Chapter II Problems Solutions

1. • a) Prove that the optimum choice of the threshold V_T when the outputs are Gaussian variables centered at s_{o1} and s_{02} is given by

$$V_T = \frac{s_{o1} + s_{o2}}{2}$$

b) Prove that $|\rho| \leq 1$ in Eq.(II.2.41)

• a)Suppose we choose any level V_T as the threshold. Yhen the probability of error is :

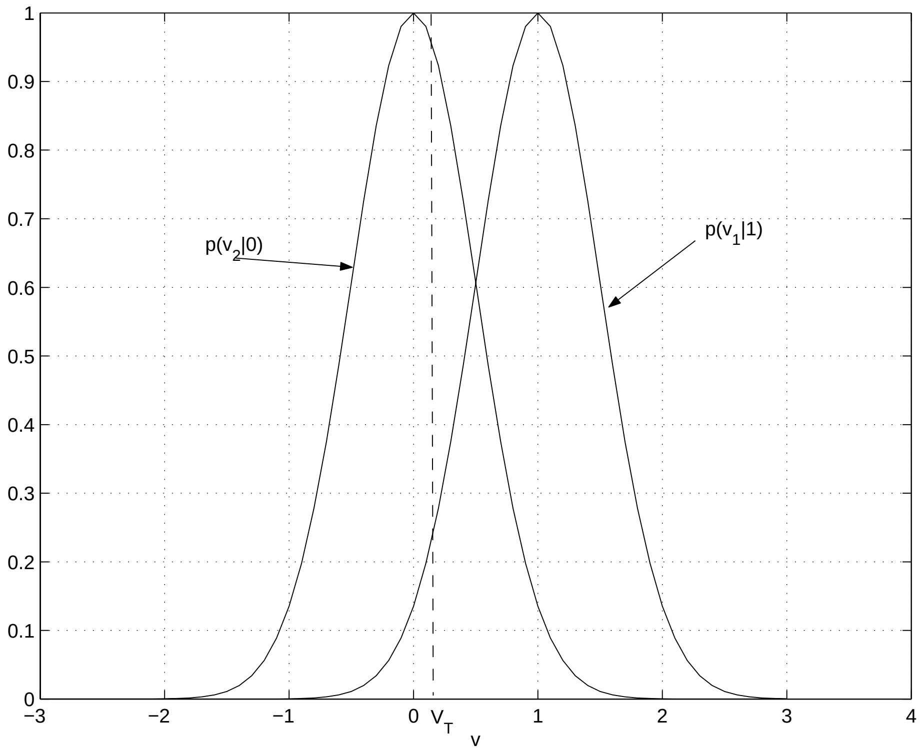


Figure 1:

$$P_e = \int_{-\infty}^{V_T} P(1)p(v_1|1)dv_1 + \int_{V_T}^{\infty} P(0)p(v_2|0)dv_2$$

$$= \frac{1}{2}\left[\int_{-\infty}^{V_T} p(v_1|1)dv_1 + \int_{V_T}^{\infty} p(v_2|0)dv_2\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{V_T} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v_1-s_{o1})^2}{2\sigma^2}} dv_1 + \int_{V_T}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v_2-s_{o2})^2}{2\sigma^2}} dv_2\right]$$

To obtain the optimum V_T, we find $\frac{\partial P_e}{\partial V_T}$ and equate it to zero.

$$\frac{\partial P_e}{\partial V_T} = \frac{1}{2}[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(V_T-s_{o1})^2}{2\sigma^2}} - \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(V_2-s_{o2})^2}{2\sigma^2}}] = 0$$

Therefore,

$$(V_{Top} - s_{o1})^2 = (V_{Top} - s_{o2})^2 \implies V_{Top} = \frac{s_{o1} + s_{o2}}{2}$$

b)

$$\rho = \frac{\int_0^{T_b} s_1(t)s_2(t)dt}{\frac{1}{2}\int_0^{T_b}[s_1^2(t) + s_2^2(t)]dt}$$

Consider the integral

$$I = \int_0^{T_b} [s_1(t) - s_2(t)]^2 dt$$

Since the integrand $[s_1(t) - s_2(t)]^2$ is always positive, then,

$$I = \int_0^{T_b} [s_1(t) - s_2(t)]^2 dt \geq 0$$

i.e,

$$\int_0^{T_b} [s_1^2(t) + s_2^2(t)]dt - \int_0^{T_b} 2s_1(t)s_2(t)dt \geq 0$$

$$\implies \int_0^{T_b} [s_1^2(t) + s_2^2(t)]dt \geq \int_0^{T_b} 2s_1(t)s_2(t)dt$$

$$\implies |\rho| = \frac{|\int_0^{T_b} s_1(t)s_2(t)dt|}{\frac{1}{2}\int_0^{T_b}[s_1^2(t) + s_2^2(t)]dt} \leq 1$$

2. • $n(t)$ is a zero mean Gaussian white noise with a psd of $\eta/2$. $n_0(T_b)$ is related to $n(t)$ by

$$n_0(T_b) = \int_0^{T_b} s(t)n(t)dt$$

where $s(t) = 0$ outside the interval $[0, T_b]$ and

$$\int_0^{T_b} s^2(t)dt = E_s$$

Show that

$$\mathrm{E}\{n_0(T_b)\} = 0 \text{ and } \mathrm{E}\{n_0^2(T_b)\} = \eta E_s/2$$

•

$$\mathrm{E}\{n_0(T_b)\} = \mathrm{E}\{\int_0^{T_b} n(t)s(t)dt\} = \int_0^{T_b} \mathrm{E}\{n(t)\}s(t)dt = 0, \text{ since } \mathrm{E}\{n(t)\} = 0$$

$$\mathrm{E}\{n_0^2(T_b)\} = \mathrm{E}\{\int_0^{T_b} s(t_1)n(t_1)dt_1 \int_0^{T_b} s(t_2)n(t_2)dt_2\}$$

$$= \int_0^{T_b}\int_0^{T_b} \mathrm{E}\{n(t_1))n(t_2)\}s(t_1)s(t_2)dt_1dt_2$$

But,

$$\mathrm{E}\{n(t_1))n(t_2)\} = \frac{\eta}{2}\delta(t_1 - t_2) \quad \text{(White Noise)}$$

Therefore,

$$\mathrm{E}\{n_0^2(T_b)\} = \frac{\eta}{2}\delta(t_1 - t_2)s(t_1)s(t_2)dt_1dt_2$$

$$= \frac{\eta}{2}\int_0^{T_b} s(t_1)\{\int_0^{T_b} s(t_2)\delta(t_1 - t_2)dt_2\}dt_1$$

$$= \frac{\eta}{2}\int_0^{T_b} s(t_1)s(t_1)dt_1 = \eta E_s/2$$

3. • A statistically independent sequence of equiprobable binary digits is transmitted over a channel having finite bandwidth using rectangular signaling waveform shown in Fig. 2. The bit rate is r_b and the channel noise is white Gaussian with a psd of $\eta/2$.

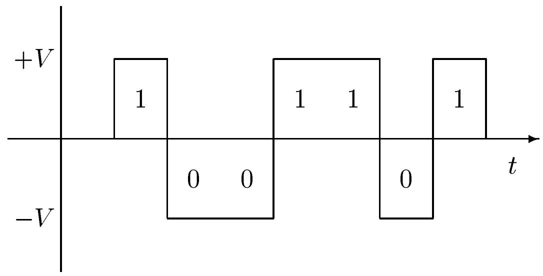


Figure 2:

(a) Derive the structure of an optimum receiver for this signaling scheme.

(b) Derive an expression for the probability of error.

• Figure 3 shows the optimum receiver. Since the multiplier is constant, the receiver is essentially an integrator whose output is sampled at time instants nT_b.

Neglecting the multiplier,

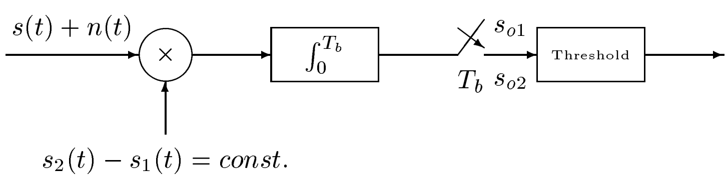


Figure 3:

$$s_o1(T_b) = \int_0^{T_b} s_1(t)dt = -VT_b$$

Similarly,

$$s_o2(T_b) = \int_0^{T_b} s_1(t)dt = VT_b$$

$$\implies [s_o2(T_b) - s_o1(T_b)]^2 = 4V^2T_b^2$$

From problem II.2,

$$\sigma_{n_o}^2 = \frac{\eta}{2} E_s = \frac{\eta}{2} T_b$$

Therefore, using Eq.(II.2.16), we get

$$P_e = \Psi_c(\sqrt{\frac{V^2 T_b^2}{\eta T_b/2}}) = \Psi_c(\sqrt{\frac{2V^2 T_b}{\eta}})$$

4. • In Problem II.3, assume that the channel noise has a psd $S_n(\omega)$ given by

$$S_n(\omega) = \frac{A}{1 + (\omega/\omega_1)^2}$$

(a) Find the transfer function of the optimum receiver and calculate P_e.

(b) If an integrate and dump receiver is used instead of the optimum receiver, find P_e and compare with the P_e for the optimum receiver.

• a) The power spectral density of noise is given by

$$S_n(\omega) = \frac{A}{1 + (\omega/\omega_1)^2}$$

According to Eq.(II.2.29), the optimum filter has the transfer function

$$H(\omega) = k\frac{[S_2^*(\omega) - S_2^*(\omega)]e^{-j\omega T_b}}{S_n(\omega)}$$

$$= k\frac{[S_2^*(\omega) - S_2^*(\omega)]e^{-j\omega T_b}}{A}.\{1 + \left(\frac{\omega}{\omega_1}\right)^2\}$$

Notice that this filter is made up of two parts,

$$H_1(\omega) = \frac{1}{A}.\{1 + \left(\frac{\omega}{\omega_1}\right)^2\}$$

and

$$H_2(\omega) = k[S_2^*(\omega) - S_2^*(\omega)]e^{-j\omega T_b}$$

The function $H_1(\omega)$ is to convert the non-white noise to white noise and $H_2(\omega)$, is by comparison to Eq.(II.2.32), a matched filter. Hence the optimum filter will consist of a pre-whitening filter $H_1(\omega)$ in cascade with a matched filter. The optimum threshold is set at $V_{Top} = 0$,

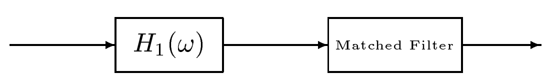


Figure 4: Optimum Filter

From Eq.(II.2.30) we have,

$$\gamma_{max}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|S_2^*(\omega) - S_1^*(\omega)|^2}{S_n(\omega)}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1+\left(\frac{\omega}{\omega_1}\right)^2}{A}|S_2^*(\omega)-S_1^*(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi A}\int_{-\infty}^{\infty}|S_2^*(\omega)-S_1^*(\omega)|^2 d\omega + \underbrace{\frac{1}{2\pi A}\int_{-\infty}^{\infty}\left(\frac{\omega}{\omega_1}\right)^2|S_2^*(\omega)-S_1^*(\omega)|^2 d\omega}_{Differentiator}$$

$$= \frac{4VT_b}{A}+\infty$$

Hence $\gamma_{max}^2 \to \infty$, and $P_e \to 0$.
b) If we use an integrate and dump circuit, then γ will not be maximum.

$$\gamma = \frac{s_{o2}(T_b)-s_{o1}(T_b)}{\sqrt{N_o}} = \frac{1}{\sqrt{N_o}}\int_0^{T_b}(s_2(t)-s_1(t))dt = \frac{2VT_b}{\sqrt{N_o}}$$

To evaluate N_o, we need to find the transfer function of the integrate-and-dump circuit.
Consider a filter with impulse response

$$h(t) = u(t) - u(t-T_b)$$

The output of this filter is

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)[u(t-\tau)-u(t-\tau-T_b)]d\tau$$

Now,

$$y(T_b) = \int_{-\infty}^{\infty} x(\tau)[u(T_b-\tau)-u(-\tau)]d\tau = \int_0^{T_b} x(\tau)d\tau$$

This is the operation of the integrate and dump circuit. hence we can conclude that the impulse response of an integrate and dump circuit to be

$$h(t) = u(t) - u(t-T_b)$$

Hence,

$$H(\omega) = T_b\frac{\sin\omega T_b/2}{\omega T_b/2}e^{-j\omega T_b/2} \implies |H(\omega)|^2 = T_b^2\frac{\sin^2\omega T_b/2}{\omega^2 T_b^2/4}$$

$$\implies N_o = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(\omega)|H(\omega)|^2 d\omega = \frac{AT_b^2}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\left(\frac{\omega}{\omega_1}\right)^2}\frac{\sin^2\omega T_b/2}{\omega^2 T_b^2/4}d\omega$$

This integral is not easy to calculate. But we see that the integral is greater than zero and is finite.I.e.

$$0 < N_o < \infty \implies \gamma = \frac{2VT_b}{\sqrt{N_o}} < \infty \implies P_e > 0$$

5. • A received signal is $\pm$ mV for T_b second interval with equal probability. The signal is accompanied by white Gaussian noise with a psd of 10^{10} watt/Hz. The receiver integrates the signal plus noise synchronously for T_b second duration and decodes the signal by comparing the integrator output with 0.
a) Find the maximum signaling rate (minimum value of T_b) such that $P_e = 10^{-4}$.
b) If actual signaling takes place at 1/2 the rate found in (a), what is the signal amplitude required to maintain $P_e - 10^{-4}$?

• (a) $V = 10^{-3}$, $\eta/2 = 10^{-10}$
From from problem (II.3),

$$P_e = \Psi_c(\sqrt{\frac{2V^2T_b}{\eta}}) \leq 10^{-4}$$

From table,

$$\sqrt{\frac{2V^2T_b}{\eta}} \geq 3.7 \quad \Longrightarrow \quad r_b = \frac{1}{T_b} \leq 730 \; bits/sec$$

(b) With $r_b = 365 \; bits/sec$, and $P_e \leq 10^{-4}$, we have,

$$\frac{2V^2}{\eta}(\frac{1}{365}) \geq (3.7)^2 \quad \Longrightarrow \quad V = 0.7mV$$

6. • Refering to Eq.(II.2.38), we have the signal-to-noise power ratio at the output of a matched filter receiver as:

$$\gamma_{max}^2 = \frac{2}{\eta}\int_0^{T_b}[s_1(t) - s_2(t)]^2 dt$$

Now suppose that we want $s_1(t)$ and $s_2(t)$ to have the same signal energy. Show that the optimum choice of $s_2(t)$ is: $s_2(t) = -s_1(t)$. With this choice of $s_2(t)$, show that

$$\gamma_{max}^2 = \frac{8}{\eta}\int_0^{T_b} s_1(t)dt$$

•

$$\gamma_{max}^2 = \frac{2}{\eta}\int_0^{T_b}[s_1^2(t) - 2s_1(t)s_2(t) + s_2(t)^2]dt$$

$$= k[2E_s - 2\int_0^{T_b} s_1(t)s_2(t)dt]$$

To maximize γ_{max}^2, we need to minimize $\int_0^{T_b} s_1(t)s_2(t)dt$. Using Schwartz inequality,

$$\left|\int_0^{T_b} s_1(t)s_2(t)dt\right| \leq \sqrt{\int_0^{T_b} s_1^2(t)dt \int_0^{T_b} s_2^2(t)dt}$$

Since

$$\int_0^{T_b} s_1^2(t)dt = \int_0^{T_b} s_2^2(t)dt = E_s$$

$$\Longrightarrow \quad \left|\int_0^{T_b} s_1(t)s_2(t)dt\right| \leq E_s$$

Equality holds when $s_1(t) = k_1 s_2(t)$. But $s_1(t)$ and $s_2(t)$ have equal energy. Therefore $K_1 = \pm 1$ Thus the minimum value of $\int_0^{T_b} s_1(t)s_2(t)dt$ is achieved when $s_1(t) = -s_2(t)$. Therefore,

$$\gamma_{max}^2 = \frac{2}{\eta}[4E_s] = \frac{8}{\eta}E_s$$

7. • An on-off binary system uses the following waveforms:

$$s_2(t) = \begin{cases} \frac{2t}{T_b} & 0 < t < \frac{T_b}{2} \\ 2 - \frac{2t}{T_b} & \frac{T_b}{2} \le t < T_b \end{cases} \qquad s_1(t) = 0$$

Assume that $T_b = 20\ \mu sec$, and the noise psd is $\frac{\eta}{2} = 10^{-7}$ watt/Hz. FInd P_e for the optimum receiver assuming

$$P(0\ sent) = \frac{1}{4},\ P(1\ sent) = \frac{3}{4}$$

• A matched filter is an optimum receiver.Therefore, $s_{o1}(T_b) = 0$

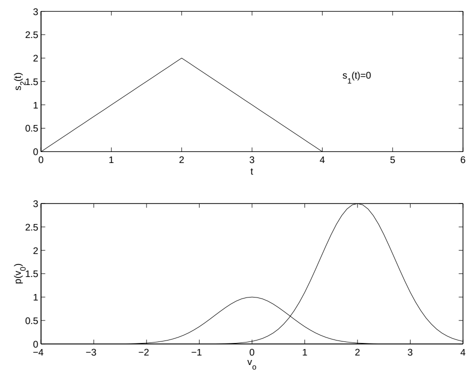


Figure 5:

$$s_{o2}(T_b) = \int_0^{T_b} s_2^2(t)dt = 2\int_0^{T_b/2} (2t/T_b)^2 dt = \frac{T_b}{3}$$

From Problem (II.2), we know that:

$$N_o = \sigma_{n_o}^2 = \frac{\eta}{2}E_s = \frac{\eta T_b}{6}$$

The output v_o of the matched filter is a Gaussian random variable such that:

$$p_{V_{o1}}(v_{o1}|s_1) = \frac{1}{\sqrt{2\pi N_o}} e^{-v_{o1}^2/2N_o}$$

Similarly,

$$p_{V_{o2}}(v_{o2}|s_1) = \frac{1}{\sqrt{2\pi N_o}} e^{-(v_{o2}-\frac{T_b}{3})^2/2N_o}$$

Therefore,

$$p_{V_{o1}}(v_{o1}) = p_{V_{o1}}(v_{o1}|s_1)P(s_1) = \frac{1}{4}\frac{1}{\sqrt{2\pi N_o}} e^{-v_{o1}^2/2N_o}$$

and,

$$p_{V_{o2}}(v_{o2}) = p_{V_{o2}}(v_{o2}|s_2)P(s_2) = \frac{3}{4}\frac{1}{\sqrt{2\pi N_o}}e^{-(v_{o2}-\frac{T_b}{3})^2/2N_o}$$

These two curves are shown in figure (5). The optimum threshold V_{Topt} can be shown to be when the two curves cross. That is:

$$\frac{1}{\sqrt{2\pi N_o}}e^{-V_{Topt}^2/2N_o} = \frac{3}{4}\frac{1}{\sqrt{2\pi N_o}}e^{-(V_{Topt}-\frac{T_b}{3})^2/2N_o}$$

$$\Longrightarrow \quad V_{Topt} = \frac{3N_o}{T_b}[\ln(1/3) + \frac{(T_b/3)^2}{2N_o}] = \frac{T_b}{6} - \frac{\eta}{2}\ln 3 \simeq \frac{T_b}{6}$$

$$P_e = \int_{V_{Topt}}^{\infty} \frac{1}{4\sqrt{2\pi N_o}}e^{-v_{o1}^2/2N_o}dv_{o1} + \int_{-\infty}^{V_{Topt}} \frac{3}{4\sqrt{2\pi N_o}}e^{-(v_{o2}-\frac{T_b}{3})^2/2N_o}dv_{o2}$$

$$= \frac{1}{4}\Psi_c(\frac{T_b/6}{\sqrt{N_o}}) + \frac{3}{4}\Psi_c(\frac{T_b/6}{\sqrt{N_o}}) = \Psi_c(\frac{T_b/6}{\sqrt{N_o}}) = \Psi_c(4.08) \simeq 2\times 10^{-5} \text{ (from table)}$$

8. • In a binary scheme using correlation receiver, the local carrier waveform is $A\cos(\omega_c t+\phi)$ instead of $A\cos(\omega_c t)$ due to poor synchronization. Deriver an expression for the probability of error and compute the increase in error probability when $\phi = 15°$ and $A^2T_b/\eta = 10$.

•

$$s_{o1}(T_b) = -\int_{t=0}^{T_b}[A\cos(\omega_c t+\phi)]A\cos(\omega_c t)dt = -\frac{A^2T_b\cos(\phi)}{2}$$

Also

$$s_{o2}(T_b) = \frac{A^2T_b\cos(\phi)}{2}$$

where $\omega_c = \frac{2n\pi}{T_b}$

$$\sigma_{n_o}^2 = \frac{\eta}{2}\int_0^{T_b} A^2\cos^2(\omega_c t+\phi)dt = \frac{\eta A^2 T_b}{2.2}$$

$$\gamma_{max}^2 = \frac{(s_{o2}-s_{o1})^2}{\sigma_{n_o}^2} = \frac{4A^2T_b\cos^2(\phi)}{\eta}$$

$$P_e = \Psi_c(\sqrt{\frac{\gamma_{max}}{2}}) = \Psi_c(\sqrt{\frac{A^2T_b\cos^2(\phi)}{\eta}})$$

For $\frac{A^2T_b}{\eta} = 10$,

$$\phi = \mathbf{0}, \qquad P_e \simeq 0.0008$$

$$\phi = \mathbf{15}°, \qquad P_e \simeq 0.0014$$

9. • In a coherent binary PSK system, the peak carrier amplitude at the receiver, A, varies slowly due to fading. Assume that A has a *pdf*:

$$p_A(a) = \frac{a}{A_0^2}\exp(-\frac{a^2}{2A_0^2}), \quad a \geq 0$$

(a) Find the mean and standard deviation of A.

(b) Find the average probability of error P_e.

• (a) The mean of a is

$$\mathrm{E}\{a\} = \int_0^\infty a p_A(a) da = \int_0^\infty \frac{a^2}{A_0^2} e^{-a^2/2A_0^2} da$$

Let $z = a/A_0$, then $da = A_0 dz$, hence,

$$\mathrm{E}\{a\} = A_0 \int_0^\infty z^2 e^{-z^2/2} dz = A_0 \int_0^\infty z(ze^{-z^2/2})dz$$

Noting that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$

Integrating by parts we get,

$$\mathrm{E}\{a\} = A_0 \sqrt{\frac{\pi}{2}}$$

The mean square of a is:

$$\mathrm{E}\{a^2\} = \int_0^\infty a^2 p_A(a) da = \int_0^\infty \frac{a^3}{A_0^2}\exp(-\frac{a^2}{2A_0^2}) da$$

Again, let $z = a/A_0$, therefore,

$$\mathrm{E}\{a^2\} = A_0^2 \int_0^\infty z^2(ze^{-z^2/2})dz = 2A_0^2$$

Hence the variance of a is

$$\sigma_a^2 = \mathrm{E}\{a^2\} - \mathrm{E}^2\{a\} = \frac{4-\pi}{2} A_0^2$$

(b) Given $A = a$, then the probability of error using a matched filter is given by:

$$P(e|A=a) = \Psi_c(\sqrt{\frac{a^2 T_b}{\eta}}) = \frac{1}{2\pi} \int_{\sqrt{\frac{a^2 T_b}{\eta}}}^\infty e^{-z^2/2} dz \simeq \frac{1}{2\pi} \frac{1}{\sqrt{\frac{a^2 T_b}{\eta}}} e^{-a^2 T_b/2\eta}$$

The average probability of error is given by:

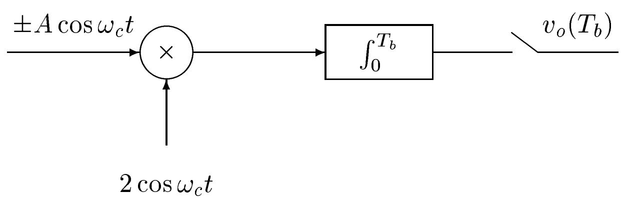
$$P_e = \mathrm{E}\{[P(e|A=a)\} = \int_0^\infty \frac{1}{2\pi}\sqrt{\frac{\eta}{T_b}} e^{-\frac{a^2}{2}(\frac{T_b}{\eta} + \frac{1}{A_0^2})} da$$

$$= \frac{1}{2A_0^2}\sqrt{\frac{\eta}{T_b}}(\frac{T_b}{\eta} + \frac{1}{A_0^2})^{-1/2}$$

10. • In a coherent binary PSK system, the symbol probabilities are $P(0\text{sent}) = p$ and $P(1\text{sent}) = 1-p$. The receiver is operating with signal-to-noise ratio $A^2T_b/\eta = 4$.

(a) Find the optimum threshold setting for $p = 0.4, 0.5$ and 0.6 and find theprobability of error P_e for $p = 0.4, 0.5$ and 0.6.

(b) Suppose that the receiver threshold setting was set at 0 for $p = 0.4, 0.5$ and 0.6. Find P_e and compare it with P_e obtained in part (a).



•

(a)

$$s_1(t) = A\cos\omega_c t \qquad 0 \le t \le T_b$$

$$s_2(t) = -A\cos\omega_c t \qquad 0 \le t \le T_b$$

$s_1(t)$ is sent if $b_k = 0$ and $s_2(t)$ is sent if $b_k = 1$

$$s_{o1}(T_b) = +A^2T_b \quad , \quad s_{o2}(T_b) = -A^2T_b \quad \text{and} \quad N_o = \eta A^2 T_b$$

The output v_o of the optimum receiver is a Gaussian variable such that:

$$p_{v_o}(v_o|s_1) = \frac{1}{\sqrt{2\pi N_o}} e^{-(v_o - s_{o1})^2/2N_o}$$

and

$$p_{v_o}(v_o|s_2) = \frac{1}{\sqrt{2\pi N_o}} e^{-(v_o - s_{o2})^2/2N_o}$$

Hence

$$p_{v_{o1}}(v_{o1}) = p.\frac{1}{\sqrt{2\pi N_o}} e^{-(v_o - s_{o1})^2/2N_o}$$

$$p_{v_{o2}}(v_{o2}) = (1-p).\frac{1}{\sqrt{2\pi N_o}} e^{-(v_o - s_{o2})^2/2N_o}$$

The optimum threshold is when the two probability functions are equal. I.e.

$$p.\frac{1}{\sqrt{2\pi N_o}} e^{-(v_o - s_{o1})^2/2N_o} = (1-p).\frac{1}{\sqrt{2\pi N_o}} e^{-(v_o - s_{o2})^2/2N_o}$$

$$\Longrightarrow \quad p.e^{-2v_oA^2T_b/2No} = (1-p).e^{2v_oA^2T_b/2No}$$

Hence

$$v_{Top} = \frac{N_o}{2A^2T_b}\ln(\frac{p}{1-p}) = \frac{\eta}{2}\ln(\frac{p}{1-p})$$

$$P_e = P(0).\int_{v_{Top}}^{\infty} p_{v_{o1}}(v_{o1})dv_{o1} + P(1).\int_{v-\infty}^{Top} p_{v_{o2}}(v_{o2})dv_{o2}$$

$$= p.\int_{v_{Top}}^{\infty}\frac{1}{\sqrt{2\pi N_o}}e^{-(v_o+A^2T_b)^2/2N_o}dv_o + (1-p).\int_{v-\infty}^{Top}\frac{1}{\sqrt{2\pi N_o}}e^{-(v_o-A^2T_b)^2/2N_o}dv_o$$

Let $z = \frac{v_o+A^2T_b}{\sqrt{N_o}}$ in the first integral,
and $z = \frac{v_o-A^2T_b}{\sqrt{N_o}}$ in the second integral. Then,

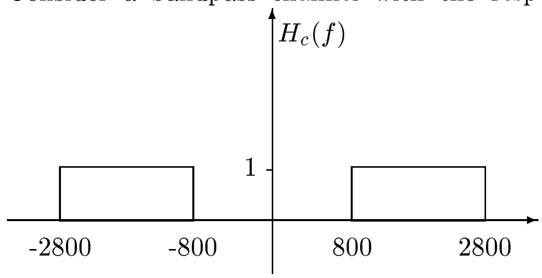
$$P_e = p.\Psi_c(\frac{v_{Top}+A^2T_b}{\sqrt{N_o}}) + (1-p).\Psi_c(\frac{A^2T_b-v_{Top}}{\sqrt{N_o}})$$

For the given values of p, find the corresponding v_{Top} and hence the corresponding P_e.

(b) When the receiver sets the threshold at 0-level, the probability of error is given by

$$P_e = \Psi_c(\frac{A^2T_b}{\sqrt{N_o}})$$

11. • Consider a bandpass channel with the response shown in figure.



(a) Binary data is transmitted over this channel at a rate of $300 bits/sec$ using a noncoherent FSK signalling scheme with tone frequencies of 1070 and 1270 Hz. Calculate P_e assuming $A^2/\eta = 8000$.

(b) How fast can a PSK signalling scheme operate over this channel? Find P_e for the PSK scheme assuming coherent demodulation

• The probability of error in a non-coherent FSK system is given by

$$T_b = \frac{1}{300}, \quad \frac{A^2}{\eta} = 8000 \implies P_e = \frac{1}{2} e^{-A^2 T_b / 4\eta}$$

• Bandwidth of channel $B = 2800 - 800 = 2000 Hz$. Thus the channel can pass a bit with bit period T_b such that $T_b \simeq \frac{1}{\frac{1}{2}B} \simeq \frac{1}{2} \times 10^{-3}$ For PSK,

$$P_e = \Psi_c(\sqrt{\frac{A^2(0.5 \times 10^{-3})}{\eta}}) = \Psi_c(\sqrt{4}) \simeq 0.0228$$

12. • The bit stream 1 1 0 1 1 1 0 0 1 0 1 is to be transmitted using DSPK. Determine the encoded sequence and the transmitted phase sequence. Show that the phase comparison scheme described in Section II.4 can be used for demodulating the signal.

• For encoding using DPSK,

$$d_k = b_k \circ d_{k-1} \oplus \bar{b}_k \circ \bar{d}_{k-1}$$

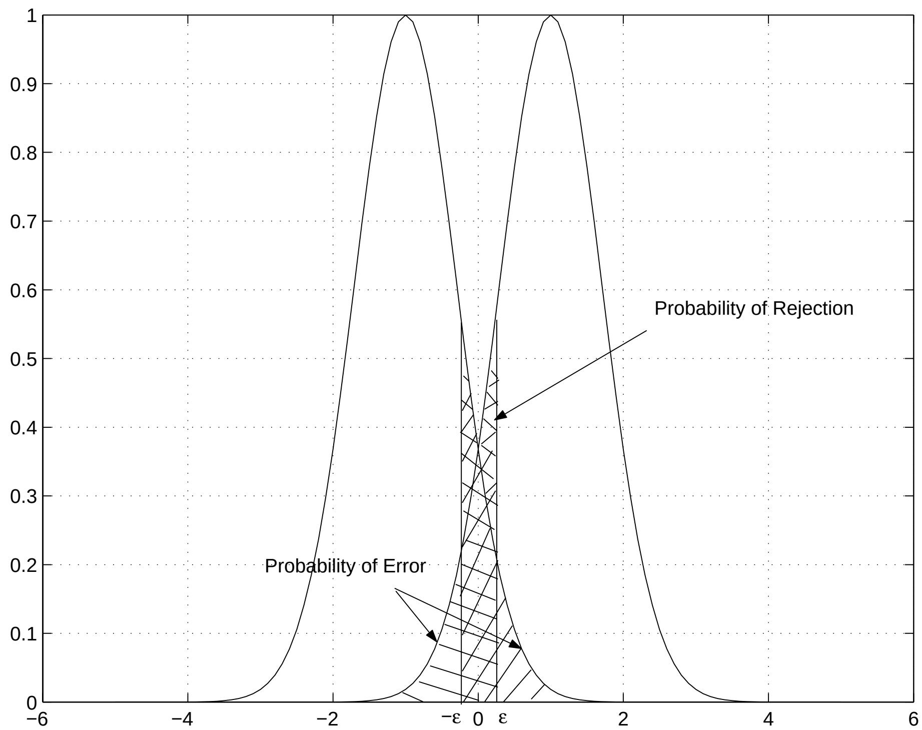
b_k		1	1	0	1	1	1	0	0	1	0	1
d_k	1	1	1	0	0	0	0	1	0	0	1	1
c_k		1	1	-1	-1	-1	-1	1	-1	-1	1	1
Phase		0	0	π	π	π	π	0	π	π	0	0

13. In some threshold devices, a no-decision zone centered at the optimum threshold level is used such that if the input Y to the threshold device falls in this region, no decision is made, that is, the output is 0 if say $y \leq V_1$ and 1 if $y > V_2$ and no decision is made if $V_1 \leq y \leq V_2$. Assuming that

$$p_Y(y|1\ sent) = \frac{1}{2\sqrt{\pi}} \exp(\frac{(y-1)^2}{4}), \quad \infty < y < \infty$$

$$p_Y(y|0\ sent) = \frac{1}{2\sqrt{\pi}} \exp(\frac{(y+1)^2}{4}), \quad \infty < y < \infty$$

$P(1\ sent) = P(0\ sent) = 0.5$, $V_1 = -\epsilon$, $V_2 = \epsilon$.
Sketch P_e and the probability of no decision versus ϵ. (Use ϵ =0.1, 0.2, 0.3, 0.4 and 0.5)

- 

$N_o = 2$
$P_e \equiv$ Probability of Error

$$P_e = \frac{1}{2}\int_{\epsilon}^{\infty} \frac{1}{\sqrt{2\pi N_o}} e^{-\frac{(y+1)^2}{2N_o}} dy + \frac{1}{2}\int_{-\infty}^{-\epsilon} \frac{1}{\sqrt{2\pi N_o}} e^{-\frac{(y-1)^2}{2N_o}} dy = \int_{\epsilon}^{\infty} \frac{1}{\sqrt{2\pi N_o}} e^{-\frac{(y+1)^2}{2N_o}} dy$$

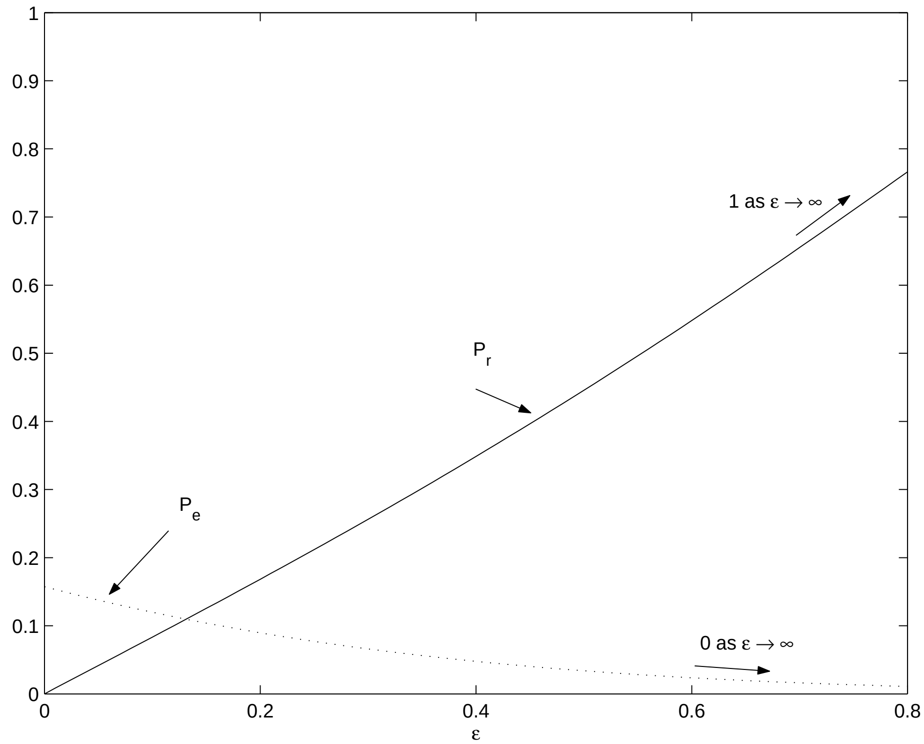
$$\Longrightarrow \quad P_e = \Psi_c(\frac{1+\epsilon}{\sqrt{2}})$$

$P_r \equiv$ Probability of Rejection

$$P_r = \frac{1}{2}\int_{-\epsilon}^{\epsilon} \frac{1}{\sqrt{2\pi N_o}} e^{-\frac{(y+1)^2}{2N_o}} dy + \frac{1}{2}\int_{-\epsilon}^{\epsilon} \frac{1}{\sqrt{2\pi N_o}} e^{-\frac{(y-1)^2}{2N_o}} dy = \int_{-\epsilon}^{\epsilon} \frac{1}{\sqrt{2\pi N_o}} e^{-\frac{(y+1)^2}{2N_o}} dy$$

$$\Longrightarrow \quad P_r = \int_{-\epsilon}^{\infty} \frac{1}{\sqrt{2\pi N_o}} e^{-\frac{(y+1)^2}{2N_o}} dy - \int_{\epsilon}^{\infty} \frac{1}{\sqrt{2\pi N_o}} e^{-\frac{(y+1)^2}{2N_o}} dy = \Psi_c(\frac{1-\epsilon}{\sqrt{2}}) - \Psi_c(\frac{1+\epsilon}{\sqrt{2}})$$

The values of Ψ_c for P_e and P_r can be looked up from tables for various values of ϵ and P_e and P_r can be plotted as shown.



14. • Let $n(t)$ be a stationary zero mean Gaussian white noise and let

$$n_{o1}(T_b) = \int_0^{T_b} n(t)\cos(\omega_c t + \omega_d t)dt$$

$$n_{o2}(T_b) = \int_0^{T_b} n(t)\cos(\omega_c t - \omega_d t)dt$$

Show that $n_{o1}(T_b)$ and $n_{o2}(T_b)$ are independent if $\omega_c = 2\pi k/T_b$ and $\omega_d = m\pi/2T_b$, where k and m are (arbitrary) positive integers ($k \gg m$)

•

$$n_{o1}(T_b) = \int_0^{T_b} n(t)\cos(\omega_c t + \omega_d t)dt$$

$$n_{o2}(T_b) = \int_0^{T_b} n(t)\cos(\omega_c t - \omega_d t)dt$$

Since $n(t)$ is zero mean, thus

$$\mathrm{E}\{n_{o1}(T_b)\} = \mathrm{E}\{n_{o2}(T_b)\} = \int_0^{T_b} \mathrm{E}\{n(t)\}\cos(\omega_c t + \omega_d t)dt = 0$$

$$\mathrm{E}\{n_{o1}(T_b)n_{o2}(T_b)\} = \int_0^{T_b}\int_0^{T_b} \mathrm{E}\{n(t_1)n(t_2)\}\cos(\omega_c t_1+\omega_d t_1)\cos(\omega_c t_2-\omega_d t_2)dt_1 dt_2$$

$$= \int_0^{T_b}\int_0^{T_b} \Phi_{nn}(t_1,t_2)\cos(\omega_c t_1 + \omega_d t_1)\cos(\omega_c t_2 - \omega_d t_2)dt_1 dt_2$$

Now, for white noise, $\Phi_{nn}(t_1,t_2) = \frac{\eta}{2}\delta(t_1 - t_2)$
Therefore,

$$\mathrm{E}\{n_{o1}(T_b)n_{o2}(T_b)\} = \frac{\eta}{2}\int_0^{T_b} \cos(\omega_c t_1 + \omega_d t_1)\cos(\omega_c t_1 - \omega_d t_1)dt_1$$

$$= \frac{\eta}{4}\underbrace{\int_0^{T_b} \cos(2\omega_c t_1)dt_1}_{=0} + \frac{\eta}{4}\int_0^{T_b} \cos(2\omega_d t_1)dt_1$$

$$= \frac{\eta}{8\omega_d}\sin(2\omega_d t_1)\Big|_0^{T_b} = 0 \quad \text{because } 2\omega_d T_b = m\pi$$

Hence $n_{o1}(T_b)$ and $n_{o2}(T_b)$ are uncorrelated and thus independent because both $n_{o1}(T_b)$ and $n_{o2}(T_b)$ are Gaussian.

15. • Consider the channel described in Problem II.11,

(a) Compute the fastest rate at which data can be transmitted over this channel using four-phase PSK signalling schemes.

(b) Compute P_e for QPSK and differential QPSK.

• (a) Available bandwidth $=2000Hz$ Let r_s be the symbol rate of the transmission system. For both QPSK and DQPSK, the bandwidth required is:

$$BW \simeq 2r_s = 2000Hz \quad \Longrightarrow \quad r_s \simeq 1000 \; symbols/sec$$

(b) Assume $\frac{A^2}{\eta} = 20000$,

$$P_e|_{QPSK} = 2\Psi_c(\sqrt{\frac{A^2 T_s}{2\eta}}) = 2\Psi_c(\sqrt{10}) \simeq 0.16$$

Also,

$$P_e|_{DQPSK} = 2\Psi_c(\sqrt{\frac{A^2 T_s}{\eta}\sin^2(\pi/M)}) \qquad M = 4$$

$$[P_e|_{DQPSK} = 2\Psi_c(\sqrt{\frac{0.146A^2 T_s}{\eta}}) \simeq 0.872$$

Note that the above P_e's are symbol (4-level) error probabilities.

16. • A microwave channel has a usable bandwidth of 10 MHz. Data had to be transmitted over this channel at a rate of $(1.5)(10^6)$ bits/sec. The channel noise is zero mean Gaussian with a psd of $\eta/2 = 10^{-14}$ watt/Hz.

(a) Design a wideband FSK signalling scheme operating at $P_e = 10^{-5}$ for this problem, that is, find a suitable value of M and $\frac{A^2}{2}$.

(b) If a binary differential PSK signalling schme is used for this problem, find its power requirement.

• (a) Let the number of signals transmitted in the FSK system be M. For M signals, Bandwidth is (Eq. II.6.20)

$$B \simeq (\frac{M\pi}{T_s})\frac{1}{2\pi} = \frac{M}{2T_s}Hz$$

Now, T_s is the duration of one symbol and there are M of these symbols. Let $M = 2^k$, then we need k bits to represent one symbol.
The problem specifies that bit rate $r_b = 1.5 \times 10^6$ bits/sec.
Therefore, $T_b = 1/1.5 \times 10^6$ seconds
But each symbol has k bits, thus $T_s = kT_b$
Therefore,

$$B = \frac{M}{2kT_b} = \frac{2^k}{2kT_b} \leq 10MHz$$

or

$$\frac{2^k}{k} \leq 13.33 \qquad \text{k has to be an integer} \quad \Longrightarrow k = 6 \text{ or } \quad M = 64$$

Note that this will be a complex sytem-almost impractical. From figure (II.6.7), $\frac{S_{av}}{\eta r_b} \simeq 4$ for $P_e = 10^{-4}$ and $M = 64$

$$\Longrightarrow \quad S_{av} = 12 \times 10^{-8}$$

(b)

$$P_e = 10^{-5} \qquad \Longrightarrow \qquad \frac{1}{2}e^{-A^2T_b/2\eta} = 10^{-5}$$

Therefore

$$\frac{A^2}{2} \simeq 32.46 \times 10^{-8}$$

17. • If the value of M obtained in problem II.16 is doubled, how will it affect the bandwidth and power requirements if P_e is to be maintained at 10^{-5}?

•

$$M = 2 \times 64 = 128$$

$$\Longrightarrow \quad \text{Bandwidth} = B \simeq \frac{M}{2 \times 7 \times T_b} = 13.71 MHz$$

For $P_e = 10^{-5}$ and from figure (II.6.7), signal power requirements are not much different.