Mathematics for Linear Systems

Jian-Kang Zhang & Jun Chen

Department of Electrical and Computer Engineering

McMaster University

COURSE ORGANIZATION

Webpage:

http://www.ece.mcmaster.ca/~jkzhang/Course_3ck3_2008.htm

http://www.ece.mcmaster.ca/~junchen/EE3CK3.htm

Assessment:

- Two Assignments: 10% (each 5%)
- Tutorial Attendance: 4% (Random check)
- Two Midterms: 36% (each 18%)
- Final exam: 50%

IMPORTANT INFORMATION

- *Students must pass the combined midterm/exam component separately to get a pass in the course. The midterm and exam will be combined with the weighting 36% on the midterm and 50% on the final. A grade of 50% in this combination must be attained to pass. Statistical adjustments (such as bell curving) will not normally be used.*
- *Please note that students who miss the midterm, and who have a valid excuse, will be subjected to an oral makeup test or a written test, at the discretion of the instructor. Those who do not have a valid excuse will be assessed zero for the midterm component of the final grade.*

COURSE ORGANIZATION

Teaching assistants:

- Min Huang (turorial), ITB A202, ext. 23151,
  Email: huangm2@mcmaster.ca
- Amin Behnad (turorial), ITB A103, ext. 26112,
  Email: behnad@grads.ece.mcmaster.ca
- Lin Song (grading)

SYLLABUS

- Complex Variables and Contour Integration
- The Laplace Transform and Its Inversion
- The Fourier Transform and Applications
- Discrete Transforms
- Linear Algebra and State Variables (if time permits)

COURSE TEXTBOOK

- Shlomo Karni and William J. Byatt
  Mathematical Methods in Continuous and Discrete Systems
  NY: Holt, Rinehart and Winston, 1982.
  ISBN: 0-03-057038-7

COMPLEX ANALYSIS

- The shortest route between two truths in the real domain passes through the complex domain.

  Jacques Salomon Hadamard (1865-1963)

- Complex analysis is beautiful, real analysis is dirty.

  André Weil (1906-1998)

1 ARITHMETIC OPERATIONS OF COMPLEX VARIABLES

1.1 Complex Variables

- Imaginary unit: $j = \sqrt{-1}$
- Complex variable: z
  - Rectangular form: $z = x + jy$ Re$(z) = x$, Im$(z) = y$
  - Exponential form: $z = re^{j\theta}$

    Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$

    $\Rightarrow x = r\cos\theta$, $y = \sin\theta$, $x^2 + y^2 = r^2$

◇ Example: $e^{j\pi/2} = j$, $e^{j\pi} = -1$, $e^{2n\pi j} = 1$ (n any integer)

◇ Example: $z = 1 - j \Leftrightarrow z = \sqrt{2}e^{-j\pi/4}$

1.2 Arithmetic Operations

Rectangular form: $z_1 = x_1 + jy_1$, $z_2 = x_2 + jy_2$

Exponential form: $z_1 = r_1 e^{j\theta_1}$, $z_2 = r_1 e^{j\theta_2}$

- Addition: $z_1 + z_2 = x_1 + x_2 + j(y_1 + y_2)$
- Subtraction: $z_1 - z_2 = x_1 - x_2 + j(y_1 - y_2)$
- Multiplication: $z_1 z_2 = x_1 x_2 - y_1 y_2 + j(y_1 x_2 + y_2 x_1)$ (rectangular form)
  $z_1 z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}$ (exponential form)

◇ Example: $z_1 = 4 + j3$, $z_2 = 1 - j$

$\Rightarrow z_1 + z_2 = 5 + j2$, $z_1 - z_2 = 3 + j4$, $z_1 z_2 = 7 - j$

- Complex conjugate: $z = x + jy$, $z^* = x - jy$
  $zz^* = x^2 + y^2 = r^2$, $\quad |z| = \sqrt{zz^*} = r$

- Division: $\frac{z_1}{z_2} = \frac{x_1x_2+y_1y_2+j(y_1x_2-y_2x_1)}{x_2^2+y_2^2}$ (rectangular form)
  $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}$ (exponential form)
- Power: $z = re^{j\theta}$
  $z^n = r^n e^{jn\theta}$ Re$(z^n) = r^n \cos n\theta$, Im$(z^n) = r^n \sin n\theta$
- Fractional power: $z = re^{j\theta}$
  $z^{1/n} = r^{1/n}e^{j(\theta+2\pi p)/n}$, $p = 0, 1, \cdots, n-1$

◇ Example: $z = 3 + j4 = 5e^{j\theta}$ with $\theta = \tan^{-1}\frac{4}{3}$
$\Rightarrow (3 + j4)^{1/2} = \sqrt{5}e^{j(\theta/2+\pi p)}$, $p = 0, 1$

1.3 Functions of a Complex Variable

$$f(z) = f(x + jy) = u(x, y) + jv(x, y)$$

◇ Example: $f(z) = e^{\pm z}$

$$e^{\pm z} = e^{\pm(x+jy)} = e^{\pm x}(\cos y \pm j \sin y)$$

$$\Rightarrow u(x, y) = e^{\pm x} \cos y, \; v(x, y) = \pm e^{\pm x} \sin y$$

◇ Example: $f(z) = \sin z$

$$\sin(x + jy) = \sin x \cos(jy) + \cos x \sin(jy)$$

$$\cos jy = \cosh y, \; \sin(jy) = j \sinh y$$

$$\Rightarrow u(x, y) = \sin x \cosh y, \; v(x, y) = \cos x \sinh y$$

◇ Example: $f(z) = \ln z$

$\ln z = \ln(re^{j\theta}) = \ln(re^{j(\theta \pm 2n\pi)}) = \ln r + j(\theta \pm 2n\pi)$ (n any integer)

$$\Rightarrow u(x, y) = \ln r), \; v(x, y) = \theta \pm 2n\pi$$

1.4 Derivatives of a Complex Function

$f(z) = u(x, y) + jv(x, y)$

- Definition (derivative): $\left.\frac{df}{dz}\right|_{z=z_0} = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$
- Definition (Cauchy-Riemann conditions): $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

⋆ Theorem (sufficient conditions for differentiability):

1. the first-order partial derivatives of the functions $u(x, y)$and $v(x, y)$ with respect to x and y exist everywhere in the neighborhood of $z_0 = x_0 + jy_0$;
2. those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy-Riemann conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ at (x_0, y_0).

Then $\left.\frac{df}{dz}\right|_{z=z_0}$ exists, its value being

$$\left.\frac{df}{dz}\right|_{z=z_0} = \left.\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}\right|_{(x,y)=(x_0,y_0)}$$

- Definition (analytical function): $f(z)$ is analytic at a point z_0 if it has a derivative at each point in some neighborhood of z_0. It follows that if f is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0.

◇ Example: $f(z) = e^{-z}$

$u(x,y) = e^{-x}\cos y$, $v(x,y) = -e^{-x}\sin y$

$\frac{\partial u}{\partial x} = -e^{-x}\cos y = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -e^{-x}\sin y = -\frac{\partial v}{\partial x}$

The Cauchy-Riemann conditions are satisfied.

◇ Example: $f(z) = \ln z$

$u(x,y) = \frac{1}{2}\ln(x^2+y^2)$, $v(x,y) = \tan^{-1}(y/x) + 2n\pi$
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2+y^2} = \frac{\sin\theta}{r}$
The Cauchy-Riemann conditions are satisfied at all finite points other than $r = 0$ $(x = y = 0)$. The origin $x = y = 0$ is called a singular point for $f(z) = \ln z$.

- Definition (singularity): A point in the z-plane at which $f(z)$ is not analytic is called a singular point (or a singularity) of $f(z)$. There are several types of singularities. We say that $f(z)$ has an isolated singularity at $z = z_0$ if in the neighborhood of $z = z_0$, no matter how small, there are no other singularities. In other words, $f(z)$ is analytic throughout the neighborhood of $z = z_0$ except at $z = z_0$.
  The function $f(z)$ has a pole of order n at $z = z_0$ (also called a removable singularity) if $(z - z_0)^n f(z)$ is analytic at z_0. If no integer n

can be found, then $z = z_0$ is an essential singularity.

◇ Example: $f(z) = \frac{z-2}{z^2(z+1)}$ has isolated singularities at $z = 0$ and at $z = 1$. The singularity at $z = 0$ is a pole of order 2, and the singularity at $z = -1$ is a pole of order 1 (simple pole).

1.5 Laplace's Equation

- Cauchy-Riemann conditions: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$
  $\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$
  $\Rightarrow \nabla^2 u = 0$ with $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
  Similarly, $\nabla^2 v = 0$
- Definition (Laplace's equation): $\nabla^2 H = 0$

⋆ Theorem: If a function $f(z) = u(x, y) + jv(x, y)$ is analytic in some region of the complex plane, both u and v satisfy Laplace's equation throughout that same region.

1.6 Integration in the Complex Plane

- Definition (Contour): A contour, or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs joined end to end. When only the initial and final values are the same, a contour C is called a simple closed contour. A contour is positively oriented when it is in the counterclockwise direction.

⋆ Theorem (Cauchy's first integral theorem): If a function $f(z)$ is analytic all all points interior to and on a simple closed contour C, then

$$\oint_C f(z)dz = 0$$

⋆ Theorem (Cauchy's second integral theorem): Let $f(z)$ be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi j} \oint_C \frac{f(z)}{z - z_0} dz$$

Extension:

$$f^{(n)}(z_0) = \frac{n!}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

where

$$f^{(n)}(z_0) = \left. \frac{d^n f}{dz^n} \right|_{z=z_0}$$

1.7 The Taylor Series

⋆ Theorem: Suppose that a function $f(z)$ is analytic throughout a disk $|z - z_0| < R_0$, centered at z_0 and with radius R_0. Then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R_0)$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (n = 0, 1, 2, \cdots)$$

and the contour C is inside the disk.

◇ Example: The Taylor series expansion of $\cos z$ about the point

$z = z_0 = \pi/2$.

$$\begin{aligned}
\cos z &= \cos\frac{\pi}{2} + \frac{d}{dz}(\cos z)_{z=\pi/2}(z - \frac{\pi}{2}) \\
&\quad + \frac{1}{2!}\frac{d^2}{dz^2}(\cos z)_{z=\pi/2}(z - \frac{\pi}{2})^2 \\
&\quad + \frac{1}{3!}\frac{d^3}{dz^3}(\cos z)_{z=\pi/2}(z - \frac{\pi}{2})^3 + \cdots \\
&= -(z - \frac{\pi}{2}) + \frac{1}{6}(z - \frac{\pi}{2})^3 + \cdots
\end{aligned}$$

1.8 The Laurent Expansion

- Theorem: Suppose that a function $f(z)$ is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (R_1 < |z - z_0| < R_2)$$

where

$$a_n = \frac{1}{2\pi j} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad (n = 0, 1, 2, \cdots)$$

and

$$b_n = \frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}dz \quad (n = 1, 2, \cdots)$$

◇ Example: Find the Laurent expansion of $f(z) = (z-2)^{-1}$ for $|z| < 2$.

$$f(z) = \frac{-1}{2(1-z/2)} = \sum_{n=0}^{\infty} -2^{-(n+1)} z^n$$

◇ Example: Find the Laurent expansion of $f(z) = (z-2)^{-1}$ for $|z| > 2$.

$$f(z) = \frac{1}{z}(1-2/z)^{-1} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

1.9 Cauchy's Residue Theorem

- Definition (Residues): When z_0 is an isolated singular point of $f(z)$, there is a positive number R_2 such that $f(z)$ is analytic at each point z for which $0 < |z - z_0| < R_2$. Let C be any positively oriented simple closed contour around z_0 that lies in the punctured disk $0 < |z - z_0| < R_2$. Define

$$\text{Res}_{z=z_0} f(z) = \frac{1}{2\pi j} \oint_C f(z) dz$$

which is called the residue of $f(z)$ at the isolated singular point z_0. Remark: The residues can often be calculated using Cauchy's second integral theorem.

⋆ Theorem: Let C be a simple closed contour, described in the positive sense. If a function $f(z)$ is analytic inside and on C except for a finite number of singular points z_k $(k = 1, 2, \cdots, n)$ inside C, then

$$\oint_C f(z)dz = 2\pi j \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$$

◇ Example: Find the residue of

$$f(z) = \frac{\sin z}{(z - \pi/2)^3}$$

at $z = \pi/2$. The residue can be found by calculating

$$\frac{1}{2!} \left. \frac{d^2 \sin z}{dz^2} \right|_{z=\pi/2} = -\frac{1}{2}$$

2.0 The Evaluation of Real Definite Integrals

◇ Example: Consider the integral I, defined by

$$I(a, b, \pi) = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$$

where a and b are real, and $b < a$. Set $z = e^{j\theta}$. The contour C of integration in the complex plane will, then, be a circle of unit radius. Since $\cos\theta = (e^{j\theta} + e^{-j\theta})/2$, we have $\cos\theta = (z + z^{-1})/2 = (z^2 + 1)/2z$. Further, with $z = e^{j\theta}$, $dz = je^{j\theta}d\theta$, so that $d\theta = dz/jz$. The integral I becomes

$$I = \oint_C \frac{2dz}{j[2az + b(z^2 + 1)]} = \frac{2}{jb}\oint_C \frac{dz}{(z - z_+)(z - z_-)}$$

where the poles of the integrand are at the points

$$z_+ = -\frac{a}{b} + \sqrt{(\frac{a}{b})^2 - 1}$$
$$z_- = -\frac{a}{b} - \sqrt{(\frac{a}{b}) - 1}$$

Since $b < a$ by assumption, both poles are real, and $|z_+| < 1$, $|z_-| > 1$. Thus only the root z_+ is within a circle of unit radius. Therefore the application of the Cauchy's residue theorem leads to the result

$$I = \frac{2}{jb} 2\pi j \text{Res}_{z=z_+} \frac{1}{z - z_-} = \frac{4\pi}{b} \frac{1}{z_+ - z_-}$$

On inserting the expression for z_+ and z_-, the answer is

$$I = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

◇ Example: Consider the integral

$$I(\omega) = \int_0^\infty \frac{\sin \omega t}{t} dt$$

By setting $\omega t = x$, we have

$$I(\omega) = \int_0^\infty \frac{\sin x}{x} dx$$

It is easy to see the value of I is be independent of ω. Now, $\sin x/x$ is an even function of x. Thus we can write

$$I(\omega) = \frac{1}{2} \text{Im} \int_{-\infty}^\infty \frac{e^{jx}}{x} dx$$

To evaluate this integral, consider the associated integral

$$J = \oint \frac{e^{jz}}{z} dz$$

Here the integrand has a pole at the point $z = 0$. To exclude the point $z = 0$, we choose the contour C shown in Fig. 2.15 (p. 91).

By Cauchy's first integral theorem, we have

$$J = 0 = \oint_C \frac{e^{jz}}{z} dz$$

The contributions from the four parts of C must now be found. We have

$$0 = \int_{-R}^{-\rho} \frac{e^{jx}}{x} dx + j \int_{\pi}^{0} \frac{e^{j\rho e^{j\theta}} \rho e^{j\theta}}{\rho e^{j\theta}} d\theta + \int_{\rho}^{R} \frac{e^{jx}}{x} dx + j \int_{0}^{\pi} \frac{e^{jRe^{j\theta}} Re^{j\theta}}{Re^{j\theta}} d\theta$$

The value of the second term of the right-hand side, as $\rho \to 0$, is $-j\pi$; the first and third terms are combined, so that

$$0 = -j\pi + \int_{-R}^{R} \frac{e^{jx}}{x} dx + j \int_{0}^{\pi} e^{jRe^{j\theta}} d\theta$$

The absolute value of the integral over θ satisfies the inequality

$$\left| j \int_0^{\pi} e^{jRe^{j\theta}} d\theta \right| \leq \int_0^{\pi} e^{-R\sin\theta} d\theta$$

Further, since $\sin\theta$ is an even function about $\pi/2$, we have

$$\int_0^{\pi} e^{-R\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta} d\theta$$

and $\sin\theta \geq 2\theta/\pi$ for all θ in $0 \leq \theta \leq \pi/2$. Thus,

$$2\int_0^{\pi/2} e^{-R\sin\theta} d\theta \leq 2\int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{\pi}{R}(1 - e^{-R})$$

Clearly, as $R \rightarrow \infty$, the last result approaches zero. Thus,

$$I(\omega) = \int_0^{\infty} \frac{\sin\omega t}{t} dt = \frac{1}{2} \mathrm{Im} \int_{-\infty}^{\infty} \frac{e^{jx}}{x} dx = \frac{\pi}{2}$$