Optical Fibers- I

Maxwell's Equations

Differential Form

$$\nabla.\mathbf{D} = \rho \qquad (1)$$

$$\nabla.\mathbf{B} = 0 \qquad (2)$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad (3)$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t} \qquad (4)$$

Integral Form

$$\oint \mathbf{D}.d\mathbf{S} = \int \rho dV$$

$$\oint \mathbf{B}.d\mathbf{S} = 0$$

$$\oint \mathbf{E}.dl = -\frac{\partial}{\partial t}\int \mathbf{B}.d\mathbf{S}$$

$$\oint \mathbf{H}.dl = \mathbf{I} + \frac{\partial}{\partial t}\int \mathbf{D}.d\mathbf{S}$$

Electrical and Magnetic flux density, $\mathbf{D}$ and $\mathbf{H}$ are

$\mathbf{D} = \varepsilon\mathbf{E}$, ε = permittivity

$\mathbf{B} = \mu\mathbf{H}$, μ = permeability

Wave Equation

Using the vector identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla.\mathbf{E}) - \nabla^2 \mathbf{E} \qquad (5)$$

and using Eqs. (3) and (4),

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu \frac{\partial}{\partial t}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \qquad (6)$$

In the source free region, $\rho = \mathbf{J} = 0$. From (5) and (6),

$$\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Similar calculation for magnetic field yields

$$\nabla^2 \mathbf{H} = \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

Wave Equation

The permittivity ε and permeability μ can be written as

$\varepsilon = \varepsilon_0 \varepsilon_r$ $\qquad$ $\mu = \mu_0 \mu_r$

$\varepsilon_0 =$ free space permittivity. $\mu_0 =$ free space permeability.

$\varepsilon_r = n^2 =$ relative permittivity, which is medium dependent.

$n =$ refractive index.

$\mu_r =$ relative permeability $= 1$ for dielectrics.

Velocity of light in free space, c is

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Wave Equation

The wave equation for electric field can be rewritten as

$$\nabla^2 \mathbf{E} = \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad (8)$$

Eq. (8) is a vector equation for components E_x, E_y and E_z.

These equations should be solved with the boundary conditions that tangential electric and magentic fields be continuous at the boundary.

But if the refractive index difference at the boundary is small, one could use scalar wave approximation, i.e.

$$\nabla^2 \psi = \frac{n^2(r)}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

ψ could represent a transeverse electric field component E_x or E_y.

Linearly Polarized (LP) Modes of Optical Fiber

Let

$$\psi(r,\theta,z,t)=\phi(r,\theta)\exp[i(\omega t-\beta z)] \qquad (9)$$

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+[k_0^2n^2(r)-\beta^2]\phi=0 \; (10)$$

$$\frac{\omega}{c}=k_0=\frac{2\pi}{\lambda_0}$$

$\lambda_0 = \text{free space wavelength}$

LP Modes

Using the method of separation of variable

$$\phi = R(r)\Theta(\theta) \qquad (11)$$

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+r^2[k_0^2n^2(r)-\beta^2]=-\frac{1}{\Theta}\frac{d^2\Theta}{dr^2}=l^2 \quad (12)$$

$l = \text{const.}$

$$\frac{d^2\Theta}{dr^2}+l^2\Theta=0 \qquad (13)$$

$$\Theta(\theta)=A\cos(l\theta)+B\sin(l\theta)$$

Since $\Theta(\theta+2\pi)=\Theta(\theta)$

$$= A\cos[l(\theta+2\pi)]+B\sin[l(\theta+2\pi)]$$

$$= A\cos(l\theta)+B\sin(l\theta) \quad (14)$$

Eq. (14) would be true only if l is an integer, 0,1,2...

We could choose either the form $\cos(l\theta)$ or $\sin(l\theta)$ for Θ.

LP Modes

From Eq.(14),

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[k_0^2 n^2(r) - \beta^2 - \frac{l^2}{r^2} \right] R = 0$$

For step index fiber

$$n^2(r) = n_1^2 \quad \text{for} \quad r < a \qquad \text{(core)}$$

$$= n_2^2 \quad \text{for} \quad r >= a \qquad \text{(cladding)}$$

Guided Modes $: (k_0^2 n_2^2 < \beta^2 < k_0^2 n_1^2)$

$$U = a(k_0^2 n_1^2 - \beta^2)^{1/2}$$

$$W = a(\beta^2 - k_0^2 n_2^2)^{1/2}$$

$$V = (U^2 + W^2)^{1/2} = k_0 a (n_1^2 - n_2^2)^{1/2}$$

LP Modes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[\frac{U^2}{a^2} - \frac{l^2}{r^2}\right]R = 0 \qquad r < a$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left[\frac{W^2}{a^2} + \frac{l^2}{r^2}\right]R = 0 \qquad r >= a$$

For $r < a$,

$R(r) = CJ_l(Ur/a) + DY_l(Ur/a)$

J_l and Y_l are Bessel functions of first and second kind, resply. $Y_l(0) = -\infty$, so we can not have the term with Y_l $.i.e.D = 0.$

$R(r) = CJ_l(Ur/a)$

For $r >= a$

$R(r) = EK_l(Wr/a) + FI_l(Wr/a)$

K_l and I_l are modified Bessel functions of first and second kind, resply. I_l is rejected because $I_l(\infty) = \infty$.

LP Modes

For $r >= a$

$R(r) = EK_l(Wr/a)$

The field ψ and its derivative $\partial\psi/\partial r$ should continuous at the boundary $r = a$. If $\partial\psi/\partial r$ is not continuous, $\partial^2\psi/\partial r^2$ would be a Dirac Delta function and from Eq. (10), we see that $n^2(r)$ would be a Dirac Delta function which is not consistent.

In our example, $n^2(r)$ is a step function.

$$R(a) = CJ_l(U) = EK_l(W)$$

$$\left.\frac{dR}{dr}\right|_{r=a} = UCJ_l'(U) = WEK_l'(W)$$

$$\frac{UJ_l'(U)}{J_l(U)} = \frac{WK_l'(W)}{K_l(W)} \quad (15)$$

The transcedental equation (15) should be numerically solved for the unknown β or U.

LP Modes

For numerical calculation, it is easier if we avoid derivatives.

Using the identities:

$$\mathrm{U}J_l^{'}(U) = UJ_{l-1}(U) - lJ_l(U)$$

$$WK_l^{'}(W) = -WK_{l-1}(W) - lK_l(W)$$

From Eq.(15), we obtain

$$\frac{UJ_{l-1}(U)}{J_l(U)} = -\frac{WK_{l-1}(W)}{K_l(W)} \quad (16)$$

It is convenient to use normalized prop. const. b and normalized parameter V.

$$b = \frac{W^2}{V^2} = \frac{\beta^2 - k_0^2 n_2^2}{k_0^2(n_1^2 - n_2^2)} \quad (17)$$

$$U^2 = V^2(1-b)$$

Solve Eq.(16) with b as unknown. Since $k_0^2 n_2^2 < \beta^2 < k_0^2 n_1^2$, we must have $0 < b < 1$.

LP modes

Mode Classification :

The modes are designated as LP_{lm} where l = mode order and

m = radial mode number.

l is the index corresponding to the variations along θ.

m is the index corresponding to the variations along r.

When $l = 0$, the field does not vary as a function of θ.

Mode Cutoff :

At cutoff $\beta = k_0 n_2$ and therefore,

$b = 0, W = 0, U = V \equiv V_c$

For the $\mathrm{l} = 1$ mode, from Eq.(16), we have

$V_c \; J_0(V_c) = 0 \quad (18)$

Solving eq.(18), we have $V_c = 2.408$ which is the cutoff

condition for LP11 mode.

Mode Cutoff

For a step index fiber with

$0 < V < V_c \ (=2.4048)$

we will have only one guided mode namely LP01 mode.

Such a fiber is referred to as single mode fibers and

is of tremendous importance in optical communication

systems.

Since V-number is inversely proportional to wavelength,

above condition implies that the fiber will remain single moded if

$$\lambda > \lambda_c = \frac{2\pi a(n_1^2 - n_2^2)^{1/2}}{V_c}$$

λ_c is called cut-off wavelength. The fiber will support more than

one mode if the operating wavelength is less than λ_c.

Effective Index

- When an optical mode is interpreted as a ray, we could define the refractive index "seen" by the ray as if it is propagating in a uniform medium. Mathematically, effective refractive index is defined as

$$n_{eff}=\beta/k_0$$

$k_0=2\pi/\lambda=$ free space prop. const.

Problem #1

The single mode step-index fiber to be designed should have a cutoff wavelength of 1.2 microns or less. R.I. of the core and cladding are 1.55 and 1.545, respectively. The free space wavelength = 1.55 microns. What is the maximum core radius?

Problem #2

For a step-index fiber, R.I. of the cladding=1.52. Δ=0.3%. Wavelength = 1.55 and core radius = 10 microns. Calculate V number. Is this fiber a single mode fiber?