the best known (ℓ,m) system and make sensible comparisons with that of the optimal centralized system in the eigenspace.

B. Known Results

A complete characterization of $r^{(\ell,m)}(d)$ was previously known only for $m=\ell$ and $m=1$. It is instructive to review the relevant results for these two extreme cases since they provide the necessary background and useful motivations for the introduction of our new results. For reasons that will become clear soon, we define

$$d_c^- \triangleq 1+(\ell-1)\rho,$$
$$d_c^+ \triangleq 1-\rho,$$

and refer to them as critical distortions. It can be seen that $\min\{\lambda_1^{(\ell)},\cdots,\lambda_\ell^{(\ell)}\}$ coincides with d_c^- for $\rho\in(-\frac{1}{\ell-1},0)$ and coincides with d_c^+ for $\rho\in(0,1)$.

Now consider the case $m=\ell$. The following result is a simple consequence of the celebrated reverse water-filling formula [14, Thm. 13.3.3]. Define $D^{(\ell,\ell)}$ and $d_i^{(\ell,\ell)}$, $i=1,\cdots,\ell$, according to (3), (5), and (6) with

$$\theta^{(\ell,\ell)} \triangleq \begin{cases} 0, & d\in(0,d_c^-), \\ \frac{1-d}{\ell-1}+\rho, & d\in[d_c^-,1), \end{cases} \quad \rho\in(-\frac{1}{\ell-1},0), \tag{8}$$
$$\theta^{(\ell,\ell)} \triangleq \begin{cases} 0, & d\in(0,d_c^+), \\ d-1+\rho, & d\in[d_c^+,1). \end{cases} \quad \rho\in(0,1).$$

Proposition 1: For $\rho\in(-\frac{1}{\ell-1},0)\cup(0,1)$,

$$r^{(\ell,\ell)}(d) = \frac{1}{2}\log\frac{\det(\Sigma^{(\ell)})}{\det(D^{(\ell,\ell)})} = \sum_{i=1}^{\ell}\frac{1}{2}\log\frac{\lambda_i^{(\ell)}}{d_i^{(\ell,\ell)}}, \quad d\in(0,1).$$

It is easy to show that

$$r^{(\ell,\ell)} \ge \underline{r}^{(\ell)}(d) \triangleq \frac{1}{2}\log\frac{(1-\rho)^{\ell-1}(1+(\ell-1)\rho)}{d^\ell}.$$

We shall refer to $\underline{r}^{(\ell)}(d)$ as the Shannon lower bound. Note that $r^{(\ell,\ell)}(d)$ coincides with $\underline{r}^{(\ell)}(d)$ when $d\in(0,d_c^-]$ for $\rho\in(-\frac{1}{\ell-1},0)$, and when $d\in(0,d_c^+]$ for $\rho\in(0,1)$.

Next consider the other extreme case $m=1$. The following result was first proved in [6] for $\rho\in(0,1)$ and then in [7] for $\rho\in(-\frac{1}{\ell-1},1)$. Define $D^{(\ell,1)}$ and $d_i^{(\ell,1)}$, $i=1,\cdots,\ell$, according to (3), (5), and (6) with

$$\theta^{(\ell,1)} \triangleq \frac{\rho d\gamma^{(\ell,1)}}{\gamma^{(\ell,1)}+(1-\rho)(1+(\ell-1)\rho)},$$

where

$$\gamma^{(\ell,1)} \triangleq \frac{-\xi+\sqrt{\xi^2+4(1-\rho)(1+(\ell-1)\rho)d(1-d)}}{2(1-d)},$$
$$\xi \triangleq (1+(\ell-1)\rho)(1-\rho-d)-(1-\rho)d.$$

Proposition 2: For $\rho \in (-\frac{1}{\ell-1}, 0) \cup (0,1)$,

$$r^{(\ell,1)}(d) = \frac{1}{2}\log\frac{\det(\Sigma^{(\ell)})}{\det(D^{(\ell,1)})} = \sum_{i=1}^{\ell}\frac{1}{2}\log\frac{\lambda_i^{(\ell)}}{d_i^{(\ell,1)}}, \quad d \in (0,1).$$

It can be verified that, for $\rho \in (-\frac{1}{\ell-1}, 0) \cup (0,1)$, we have $\theta^{(\ell,1)} \neq \theta^{(\ell,\ell)}$, and consequently

$$r^{(\ell,1)}(d) > r^{(\ell,\ell)}(d), \quad d \in (0,1). \tag{9}$$

III. Main Results

One might be inclined to expect that (9) continues to hold with $r^{(\ell,1)}(d)$ replaced by $r^{(\ell,m)}(d)$ for any $m < \ell$. Somewhat surprisingly, it was shown in [13] that, in the high-resolution regime (i.e., when d is sufficiently close to zero), $r^{(\ell,m)}(d)$ coincides with $r^{(\ell,\ell)}(d)$ when $m \geq 2$. However, the high-resolution condition in [13] is not explicit. Our first main result shows that this high-resolution condition is in fact redundant when the correlation coefficient ρ is negative.

Theorem 1: For $\rho \in (-\frac{1}{\ell-1}, 0)$ and $m = 2, \cdots, \ell$,

$$r^{(\ell,m)}(d) = r^{(\ell,\ell)}(d), \quad d \in (0,1).$$

Proof: See Section V. ■

For positive ρ, we have the following result, which provides an explicit high-resolution condition under which $r^{(\ell,m)}(d)$ (with $m \geq 2$) matches $r^{(\ell,\ell)}(d)$.

Theorem 2: For $\rho \in (0,1)$ and $m = 1, \cdots, \ell$,

$$r^{(\ell,m)}(d) = r^{(\ell,\ell)}(d), \quad d \in (0, d_c^{(\ell,m)}],$$

where

$$d_c^{(\ell,m)} \triangleq 1 - \frac{(\ell-1)\rho(1+(m-1)\rho)}{(\ell-1)m\rho + (m-1)(1-\rho)}.$$

Proof: See Section VI. ■

Remark 5: We have $d_c^{(\ell,\ell)} = d_c^+$ and $d_c^{(\ell,1)} = 0$. The statement of Theorem 2 is trivial when $m = \ell$ and is void when $m = 1$.

Remark 6: $d_c^{(\ell,m)}$ is a monotonically increasing function of m for fixed ℓ and is a monotonically decreasing function of ℓ for fixed m. Moreover, we have

$$\lim_{\ell\to\infty} d_c^{(\ell,m)} = d_c^{(m)} \triangleq \frac{(m-1)(1-\rho)}{m},$$
$$\lim_{m\to\infty} d_c^{(m)} = d_c^+,$$

which implies that, for $\rho \in (0,1)$, $r^{(\ell,m)}(d)$ essentially matches $r^{(\ell,\ell)}(d)$ (and the Shannon lower bound $\underline{r}^{(\ell)}(d)$ as well) all the way up to the critical distortion d_c^+ when ℓ and m are sufficiently large (even if the ratio $\frac{m}{\ell}$ is close to zero).

It remains to understand the behavior of $r^{(\ell,m)}(d)$ when $d > d_c^{(\ell,m)}$ for $\rho \in (0,1)$ and $m \geq 2$. To simplify the analysis, we shall consider the asymptotic regime where ℓ goes to infinity with m fixed. Define

$$r_1^{(\ell,m)}(d) \triangleq \frac{\ell}{2}\log\frac{1-\rho}{d} + \frac{1}{2}\log\ell + \frac{1}{2}\log\frac{\rho}{1-\rho} + O(\frac{1}{\ell}),$$
$$r_2^{(\ell,m)}(d) \triangleq \frac{\ell}{2}\log\frac{1-\rho}{d} + \frac{1}{2}\log\ell + \frac{d-(m-1)(1-\rho-d)}{2m(1-\rho-d)} + \frac{1}{2}\log\frac{m\rho(1-\rho-d)}{(1-\rho)^2} + O(\frac{1}{\ell}),$$
$$r_3^{(\ell,m)}(d) \triangleq \frac{\sqrt{\ell}}{2\sqrt{m}} + \frac{1}{4}\log\ell + \frac{1}{2}\log\frac{\sqrt{m}\rho}{1-\rho} - \frac{1+(m-1)\rho}{4m\rho} + O(\frac{1}{\sqrt{\ell}}),$$
$$r_4^{(\ell,m)}(d) \triangleq \frac{1}{2}\log\frac{\rho}{d-1+\rho} + \frac{(1-\rho)(1-d)}{2m\rho(d-1+\rho)} + O(\frac{1}{\ell}),$$

where $g(\ell) = O(f(\ell))$ means the absolute value of $\frac{g(\ell)}{f(\ell)}$ is bounded for all sufficiently large ℓ.

Theorem 3: For $\rho \in (0,1)$ and $m \geq 1$,

$$r^{(\ell,m)}(d) \leq \begin{cases} r_1^{(\ell,m)}(d), & d \in (0, d_c^{(m)}], \\ r_2^{(\ell,m)}(d), & d \in (d_c^{(m)}, d_c^+), \\ r_3^{(\ell,m)}(d), & d = d_c^+, \\ r_4^{(\ell,m)}(d), & d \in (d_c^+, 1). \end{cases}$$

Moreover, this upper bound is tight when $m = 1$ or $d \in (0, d_c^{(m)}]$.

Proof: See Section VII. ■

Remark 7: It follows from Proposition 1 that, for $\rho \in (0,1)$,

$$r^{(\ell,\ell)}(d) = \begin{cases} \frac{\ell}{2}\log\frac{1-\rho}{d} + \frac{1}{2}\log\ell + \frac{1}{2}\log\frac{\rho}{1-\rho} + O(\frac{1}{\ell}), & d \in (0, d_c^+), \\ \frac{1}{2}\log\ell + \frac{1}{2}\log\frac{\rho}{1-\rho} + O(\frac{1}{\ell}), & d = d_c^+, \\ \frac{1}{2}\log\frac{\rho}{d-1+\rho} + O(\frac{1}{\ell}), & d \in (d_c^+, 1). \end{cases} \tag{10}$$

Combining Theorem 3 and (10) shows that, for $\rho \in (0,1)$ and $m \geq 1$,

$$\limsup_{\ell\to\infty} r^{(\ell,m)}(d) - r^{(\ell,\ell)}(d) \leq \delta^{(m)}(d), \quad d \in (0,1),$$

where

$$\delta^{(m)}(d) \triangleq \begin{cases} 0, & d \in (0, d_c^{(m)}], \\ \frac{1-\rho-m(1-\rho-d)}{2m(1-\rho-d)} + \frac{1}{2}\log\frac{m(1-\rho-d)}{1-\rho}, & d \in (d_c^{(m)}, d_c^+), \\ \infty, & d = d_c^+, \\ \frac{(1-\rho)(1-d)}{2m\rho(d-1+\rho)}, & d \in (d_c^+, 1). \end{cases}$$

Note that, as a function of d (with m fixed), $\delta^{(m)}(d)$ is monotonically increasing for $d \in (0, d_c^+)$ and monotonically decreasing for $d \in (d_c^+, 1)$; moreover, it approaches infinity as $d \to d_c^+$. For fixed d, $\delta^{(m)}(d)$ is a monotonically decreasing function of m and converges to zero (though not uniformly over d) as $m \to \infty$ except at $d = d_c^+$. Therefore, for $\rho \in (0,1)$, $r^{(\ell,m)}(d)$ is within a finite gap (depending on d) from $r^{(\ell,\ell)}(d)$ even in the limit of large ℓ when $d \neq d_c^+$; moreover, this gap diminishes as m increases. For $\rho \in (0,1)$, the gap between $r^{(\ell,m)}(d_c^+)$ and $r^{(\ell,\ell)}(d_c^+)$ can potentially approach infinity as $\ell \to \infty$, and indeed so when $m = 1$.

Remark 8: In view of Theorem 3, (10), and Remark 3, we have, for $\rho \in (0,1)$ and $m \geq 1$,

$$\lim_{\ell\to\infty}\frac{1}{\ell}r^{(\ell,m)}(d) = \begin{cases} \frac{1}{2}\log\frac{1-\rho}{d}, & d \in (0, d_c^+), \\ 0, & d \in [d_c^+, 1), \end{cases}$$

which implies that the average minimum achievable rate per encoder of an (ℓ, m) generalized multiterminal source coding system is essentially independent of m when ℓ is sufficiently large.

Remark 9: It is interesting to see that, for $\rho \in (0,1)$ and $m \geq 1$, $r^{(\ell,m)}(d)$ remains bounded (though not uniformly over d) even in the limit of large ℓ when $d \in (d_c^+, 1)$.

IV. A Special Case: $(\ell, m) = (3, 2)$

The following result can be obtained by specializing the well-known Berger-Tung upper bound [3], [4], [15] to our current setting.

Proposition 3: For any Gaussian random variables/vectors $V_{\mathcal{S}}$, $\mathcal{S} \in \mathcal{I}^{(\ell,m)}$, jointly distributed with X such that $V_{\mathcal{S}} \leftrightarrow (X_i, i \in \mathcal{S}) \leftrightarrow (X_{i'}, i' \in \{1, \cdots, \ell\} \backslash \mathcal{S}, V_{\mathcal{S}'}, \mathcal{S}' \in \mathcal{I}^{(\ell,m)} \backslash \mathcal{S})$ form a Markov chain for any $\mathcal{S} \in \mathcal{I}^{(\ell,m)}$, we have

$$\begin{aligned} & r^{(\ell,m)}(\frac{1}{\ell}\mathrm{tr}(\mathrm{cov}(X|V_{\mathcal{S}}, \mathcal{S} \in \mathcal{I}^{(\ell,m)}))) \\ & \leq \frac{1}{2}\log\frac{\det(\Sigma^{(\ell)})}{\det(\mathrm{cov}(X|V_{\mathcal{S}}, \mathcal{S} \in \mathcal{I}^{(\ell,m)}))}. \end{aligned}$$

Remark 10: In the current setting, there are $\binom{\ell}{m}$ encoders, indexed by $\mathcal{S} \in \mathcal{I}^{(\ell,m)}$. Roughly speaking, $(X_i, i \in \mathcal{S})$ is the observation of encoder $\mathcal{S}$, and $V_{\mathcal{S}}$ is the encoded version of $(X_i, i \in \mathcal{S})$. Note that

$$\begin{aligned} & \frac{1}{2}\log\frac{\det(\Sigma^{(\ell)})}{\det(\mathrm{cov}(X|V_{\mathcal{S}}, \mathcal{S} \in \mathcal{I}^{(\ell,m)}))} \\ & = I(X; V_{\mathcal{S}}, \mathcal{S} \in \mathcal{I}^{(\ell,m)}) \\ & = I((X_i, i \in \mathcal{S}), \mathcal{S} \in \mathcal{I}^{(\ell,m)}; V_{\mathcal{S}}, \mathcal{S} \in \mathcal{I}^{(\ell,m)}), \end{aligned}$$

which is the achievable sum rate of the Berger-Tung scheme, and $\frac{1}{\ell}\mathrm{tr}(\mathrm{cov}(X|V_{\mathcal{S}}, \mathcal{S} \in \mathcal{I}^{(\ell,m)}))$ is the associated achievable normalized mean squared error distortion.

It is clear that there is considerable freedom in the choice of $V_{\mathcal{S}}$, $\mathcal{S} \in \mathcal{I}^{(\ell,m)}$. The key idea underlying the proofs of Theorem 1 and Theorem 2 is to derive a Berger-Tung upper bound on $r^{(\ell,m)}(d)$ for $m \geq 2$ that (partially) coincides with $r^{(\ell,\ell)}(d)$ through a judicious construction of $V_{\mathcal{S}}$, $\mathcal{S} \in \mathcal{I}^{(\ell,m)}$. To illustrate this idea, we first consider the special case $(\ell, m) = (3, 2)$, which is further divided into two subcases $\rho \in (-\frac{1}{2}, 0)$ and $\rho \in (0, 1)$. The complete proofs of Theorem 1 and Theorem 2 can be found in Section V and Section VI, respectively.

A. $\rho \in (-\frac{1}{2}, 0)$

Recall that

$$D^{(3,3)} \triangleq \begin{pmatrix} d & \theta^{(3,3)} & \theta^{(3,3)} \\ \theta^{(3,3)} & d & \theta^{(3,3)} \\ \theta^{(3,3)} & \theta^{(3,3)} & d \end{pmatrix},$$

where

$$\theta^{(3,3)} \triangleq \begin{cases} 0, & d \in (0, d_c^-), \\ \frac{1-d}{2} + \rho, & d \in [d_c^-, 1). \end{cases}$$

It suffices to construct $V_{\{1,2\}}, V_{\{2,3\}}, V_{\{1,3\}}$ such that $\mathrm{cov}(X|V_{\{1,2\}}, V_{\{2,3\}}, V_{\{1,3\}})$ coincides with $D^{(3,3)}$, or equivalently, the covariance matrix of $\mathbb{E}[X|V_{\{1,2\}}, V_{\{2,3\}}, V_{\{1,3\}}]$ coincides with $\Sigma^{(3)} - D^{(3,3)}$. When $d \in [d_c^-, 1)$,

$$\Sigma^{(3)} - D^{(3,3)} = \begin{pmatrix} 1-d & \frac{d-1}{2} & \frac{d-1}{2} \\ \frac{d-1}{2} & 1-d & \frac{d-1}{2} \\ \frac{d-1}{2} & \frac{d-1}{2} & 1-d \end{pmatrix},$$

which is of rank 2 (this fact can also be inferred from the reverse water-filling solution). Inspired by this observation, we propose the following construction. For any $\gamma > 0$, let

$$U_{\{1,2\}}^-(\gamma) \triangleq X_1 - X_2 + \sqrt{\gamma}N_{\{1,2\}}^-, \tag{11}$$

$$U_{\{2,3\}}^-(\gamma) \triangleq X_2 - X_3 + \sqrt{\gamma}N_{\{2,3\}}^-, \tag{12}$$

$$U_{\{1,3\}}^-(\gamma) \triangleq X_3 - X_1 + \sqrt{\gamma}N_{\{1,3\}}^-, \tag{13}$$

where $N_{\{1,2\}}^-, N_{\{2,3\}}^-, N_{\{1,3\}}^-$ are mutually independent zero-mean unit-variance Gaussian random variables and are independent of X. It can be verified that

$$\begin{aligned} \hat{X}_1^-(\gamma) & \triangleq \mathbb{E}[X_1|U_{\{1,2\}}^-(\gamma), U_{\{2,3\}}^-(\gamma), U_{\{1,3\}}^-(\gamma)] \\ & = \frac{1-\rho}{\gamma + 3(1-\rho)}(U_{\{1,2\}}^-(\gamma) - U_{\{1,3\}}^-(\gamma)), \\ \hat{X}_2^-(\gamma) & \triangleq \mathbb{E}[X_2|U_{\{1,2\}}^-(\gamma), U_{\{2,3\}}^-(\gamma), U_{\{1,3\}}^-(\gamma)] \\ & = \frac{1-\rho}{\gamma + 3(1-\rho)}(U_{\{2,3\}}^-(\gamma) - U_{\{1,2\}}^-(\gamma)), \\ \hat{X}_3^-(\gamma) & \triangleq \mathbb{E}[X_3|U_{\{1,2\}}^-(\gamma), U_{\{2,3\}}^-(\gamma), U_{\{1,3\}}^-(\gamma)] \\ & = \frac{1-\rho}{\gamma + 3(1-\rho)}(U_{\{1,3\}}^-(\gamma) - U_{\{2,3\}}^-(\gamma)). \end{aligned}$$

The covariance matrix of $(\hat{X}_1^-(\gamma), \hat{X}_2^-(\gamma), \hat{X}_3^-(\gamma))^T$ is of rank 2 (note that $\hat{X}_1^-(\gamma) + \hat{X}_2^-(\gamma) + \hat{X}_3^-(\gamma) = 0$). This suggests that $(U_{\{1,2\}}^-(\gamma), U_{\{2,3\}}^-(\gamma), U_{\{1,3\}}^-(\gamma))$ might be the right candidate for the desired $(V_{\{1,2\}}, V_{\{2,3\}}, V_{\{1,3\}})$ in the high-distortion regime $[d_c^-, 1)$. Indeed, $U_{\{1,2\}}^-(\gamma), U_{\{2,3\}}^-(\gamma), U_{\{1,3\}}^-(\gamma)$ satisfy the Markov chain condition in Proposition 3, and $\mathrm{cov}(X|U_{\{1,2\}}^-(\gamma), U_{\{2,3\}}^-(\gamma), U_{\{1,3\}}^-(\gamma))$ coincides with $D^{(3,3)}$ if we set

$$\gamma = \frac{2(1-\rho)^2}{1-d} - 3(1-\rho), \quad d \in [d_c^-, 1).$$

Note that γ is a monotonically increasing function of d, and $\gamma = \gamma_c^{(3,2)} \triangleq -\frac{(1-\rho)(1+2\rho)}{\rho} > 0$ when $d = d_c^-$. For $d \in (0, d_c^-]$, $D^{(3,3)}$ is a diagonal matrix. This special structure implies that the desired construction for any d in the low-distortion regime $(0, d_c^-)$ can be obtained by superimposing a simple refinement layer on the construction targeted at the critical distortion d_c^-. Specifically, we can let

$$V_{\{1,2\}} \triangleq \left(U_{\{1,2\}}^-(\gamma_c^{(3,2)}), X_1 + \sqrt{\frac{d_c^- d}{d_c^- - d}}Z_1^-\right),$$

$$V_{\{2,3\}} \triangleq \left(U_{\{2,3\}}^-(\gamma_c^{(3,2)}), X_2 + \sqrt{\frac{d_c^- d}{d_c^- - d}}Z_2^-\right),$$

$$V_{\{1,3\}} \triangleq \left(U_{\{1,3\}}^-(\gamma_c^{(3,2)}), X_3 + \sqrt{\frac{d_c^- d}{d_c^- - d}}Z_3^-\right),$$

where Z_1^-, Z_2^-, Z_3^- are mutually independent zero-mean unit-variance Gaussian random variables and are independent of $(X, U_{\{1,2\}}^-(\gamma_c^{(3,2)}), U_{\{2,3\}}^-(\gamma_c^{(3,2)}), U_{\{1,3\}}^-(\gamma_c^{(3,2)}))$. It can be readily verified that $V_{\{1,2\}}, V_{\{2,3\}}, V_{\{1,3\}}$ satisfy the Markov chain condition in Proposition 3, and $\mathrm{cov}(X|V_{\{1,2\}}, V_{\{2,3\}}, V_{\{1,3\}})$ coincides with $D^{(3,3)}$ since

$$\begin{aligned}
&\mathrm{cov}^{-1}(X|V_{\{1,2\}}, V_{\{2,3\}}, V_{\{1,3\}})\\
&= \mathrm{cov}^{-1}(X|U_{\{1,2\}}^-(\gamma_c^{(3,2)}), U_{\{2,3\}}^-(\gamma_c^{(3,2)}), U_{\{1,3\}}^-(\gamma_c^{(3,2)}))\\
&\quad + \mathrm{cov}^{-1}\left(\sqrt{\frac{d_c^- d}{d_c^- - d}}Z_1^-, \sqrt{\frac{d_c^- d}{d_c^- - d}}Z_2^-, \sqrt{\frac{d_c^- d}{d_c^- - d}}Z_3^-\right)\\
&= \mathrm{diag}\left(\frac{1}{d_c^-}, \frac{1}{d_c^-}, \frac{1}{d_c^-}\right) + \mathrm{diag}\left(\frac{d_c^- - d}{d_c^- d}, \frac{d_c^- - d}{d_c^- d}, \frac{d_c^- - d}{d_c^- d}\right)\\
&= \mathrm{diag}\left(\frac{1}{d}, \frac{1}{d}, \frac{1}{d}\right).
\end{aligned}$$

B. $\rho \in (0,1)$

In this case, the off-diagonal entries of $D^{(3,3)}$ are all equal to

$$\theta^{(3,3)} \triangleq \begin{cases} 0, & d \in (0, d_c^+), \\ d - 1 + \rho, & d \in [d_c^+, 1). \end{cases}$$

Note that $\Sigma^{(3,3)} - D^{(3,3)}$ is of rank 3 when $d \in (0, d_c^+)$, and is of rank 1 when $d \in [d_c^+, 1)$. In view of (11), (12), and (13), it is natural to consider the following construction. For any $\gamma > 0$, let

$$U_{\{1,2\}}^+(\gamma) \triangleq X_1 + X_2 + \sqrt{\gamma}N_{\{1,2\}}^+, \tag{14}$$
$$U_{\{2,3\}}^+(\gamma) \triangleq X_2 + X_3 + \sqrt{\gamma}N_{\{2,3\}}^+, \tag{15}$$
$$U_{\{1,3\}}^+(\gamma) \triangleq X_1 + X_3 + \sqrt{\gamma}N_{\{1,3\}}^+, \tag{16}$$

where $N_{\{1,2\}}^+, N_{\{2,3\}}^+, N_{\{1,3\}}^+$ are mutually independent zero-mean unit-variance Gaussian random variables and are independent of X. It can be verified that

$$\begin{aligned}
\hat{X}_1^+(\gamma) &\triangleq \mathbb{E}[X_1|U_{\{1,2\}}^+(\gamma), U_{\{2,3\}}^+(\gamma), U_{\{1,3\}}^+(\gamma)]\\
&= \frac{(1+\rho)\gamma + 2(1-\rho)(1+2\rho)}{\gamma^2 + (5+7\rho)\gamma + 4(1-\rho)(1+2\rho)}\\
&\quad \times (U_{\{1,2\}}^+(\gamma) + U_{\{1,3\}}^+(\gamma))\\
&\quad + \frac{2\rho\gamma - 2(1-\rho)(1+2\rho)}{\gamma^2 + (5+7\rho)\gamma + 4(1-\rho)(1+2\rho)}U_{\{2,3\}}^+(\gamma),\\
\hat{X}_2^+(\gamma) &\triangleq \mathbb{E}[X_2|U_{\{1,2\}}^+(\gamma), U_{\{2,3\}}^+(\gamma), U_{\{1,3\}}^+(\gamma)]\\
&= \frac{(1+\rho)\gamma + 2(1-\rho)(1+2\rho)}{\gamma^2 + (5+7\rho)\gamma + 4(1-\rho)(1+2\rho)}\\
&\quad \times (U_{\{1,2\}}^+(\gamma) + U_{\{2,3\}}^+(\gamma))\\
&\quad + \frac{2\rho\gamma - 2(1-\rho)(1+2\rho)}{\gamma^2 + (5+7\rho)\gamma + 4(1-\rho)(1+2\rho)}U_{\{1,3\}}^+(\gamma),\\
\hat{X}_3^+(\gamma) &\triangleq \mathbb{E}[X_3|U_{\{1,2\}}^+(\gamma), U_{\{2,3\}}^+(\gamma), U_{\{1,3\}}^+(\gamma)]\\
&= \frac{(1+\rho)\gamma + 2(1-\rho)(1+2\rho)}{\gamma^2 + (5+7\rho)\gamma + 4(1-\rho)(1+2\rho)}\\
&\quad \times (U_{\{2,3\}}^+(\gamma) + U_{\{1,3\}}^+(\gamma))\\
&\quad + \frac{2\rho\gamma - 2(1-\rho)(1+2\rho)}{\gamma^2 + (5+7\rho)\gamma + 4(1-\rho)(1+2\rho)}U_{\{1,2\}}^+(\gamma).
\end{aligned}$$

The covariance matrix of $(\hat{X}_1^+(\gamma), \hat{X}_2^+(\gamma), \hat{X}_3^+(\gamma))^T$ is of rank 3. This means that such a construction is not able to achieve $D^{(3,3)}$ in the high-distortion regime $[d_c^+, 1)$, but is potentially suitable for the low-distortion regime $(0, d_c^+)$. It turns out that $\mathrm{cov}(X|U_{\{1,2\}}^+(\gamma), U_{\{2,3\}}^+(\gamma), U_{\{1,3\}}^+(\gamma))$ coincides with $D^{(3,3)}$ for a particular distortion $d_c^{(3,2)} \triangleq 1 - \frac{2\rho(1+\rho)}{1+3\rho} \in (0, d_c^+)$ at $\gamma = \frac{(1-\rho)(1+2\rho)}{\rho}$. Due to the diagonal structure of the relevant $D^{(3,3)}$, the desired construction for $d \in (0, d_c^{(3,2)})$ can be obtained by superimposing a refinement layer on the construction targeted at $d_c^{(3,2)}$.

V. PROOF OF THEOREM 1

The argument is structurally similar to that in Section IV-A. The key step is to find a generalization of the construction in (11), (12), and (13) for the special case $(\ell, m) = (3, 2)$. Let M be an $m \times m$ matrix given by

$$M \triangleq \begin{pmatrix} m-1 & -1 & \cdots & -1 \\ -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & m-1 \end{pmatrix}.$$

For any $\gamma > 0$ and $\mathcal{S} \triangleq \{i_1, \cdots, i_m\} \in \mathcal{I}^{(\ell,m)}$ with $i_1 < \cdots < i_m$, define

$$\begin{pmatrix} U_{\mathcal{S},1}^-(\gamma) \\ \vdots \\ \vdots \\ U_{\mathcal{S},m}^-(\gamma) \end{pmatrix} \triangleq M \begin{pmatrix} X_{i_1} \\ \vdots \\ \vdots \\ X_{i_m} \end{pmatrix} + \sqrt{\gamma} \begin{pmatrix} N_{\mathcal{S},1}^- \\ \vdots \\ \vdots \\ N_{\mathcal{S},m}^- \end{pmatrix},$$

where $(N_{\mathcal{S},1}^-, \cdots, N_{\mathcal{S},m}^-)^T$ is a Gaussian random vector with mean zero and covariance matrix M. Moreover, we assume that X, $(N_{\mathcal{S},1}^-, \cdots, N_{\mathcal{S},m}^-)^T$, $\mathcal{S} \in \mathcal{I}^{(\ell,m)}$, are mutually independent.

Proposition 4: We have

$$\begin{aligned}
&\mathrm{cov}(X|U_{\mathcal{S},1}^-(\gamma), \cdots, U_{\mathcal{S},m}^-(\gamma), \mathcal{S} \in \mathcal{I}^{(\ell,m)})\\
&= \begin{pmatrix} d^-(\gamma) & \theta^-(\gamma) & \cdots & \theta^-(\gamma) \\ \theta^-(\gamma) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \theta^-(\gamma) \\ \theta^-(\gamma) & \cdots & \theta^-(\gamma) & d^-(\gamma) \end{pmatrix},
\end{aligned}$$

where

$$d^-(\gamma) \triangleq 1 - \frac{\binom{\ell-2}{m-2}(\ell-1)(1-\rho)^2}{\gamma + \binom{\ell-2}{m-2}\ell(1-\rho)},$$
$$\theta^-(\gamma) \triangleq \rho + \frac{\binom{\ell-2}{m-2}(1-\rho)^2}{\gamma + \binom{\ell-2}{m-2}\ell(1-\rho)}.$$

Proof: See Appendix A. ■

Now we proceed to prove Theorem 1. It suffices to show that

$$r^{(\ell,m)}(d) \le r^{(\ell,\ell)}(d), \quad d \in (0,1), \tag{17}$$

since the other direction is trivially true (see Remark 3).

Setting $d^-(\gamma)=d$ gives

$$\gamma=\gamma^{(\ell,m)}\triangleq\frac{\binom{\ell-2}{m-2}(1-\rho)((\ell-1)(1-\rho)-\ell(1-d))}{1-d}.$$

Note that there is a one-to-one correspondence between $d\in(\frac{d_c^-}{\ell},1)$ and $\gamma^{(\ell,m)}\in(0,\infty)$. Moreover,

$$\theta^-(\gamma^{(\ell,m)})=\frac{1-d}{\ell-1}+\rho,$$

which coincides with $\theta^{(\ell,\ell)}$ in (8) for $d\in[d_c^-,1)$; in particular, $\theta^-(\gamma_c^{(\ell,m)})=0$, where

$$\gamma_c^{(\ell,m)}\triangleq-\frac{\binom{\ell-2}{m-2}(1-\rho)(1+(\ell-1)\rho)}{\rho}$$

is the value of $\gamma^{(\ell,m)}$ at $d=d_c^-$. Invoking Proposition 3 with $V_{\mathcal{S}}\triangleq(U_{\mathcal{S},1}^-(\gamma^{(\ell,m)}),\cdots,U_{\mathcal{S},m}^-(\gamma^{(\ell,m)}))^T$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$, (which satisfy the Markov chain condition in Proposition 3) proves (17) for $d\in[d_c^-,1)$.

Now consider the case $d\in(0,d_c^-)$. Let

$$W_i^-(d)\triangleq X_i+\sqrt{\frac{d_c^-d}{d_c^--d}}Z_i^-,\quad i=1,\cdots,\ell,$$

where $Z_1^-,\cdots,Z_\ell^-$ are mutually independent zero-mean unit-variance Gaussian random variables, and are independent of X, $(N_{\mathcal{S},1}^-,\cdots,N_{\mathcal{S},m}^-)^T$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$. Construct $\Omega_{\mathcal{S}}$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$, such that 1) $\Omega_{\mathcal{S}}\subseteq\mathcal{S}$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$, 2) $\Omega_{\mathcal{S}}\cap\Omega_{\mathcal{S}'}=\emptyset$, $\mathcal{S}\neq\mathcal{S}'$, 3) $\cup_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}}\Omega_{\mathcal{S}}=\{1,\cdots,\ell\}$. Such a construction always exists. For example, we can let

$$\Omega_{\mathcal{S}}\triangleq\begin{cases}\mathcal{S}, & \mathcal{S}=\{1,\cdots,m\},\\ \{i\}, & \mathcal{S}=\{i-m+1,\cdots,i\}, i=m+1,\cdots,\ell,\\ \emptyset, & \text{otherwise.}\end{cases}$$

Define $V_{\mathcal{S}}\triangleq(U_{\mathcal{S},1}^-(\gamma_c^{(\ell,m)}),\cdots,U_{\mathcal{S},m}^-(\gamma_c^{(\ell,m)}),W_i^-(d),i\in\Omega_{\mathcal{S}})^T$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$. It is clear that such $V_{\mathcal{S}}$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$, satisfy the Markov chain condition in Proposition 3. Moreover,

$$\begin{aligned}&\mathrm{cov}^{-1}(X|V_{\mathcal{S}},\mathcal{S}\in\mathcal{I}^{(\ell,m)})\\&=\mathrm{cov}^{-1}(X|U_{\mathcal{S},1}^-(\gamma_c^{(\ell,m)}),\cdots,U_{\mathcal{S},m}^-(\gamma_c^{(\ell,m)}),\mathcal{S}\in\mathcal{I}^{(\ell,m)})\\&\quad+\mathrm{cov}^{-1}\left(\left(\sqrt{\frac{d_c^-d}{d_c^--d}}Z_1^-,\cdots,\sqrt{\frac{d_c^-d}{d_c^--d}}Z_\ell^-\right)^T\right)\\&=\mathrm{diag}\left(\frac{1}{d_c^-},\cdots,\frac{1}{d_c^-}\right)+\mathrm{diag}\left(\frac{d_c^--d}{d_c^-d},\cdots,\frac{d_c^--d}{d_c^-d}\right)\\&=\mathrm{diag}\left(\frac{1}{d},\cdots,\frac{1}{d}\right),\end{aligned}$$

which implies

$$\mathrm{cov}(X|V_{\mathcal{S}},\mathcal{S}\in\mathcal{I}^{(\ell,m)})=\mathrm{diag}(d,\cdots,d).$$

Invoking Proposition 3 proves (17) for $d\in(0,d_c^-)$.

VI. Proof of Theorem 2

The argument is structurally similar to that in Section IV-B. The key step is to find a generalization of the construction in (14), (15), and (16) for the special case $(\ell,m)=(3,2)$. For any $\gamma>0$ and $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$, define

$$U_{\mathcal{S}}^+(\gamma)\triangleq\sum_{i\in\mathcal{S}}X_i+\sqrt{\gamma}N_{\mathcal{S}}^+,$$

where $N_{\mathcal{S}}^+$ is a zero-mean unit-variance Gaussian random variable. Moreover, we assume that X, $N_{\mathcal{S}}^+$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$ are mutually independent.

Proposition 5: We have

$$\begin{aligned}&\mathrm{cov}(X|U_{\mathcal{S},1}^+(\gamma),\cdots,U_{\mathcal{S},m}^+(\gamma),\mathcal{S}\in\mathcal{I}^{(\ell,m)})\\&=\begin{pmatrix}d^+(\gamma) & \theta^+(\gamma) & \cdots & \theta^+(\gamma)\\ \theta^+(\gamma) & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \theta^+(\gamma)\\ \theta^+(\gamma) & \cdots & \theta^+(\gamma) & d^+(\gamma)\end{pmatrix},\end{aligned}$$

where

$$d^+(\gamma)\triangleq1-\frac{\eta_3\gamma+\eta_1}{\gamma^2+\eta_2\gamma+\eta_1},\tag{18}$$

$$\theta^+(\gamma)\triangleq\rho-\frac{\eta_4\gamma+\eta_1\rho}{\gamma^2+\eta_2\gamma+\eta_1}\tag{19}$$

with

$$\begin{aligned}\eta_1&\triangleq\binom{\ell-1}{m-1}\binom{\ell-2}{m-1}m(1-\rho)(1+(\ell-1)\rho),\\ \eta_2&\triangleq\binom{\ell-1}{m-1}(1+(m-1)\rho)\\&\quad+\binom{\ell-2}{m-1}m(1+(\ell-2)\rho)\\&\quad+\binom{\ell-2}{m-2}((\ell-1)m\rho+(m-1)(1-\rho)),\\ \eta_3&\triangleq\binom{\ell-1}{m-1}(1+(m-1)\rho)+\binom{\ell-2}{m-1}(\ell-1)m\rho^2\\&\quad+\binom{\ell-2}{m-2}(\ell-1)\rho(1+(m-1)\rho),\\ \eta_4&\triangleq\binom{\ell-1}{m-1}\rho(1+(m-1)\rho)\\&\quad+\binom{\ell-2}{m-1}m\rho(1+(\ell-2)\rho)\\&\quad+\binom{\ell-2}{m-2}(1+(\ell-2)\rho)(1+(m-1)\rho).\end{aligned}$$

Proof: See Appendix B. ■

Now we proceed to prove Theorem 2. It suffices to show that

$$r^{(\ell,m)}(d)\le r^{(\ell,\ell)}(d),\quad d\in(0,d_c^{(\ell,m)}].\tag{20}$$

Setting $\theta^+(\gamma)=0$ gives

$$\gamma=\gamma_c^{(\ell,m)}\triangleq\frac{\binom{\ell-2}{m-2}(1-\rho)(1+(\ell-1)\rho)}{\rho}.$$

It can be verified that

$$\begin{aligned} d^{+}(\gamma_c^{(\ell,m)}) &= 1-\frac{\eta_3\gamma_c^{(\ell,m)}+\eta_1}{(\gamma_c^{(\ell,m)})^2+\eta_2\gamma_c^{(\ell,m)}+\eta_1} \\ &= 1-\frac{\eta_3\rho\gamma_c^{(\ell,m)}+\eta_1\rho}{\eta_4\gamma_c^{(\ell,m)}+\eta_1\rho} \\ &= d_c^{(\ell,m)}. \end{aligned}$$

Invoking Proposition 3 with $V_{\mathcal{S}} \triangleq U_{\mathcal{S}}^{+}(\gamma_c^{(\ell,m)})$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$, (which satisfy the Markov chain condition in Proposition 3) proves (20) for $d=d_c^{(\ell,m)}$.

Now consider the case $d\in(0,d_c^{(\ell,m)})$. We will only give a sketch of the proof here since it is similar to its counterpart in Section V. Let

$$W_i^{+}(d) \triangleq X_i+\sqrt{\frac{d_c^{(\ell,m)}d}{d_c^{(\ell,m)}-d}}Z_i^{+}, \quad i=1,\cdots,\ell,$$

where $Z_1^{+},\cdots,Z_\ell^{+}$ are mutually independent zero-mean unit-variance Gaussian random variables, and are independent of X, $N_{\mathcal{S}}^{+}$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$. Construct $\Omega_{\mathcal{S}}$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$, such that 1) $\Omega_{\mathcal{S}}\subseteq\mathcal{S}$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$, 2) $\Omega_{\mathcal{S}}\cap\Omega_{\mathcal{S}'}=\emptyset$, $\mathcal{S}\neq\mathcal{S}'$, 3) $\cup_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}}\Omega_{\mathcal{S}}=\{1,\cdots,\ell\}$. Define $V_{\mathcal{S}}\triangleq(U_{\mathcal{S}}^{+}(\gamma_c^{(\ell,m)}),W_i^{+}(d),i\in\Omega_{\mathcal{S}})^T$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$. It is clear that such $V_{\mathcal{S}}$, $\mathcal{S}\in\mathcal{I}^{(\ell,m)}$, satisfy the Markov chain condition in Proposition 3, and

$$\mathrm{cov}(X|V_{\mathcal{S}},\mathcal{S}\in\mathcal{I}^{(\ell,m)})=\mathrm{diag}(d,\cdots,d).$$

Invoking Proposition 3 proves (20) for $d\in(0,d_c^{(\ell,m)})$.

Remark 11: Setting $d^{+}(\gamma)=d$ gives

$$\begin{aligned} \gamma &= \gamma^{(\ell,m)} \\ &\triangleq \frac{\eta_3-\eta_2(1-d)+\sqrt{(\eta_2(1-d)-\eta_3)^2+4\eta_1 d(1-d)}}{2(1-d)}. \end{aligned}$$

Note that there is a one-to-one correspondence between $d\in(0,1)$ and $\gamma^{(\ell,m)}\in(0,\infty)$. The preceding argument in fact shows that, for $\rho\in(0,1)$ and $m=1,\cdots,\ell$,

$$r^{(\ell,m)}(d)\leq\overline{r}^{(\ell,m)}(d),\quad d\in(0,1), \tag{21}$$

where

$$\overline{r}^{(\ell,m)}(d)\triangleq\frac{1}{2}\log\frac{(1-\rho)^{\ell-1}(1+(\ell-1)\rho)}{(d-\theta^{(\ell,m)})^{\ell-1}(d+(\ell-1)\theta^{(\ell,m)})} \tag{22}$$

with

$$\theta^{(\ell,m)}\triangleq\begin{cases}0, & d\in(0,d_c^{(\ell,m)}],\\ \theta^{+}(\gamma^{(\ell,m)}), & d\in(d_c^{(\ell,m)},1).\end{cases}$$

The equality in (21) holds for $d\in(0,d_c^{(\ell,m)}]$. Moreover, by defining $\binom{\ell-2}{\ell-1}\triangleq 0$ and $\binom{\ell-2}{-1}\triangleq 0$, one can readily verify that $\overline{r}^{(\ell,m)}(d)$ coincides with $r^{(\ell,m)}(d)$ for $d\in(d_c^{(\ell,m)},1)$ when $m=\ell$ or $m=1$. However, it is still unknown whether $\overline{r}^{(\ell,m)}(d)=r^{(\ell,m)}(d)$ for $d\in(d_c^{(\ell,m)},1)$ when $1<m<\ell$.

VII. Proof of Theorem 3

In view of Remark 11, Remark 3, and (10), it suffices to show that, for $\rho\in(0,1)$ and $m\geq 1$,

$$\overline{r}^{(\ell,m)}(d)=\begin{cases} r_1^{(\ell,m)}(d), & d\in(0,d_c^{(m)}],\\ r_2^{(\ell,m)}(d), & d\in(d_c^{(m)},d_c^{+}),\\ r_3^{(\ell,m)}(d), & d=d_c^{+},\\ r_4^{(\ell,m)}(d), & d\in(d_c^{+},1).\end{cases}$$

First consider the case $d\in(0,d_c^{(m)})$. When ℓ is sufficiently large, we have $d\in(0,d_c^{(\ell,m)}]$ and consequently

$$\begin{aligned} \overline{r}^{(\ell,m)}(d) &= \frac{1}{2}\log\frac{(1-\rho)^{\ell-1}(1+(\ell-1)\rho)}{d^\ell} \\ &= \frac{\ell}{2}\log\frac{1-\rho}{d}+\frac{1}{2}\log\ell+\frac{1}{2}\log\frac{\rho}{1-\rho} \\ &\quad+\frac{1}{2}\log\left(1+\frac{1-\rho}{\ell\rho}\right) \\ &= r_1^{(\ell,m)}(d). \end{aligned}$$

Next we shall derive a few results that are needed for studying the remaining cases. It can be verified that

$$\begin{aligned} \eta_1 &= g_1\frac{\ell^{2m}}{((m-1)!)^2}+h_1\frac{\ell^{2m-1}}{((m-1)!)^2}+O(\ell^{2m-2}),\\ \eta_i &= g_i\frac{\ell^m}{(m-1)!}+h_i\frac{\ell^{m-1}}{(m-1)!}+O(\ell^{m-2}),\quad i=2,3,4, \end{aligned}$$

where

$$\begin{aligned} & g_1\triangleq 0,\quad g_2\triangleq m\rho,\quad g_3\triangleq m\rho^2,\quad g_4\triangleq m\rho^2,\\ & h_1\triangleq m\rho(1-\rho),\\ & h_2\triangleq(m+1)(1-\rho)+\frac{(m+4)m(m-1)\rho}{2},\\ & h_3\triangleq h_2\rho+(1-\rho)(1+(m-2)\rho),\\ & h_4\triangleq h_2\rho+(m-1)\rho(1-\rho). \end{aligned}$$

According to (18) and (19),

$$\begin{aligned} d &= \frac{(\gamma^{(\ell,m)})^2+(\eta_2-\eta_3)\gamma^{(\ell,m)}}{(\gamma^{(\ell,m)})^2+\eta_2\gamma^{(\ell,m)}+\eta_1},\\ \theta^{+}(\gamma^{(\ell,m)}) &= \frac{\rho(\gamma^{(\ell,m)})^2+(\eta_2\rho-\eta_4)\gamma^{(\ell,m)}}{(\gamma^{(\ell,m)})^2+\eta_2\gamma^{(\ell,m)}+\eta_1}, \end{aligned}$$

which implies

$$\theta^{+}(\gamma^{(\ell,m)})=\frac{(\rho\gamma^{(\ell,m)}+\eta_2\rho-\eta_4)d}{\gamma^{(\ell,m)}+\eta_2-\eta_3}. \tag{23}$$

Using the asymptotic expressions of η_2, η_3, and η_4, we can rewrite (23) as

$$\theta^{+}(\gamma^{(\ell,m)})=\frac{\rho d\gamma^{(\ell,m)}\frac{(m-1)!}{\ell^m}-\frac{(m-1)\rho(1-\rho)d}{\ell}+O(\frac{1}{\ell^2})}{\gamma^{(\ell,m)}\frac{(m-1)!}{\ell^m}+m\rho(1-\rho)+\frac{h_2-h_3}{\ell}+O(\frac{1}{\ell^2})}. \tag{24}$$

Note that

$$\begin{aligned} &\eta_3-\eta_2(1-d)\\ &\quad= m\rho(d-1+\rho)\frac{\ell^m}{(m-1)!}+(h_3-h_2(1-d))\frac{\ell^{m-1}}{(m-1)!}\\ &\qquad+O(\ell^{m-2}),\\ &(\eta_2(1-d)-\eta_3)^2+4\eta_1 d(1-d)\\ &\quad= m^2\rho^2(1-\rho-d)^2\frac{\ell^{2m}}{((m-1)!)^2}+\zeta\frac{\ell^{2m-1}}{((m-1)!)^2}\\ &\qquad+O(\ell^{2m-2}), \end{aligned}$$

where

$$\zeta \triangleq 2m\rho(1-\rho-d)(h_2(1-d)-h_3) + 4m\rho(1-\rho)d(1-d).$$

As a consequence,

$$\gamma^{(\ell,m)} = \frac{m\rho(d-1+\rho)}{2(1-d)}\frac{\ell^m}{(m-1)!} + \frac{h_3-h_2(1-d)}{2(1-d)}\frac{\ell^{m-1}}{(m-1)!} + \frac{\sqrt{m^2\rho^2(1-\rho-d)^2+\frac{\zeta}{\ell}+O(\frac{1}{\ell^2})}}{2(1-d)}\frac{\ell^m}{(m-1)!} + O(\ell^{m-2}). \quad (25)$$

Now we are in a position to study the remaining cases.

For $d \in (0, d_c^+)$ (if $m=1$) or $d \in [d_c^{(m)}, d_c^+)$ (if $m>1$), we have $1-\rho-d>0$. It follows from (25) that

$$\begin{aligned}\gamma^{(\ell,m)} &= \frac{m\rho(d-1+\rho)}{2(1-d)}\frac{\ell^m}{(m-1)!} + \frac{h_3-h_2(1-d)}{2(1-d)}\frac{\ell^{m-1}}{(m-1)!} + \frac{m\rho(1-\rho-d)\sqrt{1+\frac{\zeta}{\ell m^2\rho^2(1-\rho-d)^2}+O(\frac{1}{\ell^2})}}{2(1-d)}\frac{\ell^m}{(m-1)!} + O(\ell^{m-2})\\ &= \frac{m\rho(d-1+\rho)}{2(1-d)}\frac{\ell^m}{(m-1)!} + \frac{h_3-h_2(1-d)}{2(1-d)}\frac{\ell^{m-1}}{(m-1)!} + \frac{m\rho(1-\rho-d)\left(1+\frac{\zeta}{2\ell m^2\rho^2(1-\rho-d)^2}\right)}{2(1-d)}\frac{\ell^m}{(m-1)!} + O(\ell^{m-2})\\ &= \frac{(1-\rho)d}{1-\rho-d}\frac{\ell^{m-1}}{(m-1)!} + O(\ell^{m-2}),\end{aligned}$$

which, together with (24) and some simple calculation, gives

$$\begin{aligned}\theta^+(\gamma^{(\ell,m)}) &= \frac{\frac{\rho(1-\rho)d^2}{\ell(1-\rho-d)} - \frac{(m-1)\rho(1-\rho)d}{\ell} + O(\frac{1}{\ell^2})}{m\rho(1-\rho)+O(\frac{1}{\ell})}\\ &= \left(\frac{d(d-(m-1)(1-\rho-d))}{\ell m(1-\rho-d)} + O(\frac{1}{\ell^2})\right)\left(1+O(\frac{1}{\ell})\right)\\ &= \frac{d(d-(m-1)(1-\rho-d))}{\ell m(1-\rho-d)} + O(\frac{1}{\ell^2}).\end{aligned}$$

One can readily verify that

$$\begin{aligned}\overline{r}^{(\ell,m)}(d) &= \frac{\ell}{2}\log\frac{1-\rho}{d} + \frac{1}{2}\log\ell - \frac{\ell-1}{2}\log\left(1-\frac{\theta^+(\gamma^{(\ell,m)})}{d}\right)\\ &\quad + \frac{1}{2}\log\left(\frac{\rho d}{1-\rho}+\frac{d}{\ell}\right) - \frac{1}{2}\log(d+(\ell-1)\theta^+(\gamma^{(\ell,m)}))\\ &= r_2^{(\ell,m)}(d).\end{aligned}$$

For $d = d_c^+$, we have $1-\rho-d=0$. It follows from (25) that

$$\begin{aligned}\gamma^{(\ell,m)} &= \frac{h_3-h_2(1-d)}{2(1-d)}\frac{\ell^{m-1}}{(m-1)!} + \frac{\sqrt{\frac{4m\rho(1-\rho)d(1-d)}{\ell}+O(\frac{1}{\ell^2})}}{2(1-d)}\frac{\ell^m}{(m-1)!} + O(\ell^{m-2})\\ &= \sqrt{m}(1-\rho)\frac{\ell^{m-\frac{1}{2}}}{(m-1)!} + \frac{h_3-h_2\rho}{2\rho}\frac{\ell^{m-1}}{(m-1)!} + O(\ell^{m-\frac{3}{2}})\\ &= \sqrt{m}(1-\rho)\frac{\ell^{m-\frac{1}{2}}}{(m-1)!} + \frac{(1-\rho)(1+(m-2)\rho)}{2\rho}\frac{\ell^{m-1}}{(m-1)!} + O(\ell^{m-\frac{3}{2}}),\end{aligned}$$

which, together with (24) and some simple calculation, gives

$$\begin{aligned}\theta^+(\gamma^{(\ell,m)}) &= \frac{\frac{\sqrt{m}\rho(1-\rho)^2}{\sqrt{\ell}} + \frac{(1-\rho)^2(1+(m-2)\rho-2(m-1)\rho)}{2\ell} + O(\frac{1}{\ell^2})}{m\rho(1-\rho)+\frac{\sqrt{m}(1-\rho)}{\sqrt{\ell}}+O(\frac{1}{\ell})}\\ &= \left(\frac{1-\rho}{\sqrt{\ell m}} + \frac{(1-\rho)(1-m\rho)}{2\ell m\rho} + O(\frac{1}{\ell^2})\right)\\ &\quad\times\left(1-\frac{1}{\sqrt{\ell m}\rho}+O(\frac{1}{\ell})\right)\\ &= \frac{1-\rho}{\sqrt{\ell m}} - \frac{(1-\rho)(1+m\rho)}{2\ell m\rho} + O(\frac{1}{\ell^{\frac{3}{2}}}).\end{aligned}$$

One can readily verify that

$$\begin{aligned}\overline{r}^{(\ell,m)}(d) &= -\frac{\ell-1}{2}\log\left(1-\frac{\theta^+(\gamma^{(\ell,m)})}{1-\rho}\right) + \frac{1}{4}\log\ell\\ &\quad + \frac{1}{2}\log\left(\rho+\frac{1-\rho}{\ell}\right)\\ &\quad - \frac{1}{2}\log\left(\frac{1-\rho+(\ell-1)\theta^+(\gamma^{(\ell,m)})}{\sqrt{\ell}}\right)\\ &= r_3^{(\ell,m)}(d).\end{aligned}$$

For $d \in (d_c^+, 1)$, we have $1-\rho-d<0$. It follows from (25) that

$$\begin{aligned}\gamma^{(\ell,m)} &= \frac{m\rho(d-1+\rho)}{2(1-d)}\frac{\ell^m}{(m-1)!} + \frac{h_3-h_2(1-d)}{2(1-d)}\frac{\ell^{m-1}}{(m-1)!} + \frac{m\rho(d-1+\rho)\sqrt{1+\frac{\zeta}{\ell m^2\rho^2(1-\rho-d)^2}+O(\frac{1}{\ell^2})}}{2(1-d)}\frac{\ell^m}{(m-1)!} + O(\ell^{m-2})\\ &= \frac{m\rho(d-1+\rho)}{2(1-d)}\frac{\ell^m}{(m-1)!} + \frac{h_3-h_2(1-d)}{2(1-d)}\frac{\ell^{m-1}}{(m-1)!} + \frac{m\rho(d-1+\rho)\left(1+\frac{\zeta}{2\ell m^2\rho^2(1-\rho-d)^2}\right)}{2(1-d)}\frac{\ell^m}{(m-1)!} + O(\ell^{m-2})\\ &= \frac{m\rho(d-1+\rho)}{1-d}\frac{\ell^m}{(m-1)!} + \left(\frac{h_3-h_2(1-d)}{1-d}+\frac{(1-\rho)d}{d-1+\rho}\right)\frac{\ell^{m-1}}{(m-1)!} + O(\ell^{m-2}).\end{aligned} \quad (26)$$

Substituting (26) into (24) gives

$$\theta^+(\gamma^{(\ell,m)}) = \frac{d-1+\rho+\frac{\mu}{\ell}+O(\frac{1}{\ell^2})}{1+\frac{\nu}{\ell}+O(\frac{1}{\ell^2})},$$

where

$$\mu \triangleq \frac{h_3 - h_2(1-d)}{m\rho} + \frac{(1-\rho)d(1-d)}{m\rho(d-1+\rho)} - \frac{(m-1)(1-\rho)(1-d)}{m\rho},$$

$$\nu \triangleq \frac{h_3}{m\rho^2} + \frac{(1-\rho)(1-d)}{m\rho^2(d-1+\rho)}.$$

Clearly, we have

$$\begin{aligned}
&\theta^+(\gamma^{(\ell,m)}) \\
&= \left(d-1+\rho+\frac{\mu}{\ell}+O(\frac{1}{\ell^2})\right)\left(1-\frac{\nu}{\ell}+O(\frac{1}{\ell^2})\right) \\
&= d-1+\rho+\frac{\mu-(d-1+\rho)\nu}{\ell}+O(\frac{1}{\ell^2}) \\
&= d-1+\rho+\left(\frac{h_3-h_2(1-d)}{\ell m\rho}+\frac{(1-\rho)d(1-d)}{\ell m\rho(d-1+\rho)}\right. \\
&\quad -\frac{(m-1)(1-\rho)(1-d)}{\ell m\rho}-\frac{h_3(d-1+\rho)}{\ell m\rho^2} \\
&\quad \left.-\frac{(1-\rho)(1-d)}{\ell m\rho^2}\right)+O(\frac{1}{\ell^2}) \\
&= d-1+\rho+\left(\frac{(h_3-h_2\rho)(1-d)}{\ell m\rho^2}+\frac{(1-\rho)d(1-d)}{\ell m\rho(d-1+\rho)}\right. \\
&\quad \left.-\frac{(m-1)(1-\rho)(1-d)}{\ell m\rho}-\frac{(1-\rho)(1-d)}{\ell m\rho^2}\right) \\
&\quad +O(\frac{1}{\ell^2}) \\
&= d-1+\rho+\left(\frac{(1-\rho)(1+(m-2)\rho)(1-d)}{\ell m\rho^2}\right. \\
&\quad +\frac{(1-\rho)d(1-d)}{\ell m\rho(d-1+\rho)}-\frac{(m-1)(1-\rho)(1-d)}{\ell m\rho} \\
&\quad \left.-\frac{(1-\rho)(1-d)}{\ell m\rho^2}\right)+O(\frac{1}{\ell^2}) \\
&= d-1+\rho+\frac{(1-\rho)^2(1-d)}{\ell m\rho(d-1+\rho)}+O(\frac{1}{\ell^2}).
\end{aligned}$$

One can readily verify that

$$\begin{aligned}
\overline{r}^{(\ell,m)}(d) &= \frac{1}{2}\log\frac{1+(\ell-1)\rho}{d+(\ell-1)\theta^+(\gamma^{(\ell,m)})} \\
&\quad -\frac{\ell-1}{2}\log\frac{d-\theta^+(\gamma^{(\ell,m)})}{1-\rho} \\
&= r_4^{(\ell,m)}(d).
\end{aligned}$$

This completes the proof of Theorem 3.

VIII. Numerical Results

Some numerical examples will be provided in this section to illustrate our main results. We focus on the case $\rho > 0$ since, in view of Theorem 1, the relevant plots are not particularly interesting when $\rho \leq 0$.

First we compare $\overline{r}^{(\ell,m)}(d)$ (the best known upper bound on $r^{(\ell,m)}(d)$), $1 < m < \ell$, with $r^{(\ell,\ell)}(d)$ (the rate-distortion function in the centralized setting), $r^{(\ell,1)}(d)$ (the rate-distortion function in the distributed setting), and $\underline{r}^{(\ell)}(d)$ (the Shannon lower bound). Fig. 2 illustrates the case $\ell = 3$ with $\rho = 0.6$.

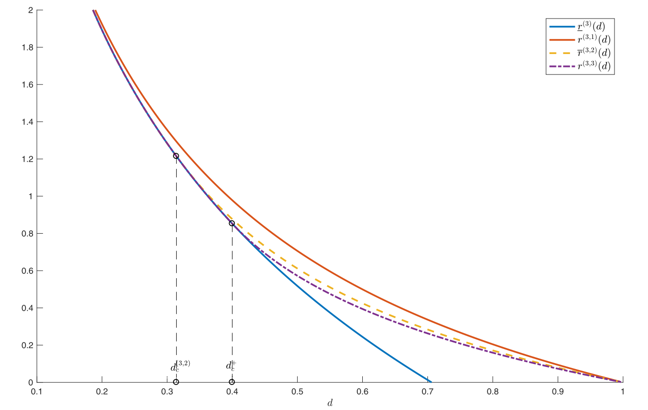

Fig. 2. An illustration of $\underline{r}^{(3)}(d)$, $r^{(3,1)}(d)$, $\overline{r}^{(3,2)}(d)$, and $r^{(3,3)}(d)$ with $\rho = 0.6$.

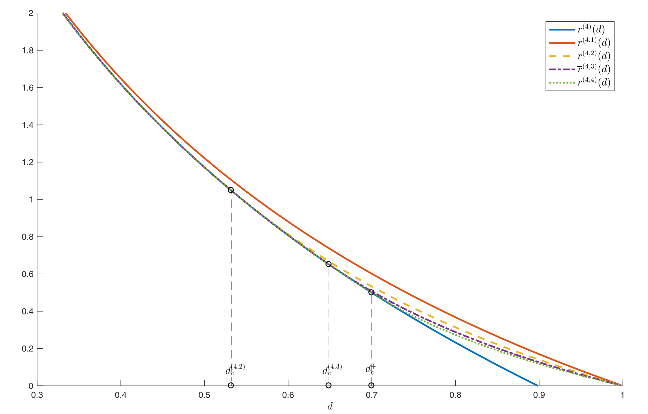

Fig. 3. An illustration of $\underline{r}^{(4)}(d)$, $r^{(4,1)}(d)$, $\overline{r}^{(4,2)}(d)$, $\overline{r}^{(4,3)}(d)$, and $r^{(4,4)}(d)$ with $\rho = 0.3$.

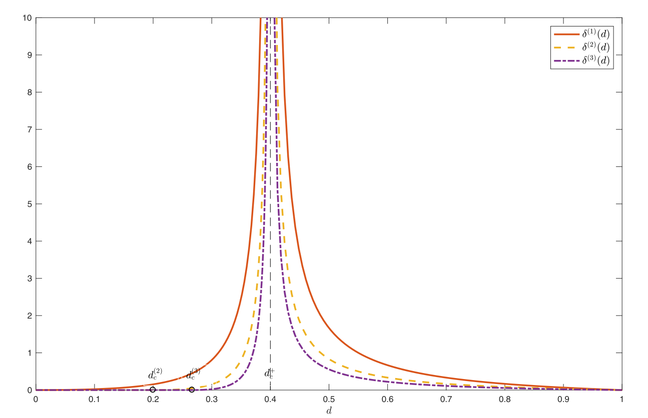

Fig. 4. An illustration of $\delta^{(1)}(d)$, $\delta^{(2)}(d)$, and $\delta^{(3)}(d)$ with $\rho = 0.6$.

It can be seen that $r^{(3,3)}(d)$ coincides with $\underline{r}^{(3)}(d)$ when $d \leq d_c^+ = 0.4$, and $\overline{r}^{(3,2)}(d)$ coincides with $r^{(3,3)}(d)$ as well as $\underline{r}^{(3)}(d)$ when $d \leq d_c^{(3,2)} = \frac{11}{35} \approx 0.314$. On the other hand, $r^{(3,1)}(d)$ is strictly above all the other curves for $d \in (0, 1)$. See a similar plot for the case $\ell = 4$ with $\rho = 0.3$ in Fig. 3, where $d_c^+ = 0.7$, $d^{(4,2)} = 0.532$, and $d^{(4,3)} = \frac{133}{205} \approx 0.649$.

Next we compare $\delta^{(m)}(d)$ for different values of m. Note that $\delta^{(m)}(d)$ indicates the asymptotic gap between $\overline{r}^{(\ell,m)}(d)$ and $r^{(\ell,\ell)}(d)$ in the large ℓ limit. Fig. 4 provides an illustration of $\delta^{(1)}(d)$, $\delta^{(2)}(d)$, and $\delta^{(3)}(d)$ with $\rho = 0.6$. It can be seen

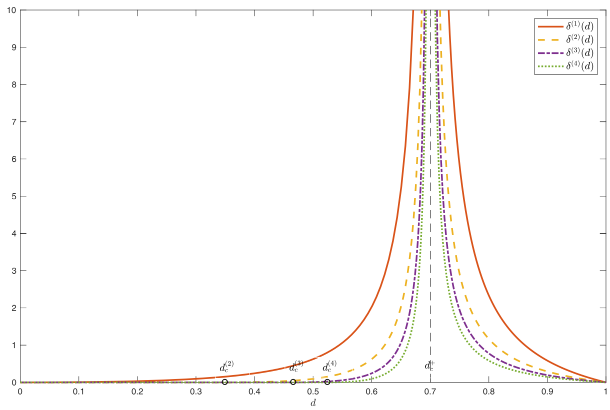

Fig. 5. An illustration of $\delta^{(1)}(d)$, $\delta^{(2)}(d)$, $\delta^{(3)}(d)$, and $\delta^{(4)}(d)$ with $\rho = 0.3$.

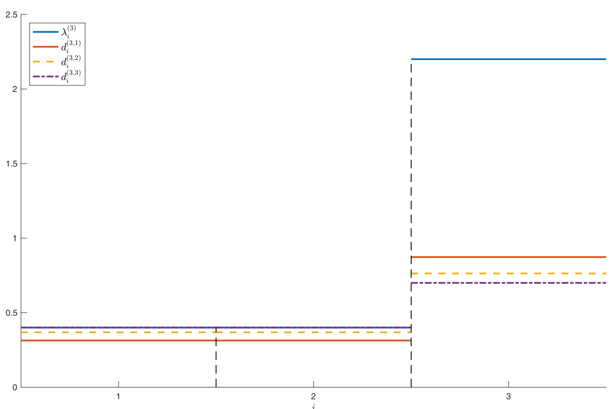

Fig. 6. An illustration of $\lambda_i^{(3)}$, $d_i^{(3,1)}$, $d_i^{(3,2)}$, and $d_i^{(3,3)}$, $i = 1, 2, 3$, with $\rho = 0.6$ and $d = 0.5$.

that all the curves blow up at at the critical distortion $d_c^+ = 0.4$. Moreover, we have $\delta^{(2)}(d) = 0$ when $d \leq d_c^{(2)} = 0.2$, and $\delta^{(3)}(d) = 0$ when $d \leq d_c^{(3)} = \frac{4}{15} \approx 0.267$. On the other hand, $\delta^{(1)}(d)$ is strictly above zero for $d \in (0, 1)$. See also a plot of $\delta^{(1)}(d)$, $\delta^{(2)}(d)$, $\delta^{(3)}(d)$, and $\delta^{(4)}(d)$ with $\rho = 0.3$ in Fig. 5, where $d_c^+ = 0.7$, $d_c^{(2)} = 0.35$, $d_c^{(3)} = \frac{7}{15} \approx 0.467$, and $d_c^{(4)} = 0.525$.

Finally we shall perform comparisons in the eigenspace. In view of (22), we have

$$\overline{r}^{(\ell,m)}(d) = \frac{1}{2} \log \frac{\det(\Sigma^{(\ell)})}{\det(D^{(\ell,m)})} = \sum_{i=1}^{\ell} \frac{1}{2} \log \frac{\lambda_i^{(\ell)}}{d_i^{(\ell,m)}},$$

where $D^{(\ell,m)}$ and $d_1^{(\ell,m)}$, $i = 1, \cdots, \ell$, are defined according to (3), (5), and (6). The discussion at the end of Section II-A suggests that $(d_1^{(\ell,m)}, \cdots, d_\ell^{(\ell,m)})$ can be interpreted as a certain form of distortion allocation in the eigenspace. In particular, $(d_1^{(\ell,\ell)}, \cdots, d_\ell^{(\ell,\ell)})$ corresponds to the celebrated reverse water-filling solution. Fig. 6 provides an illustration of $\lambda_i^{(3)}$, $d_i^{(3,1)}$, $d_i^{(3,2)}$, and $d_i^{(3,3)}$, $i = 1, 2, 3$, with $\rho = 0.6$ and $d = 0.5$. Since $d_c^+ = 0.4 < d$, the reverse water-filling solution leaves some dimensions uncoded; indeed, it can be seen that $d_i^{(3,3)} = \lambda_i^{(3)}$, $i = 1, 2$. In contrast, for $m = 1$ and $m = 2$, we have $d_i^{(3,m)} < \lambda_i^{(3)}$, $i = 1, 2, 3$, and consequently all dimensions are coded, which is suboptimal as compared to the reverse water-filling solution; nevertheless, increasing from $m = 1$ to $m = 2$ gets $(d_1^{(3,m)}, d_2^{(3,m)}, d_3^{(3,m)})$ closer to the reverse water-filling solution, resulting in an improved rate-distortion performance. Fig. 7 depicts $\lambda_i^{(4)}$, $d_i^{(4,1)}$, $d_i^{(4,2)}$, $d_i^{(4,3)}$, and $d_i^{(4,4)}$, $i = 1, 2, 3, 4$, with $\rho = 0.3$ and $d = 0.6$. Since $d_c^{(4,3)} \approx 0.649 > d$, it follows that $(d_1^{(4,3)}, d_2^{(4,3)}, d_3^{(4,3)}, d_4^{(4,3)})$ coincides with $(d_1^{(4,4)}, d_2^{(4,4)}, d_3^{(4,4)}, d_4^{(4,4)})$. That is to say, for such d, the encoders in a $(4, 3)$ generalized multiterminal source coding system can achieve the same effect as that of the reverse water-filling solution in the centralized setting even though they cannot fully cooperate.

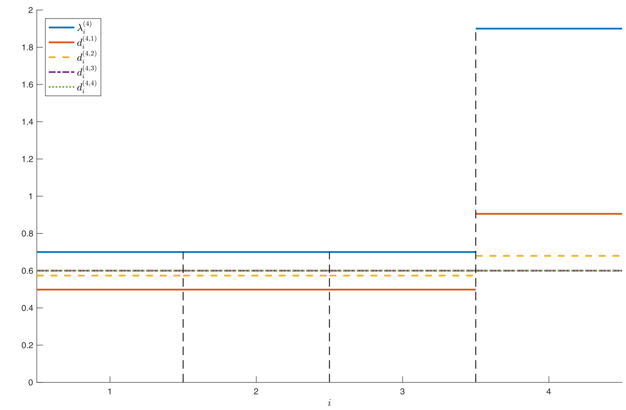

Fig. 7. An illustration of $\lambda_i^{(4)}$, $d_i^{(4,1)}$, $d_i^{(4,2)}$, $d_i^{(4,3)}$, and $d_i^{(4,4)}$, $i = 1, 2, 3, 4$, with $\rho = 0.3$ and $d = 0.6$.

IX. Conclusion

We have studied the rate-distortion limit of generalized multiterminal source coding of exchangeable Gaussian sources. Although a complete characterization of this limit has been obtained when the correlation coefficient is non-positive, a lot remains to be done for the positive correlation coefficient case. We conjecture that the upper bound established in the present work, i.e., $\overline{r}^{(\ell,m)}(d)$, is tight even when d is greater than $d_c^{(\ell,m)}$. However, a rigorous proof of this conjecture (even in the large ℓ limit) is likely to be non-trivial and may require new techniques yet to be developed.

We would like to mention that the proof of Theorems 1 and 2 was partly inspired by the consideration of the graphical model (more precisely, the Markov network) of a symmetric multivariate Gaussian distribution. It is of considerable interest to know whether a more conceptual proof can be constructed along that line. Moreover, probabilistic graphical models are expected to play an essential role in identifying the non-Gaussian counterpart of our problem and establishing the corresponding results.

Appendix A
Proof of Proposition 4

Let $\hat{X}_i^-(\gamma) \triangleq \mathbb{E}[X_i | U_{\mathcal{S},1}^-(\gamma), \cdots, U_{\mathcal{S},m}^-(\gamma), \mathcal{S} \in \mathcal{I}^{(\ell,m)}]$, $i = 1, \cdots, \ell$. We shall first prove that

$$\hat{X}_i^-(\gamma) = \kappa \sum_{\mathcal{S} \in \mathcal{I}^{(\ell,m)} : i \in \mathcal{S}} U_{\mathcal{S},\tau(i)}^-(\gamma), \quad i = 1, \cdots, \ell,$$

where $\tau(i)$ indicates the position of i in $\mathcal{S}$ when the elements of $\mathcal{S}$ are arranged in ascending order, and

$$\kappa \triangleq \frac{1-\rho}{\gamma + \binom{\ell-2}{m-2}\ell(1-\rho)}.$$

It suffices to verify that, for any $\mathcal{S}' \in \mathcal{I}^{(\ell,m)}$ and $i' \in \mathcal{S}'$,

$$\mathbb{E}\left[\left(X_i - \kappa \sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}} U^-_{\mathcal{S},\tau(i)}(\gamma)\right) U^-_{\mathcal{S}',\tau(i')}(\gamma)\right] = 0, \quad i = 1,\cdots,\ell. \tag{27}$$

Note that

$$\begin{aligned} X_i &- \kappa \sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}} U^-_{\mathcal{S},\tau(i)}(\gamma) \\ &= \left(1 - \kappa\binom{\ell-2}{m-2}\ell\right) X_i + \kappa\binom{\ell-2}{m-2}\sum_{j=1}^{\ell} X_j \\ &\quad - \kappa\sqrt{\gamma}\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}} N^-_{\mathcal{S},\tau(i)}. \end{aligned} \tag{28}$$

One can readily compute that

$$\mathbb{E}[X_i U^-_{\mathcal{S}',\tau(i')}(\gamma)] = \begin{cases} (m-1)(1-\rho), & i = i', \\ -(1-\rho), & i \in \mathcal{S}', i \neq i', \\ 0, & i \notin \mathcal{S}', \end{cases} \tag{29}$$

$$\sum_{j=1}^{\ell} \mathbb{E}[X_j U^-_{\mathcal{S}',\tau(i')}(\gamma)] = 0, \tag{30}$$

$$\begin{aligned} &\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}} \mathbb{E}[N^-_{\mathcal{S},\tau(i)} U^-_{\mathcal{S}',\tau(i')}(\gamma)] \\ &= \begin{cases} (m-1)\sqrt{\gamma}, & i = i', \\ -\sqrt{\gamma}, & i \in \mathcal{S}', i \neq i', \\ 0, & i \notin \mathcal{S}'. \end{cases} \end{aligned} \tag{31}$$

Combining (28), (29), (30), and (31) gives (27).

For $i = 1,\cdots,\ell$,

$$\begin{aligned} &\mathbb{E}[(X_i - \hat{X}^-_i(\gamma))^2] \\ &= \mathbb{E}[(X_i - \hat{X}^-_i(\gamma))X_i] - \mathbb{E}[(X_i - \hat{X}^-_i(\gamma))\hat{X}^-_i(\gamma)] \\ &= \mathbb{E}[(X_i - \hat{X}^-_i(\gamma))X_i] \qquad (32) \\ &= \left(1 - \kappa\binom{\ell-2}{m-2}\ell\right)\mathbb{E}[X_i^2] \\ &\quad + \kappa\binom{\ell-2}{m-2}\sum_{j=1}^{\ell}\mathbb{E}[X_j X_i] \\ &\quad - \kappa\sqrt{\gamma}\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}} \mathbb{E}[N^-_{\mathcal{S},\tau(i)} X_i] \qquad (33) \\ &= 1 - \kappa\binom{\ell-2}{m-2}\ell + \kappa\binom{\ell-2}{m-2}(1 + (\ell-1)\rho) \\ &= d^-(\gamma), \end{aligned}$$

where (32) and (33) are due to (27) and (28), respectively. Moreover, for $i, i' \in \{1,\cdots,\ell\}$ with $i \neq i'$,

$$\begin{aligned} &\mathbb{E}[(X_i - \hat{X}^-_i(\gamma))(X_{i'} - \hat{X}^-_{i'}(\gamma))] \\ &= \mathbb{E}[(X_i - \hat{X}^-_i(\gamma))X_{i'}] - \mathbb{E}[(X_i - \hat{X}^-_i(\gamma))\hat{X}^-_{i'}(\gamma)] \\ &= \mathbb{E}[(X_i - \hat{X}^-_i(\gamma))X_{i'}] \qquad (34) \\ &= \left(1 - \kappa\binom{\ell-2}{m-2}\ell\right)\mathbb{E}[X_i X_{i'}] \\ &\quad + \kappa\binom{\ell-2}{m-2}\sum_{j=1}^{\ell}\mathbb{E}[X_j X_{i'}] \\ &\quad - \kappa\sqrt{\gamma}\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}} \mathbb{E}[N^-_{\mathcal{S},\tau(i)} X_{i'}] \qquad (35) \\ &= \rho - \kappa\binom{\ell-2}{m-2}\ell\rho + \kappa\binom{\ell-2}{m-2}(1 + (\ell-1)\rho) \\ &= \theta^-(\gamma), \end{aligned}$$

where (34) and (35) are due to (27) and (28), respectively. This completes the proof of Proposition 4.

APPENDIX B
PROOF OF PROPOSITION 5

Let $\hat{X}^+_i(\gamma) \triangleq \mathbb{E}[X_i | U^+_{\mathcal{S}}(\gamma), \mathcal{S} \in \mathcal{I}^{(\ell,m)}]$, $i = 1,\cdots,\ell$. We shall first prove that

$$\hat{X}^+_i(\gamma) = \alpha \sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}} U^+_{\mathcal{S}}(\gamma) + \beta \sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\notin\mathcal{S}} U^+_{\mathcal{S}}(\gamma), \quad i = 1,\cdots,\ell,$$

where

$$\alpha \triangleq \frac{(1 + (m-1)\rho)\gamma + \binom{\ell-2}{m-1}m(1-\rho)(1 + (\ell-1)\rho)}{\gamma^2 + \eta_2\gamma + \eta_1},$$

$$\beta \triangleq \frac{m\rho\gamma - \binom{\ell-2}{m-2}m(1-\rho)(1 + (\ell-1)\rho)}{\gamma^2 + \eta_2\gamma + \eta_1}.$$

It suffices to verify that, for any $\mathcal{S}' \in \mathcal{I}^{(\ell,m)}$,

$$\mathbb{E}\left[\left(X_i - \alpha\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}} U^+_{\mathcal{S}}(\gamma) - \beta\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\notin\mathcal{S}} U^+_{\mathcal{S}}(\gamma)\right) U^+_{\mathcal{S}'}(\gamma)\right] = 0, \quad i = 1,\cdots,\ell. \tag{36}$$

Note that

$$\begin{aligned} X_i &- \alpha\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}} U^+_{\mathcal{S}}(\gamma) - \beta\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\notin\mathcal{S}} U^+_{\mathcal{S}}(\gamma) \\ &= \left(1 - \alpha\binom{\ell-1}{m-1} + \alpha\binom{\ell-2}{m-2} + \beta\binom{\ell-2}{m-1}\right) X_i \\ &\quad - \left(\alpha\binom{\ell-2}{m-2} + \beta\binom{\ell-2}{m-1}\right)\sum_{j=1}^{\ell} X_j \\ &\quad - (\alpha - \beta)\sqrt{\gamma}\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}} N^+_{\mathcal{S}} \\ &\quad - \beta\sqrt{\gamma}\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}} N^+_{\mathcal{S}}. \end{aligned} \tag{37}$$

One can readily compute that

$$\mathbb{E}[X_i U^+_{\mathcal{S}'}(\gamma)] = \begin{cases} 1 + (m-1)\rho, & i \in \mathcal{S}', \\ m\rho, & i \notin \mathcal{S}', \end{cases} \tag{38}$$

$$\sum_{j=1}^{\ell}\mathbb{E}[X_jU_{\mathcal{S}'}^{+}(\gamma)] = m(1+(\ell-1)\rho), \tag{39}$$

$$\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}}\mathbb{E}[N_{\mathcal{S}}^{+}U_{\mathcal{S}'}^{+}(\gamma)] = \begin{cases}\sqrt{\gamma}, & i\in\mathcal{S}',\\ 0, & i\notin\mathcal{S}',\end{cases} \tag{40}$$

$$\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}}\mathbb{E}[N_{\mathcal{S}}^{+}U_{\mathcal{S}'}^{+}(\gamma)] = \sqrt{\gamma}. \tag{41}$$

Combining (37), (38), (39), (40), and (41) gives (36).

For $i=1,\cdots,\ell$,

$$\begin{aligned}
&\mathbb{E}[(X_i-\hat{X}_i^{+}(\gamma))^2]\\
&=\mathbb{E}[(X_i-\hat{X}_i^{+}(\gamma))X_i]-\mathbb{E}[(X_i-\hat{X}_i^{+}(\gamma))\hat{X}_i^{+}(\gamma)]\\
&=\mathbb{E}[(X_i-\hat{X}_i^{+}(\gamma))X_i] \qquad (42)\\
&=\left(1-\alpha\binom{\ell-1}{m-1}+\alpha\binom{\ell-2}{m-2}+\beta\binom{\ell-2}{m-1}\right)\mathbb{E}[X_i^2]\\
&\quad-\left(\alpha\binom{\ell-2}{m-2}+\beta\binom{\ell-2}{m-1}\right)\sum_{j=1}^{\ell}\mathbb{E}[X_jX_i]\\
&\quad-(\alpha-\beta)\sqrt{\gamma}\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}}\mathbb{E}[N_{\mathcal{S}}^{+}X_i]\\
&\quad-\beta\sqrt{\gamma}\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}}\mathbb{E}[N_{\mathcal{S}}^{+}X_i] \qquad (43)\\
&=1-\alpha\binom{\ell-1}{m-1}+\alpha\binom{\ell-2}{m-2}+\beta\binom{\ell-2}{m-1}\\
&\quad-\left(\alpha\binom{\ell-2}{m-2}+\beta\binom{\ell-2}{m-1}\right)(1+(\ell-1)\rho)\\
&=d^{+}(\gamma),
\end{aligned}$$

where (42) and (43) are due to (36) and (37), respectively. Moreover, for $i,i'\in\{1,\cdots,\ell\}$ with $i\neq i'$,

$$\begin{aligned}
&\mathbb{E}[(X_i-\hat{X}_i^{+}(\gamma))(X_{i'}-\hat{X}_{i'}^{+}(\gamma))]\\
&=\mathbb{E}[(X_i-\hat{X}_i^{+}(\gamma))X_{i'}]-\mathbb{E}[(X_i-\hat{X}_i^{+}(\gamma))\hat{X}_{i'}^{+}(\gamma)]\\
&=\mathbb{E}[(X_i-\hat{X}_i^{+}(\gamma))X_{i'}] \qquad (44)\\
&=\left(1-\alpha\binom{\ell-1}{m-1}+\alpha\binom{\ell-2}{m-2}+\beta\binom{\ell-2}{m-1}\right)\mathbb{E}[X_iX_{i'}]\\
&\quad-\left(\alpha\binom{\ell-2}{m-2}+\beta\binom{\ell-2}{m-1}\right)\sum_{j=1}^{\ell}\mathbb{E}[X_jX_{i'}]\\
&\quad-(\alpha-\beta)\sqrt{\gamma}\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}:i\in\mathcal{S}}\mathbb{E}[N_{\mathcal{S}}^{+}X_{i'}]\\
&\quad-\beta\sqrt{\gamma}\sum_{\mathcal{S}\in\mathcal{I}^{(\ell,m)}}\mathbb{E}[N_{\mathcal{S}}^{+}X_{i'}] \qquad (45)\\
&=\rho-\alpha\binom{\ell-1}{m-1}\rho+\alpha\binom{\ell-2}{m-2}\rho+\beta\binom{\ell-2}{m-1}\rho\\
&\quad-\left(\alpha\binom{\ell-2}{m-2}+\beta\binom{\ell-2}{m-1}\right)(1+(\ell-1)\rho)\\
&=\theta^{+}(\gamma),
\end{aligned}$$

where (44) and (45) are due to (36) and (37), respectively. This completes the proof of Proposition 5.

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