Chapter 3 Solutions of Selected Problems

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3.1-5) (a) If $x(t)$ and $y(t)$ are orthogonal, then we have:

$$\int_{-\infty}^{\infty} |x(t) \pm y(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{\infty} |y(t)|^2 dt \pm \int_{-\infty}^{\infty} x(t) y^*(t) dt \pm \int_{-\infty}^{\infty} x^*(t) y(t) dt$$

The last two terms (integrals) are zero because of orthogonality. Then we have:

$$\int_{-\infty}^{\infty} |x(t) \pm y(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt + \int_{-\infty}^{\infty} |y(t)|^2 dt$$

or $E_{x \pm y} = E_x + E_y$ if $x(t)$ and $y(t)$ are orthogonal

(b) If $x(t)$ and $y(t)$ are orthogonal Then $x_1(t) = C_1 x(t)$ and $y_1(t) = C_2 y(t)$ are also orthogonal and we have

$$\int_{-\infty}^{\infty} x_1(t) y_1^*(t) dt = \int_{-\infty}^{\infty} x_1^*(t) y_1(t) dt = 0$$

Using part (a) we have

$$\int_{-\infty}^{\infty} |c_1 x(t) \pm c_2 y(t)|^2 dt = \int_{-\infty}^{\infty} |x_1(t) \pm y_1(t)|^2 dt$$

$$\underset{\substack{\uparrow \\ \text{using} \\ \text{Part (a)}}}{=} E_{x_1} + E_{y_1} \underset{\substack{\uparrow \\ \int_{-\infty}^{\infty} |C_1 x(t)|^2 dt = |C_1|^2 \int_{-\infty}^{\infty} |x(t)|^2 dt = |C_1|^2 E_x}}{=} |C_1|^2 E_x + |C_2|^2 E_y$$

3.2-1)

(4) $$E_{f_4} = \int_0^1 f_4^2(t)\, dt = \int_0^{0.5} (\sqrt{2})^2 dt + \int_{0.5}^1 (-\sqrt{2})^2 dt = 0.5$$

$$E_x = \int_0^1 \sin^2 2\pi t\, dt = 0.5$$

$$c = \frac{1}{\sqrt{E_x E_{f_4}}} \int_0^1 f_4(t)\, x^*(t)\, dt = \frac{1}{\sqrt{0.5 \times 0.5}} \left[\int_0^{0.5} 0.707 \sin 2\pi t - \int_{0.5}^1 0.707 \sin 2\pi t\, dt \right]$$

$$= 1.414/\pi$$

3.4-1)

$T_0 = 2$, So that $\omega_0 = 2\pi/2 = \pi$, and

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi t + b_n \sin n\pi t \qquad -1 \le t \le 1$$

where

$$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\,dt = \frac{1}{2}\int_{-1}^{1} t^2 dt = \frac{1}{3}$$

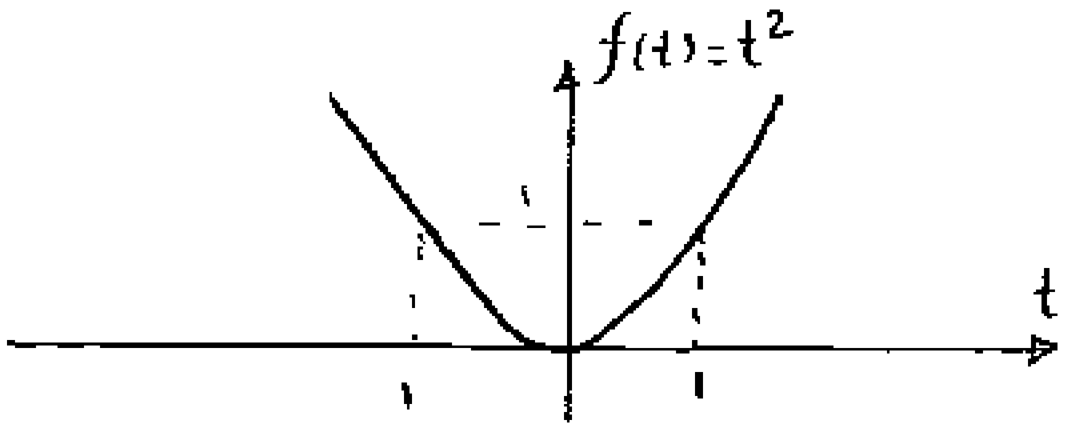
$$a_n = \frac{2}{T_0}\int_{T_0} f(t)\cos n\omega_0 t\,dt = \frac{2}{2}\int_{-1}^{1} t^2 \cos n\pi t\,dt = 2\int_0^1 t^2 \cos n\pi t\,dt$$

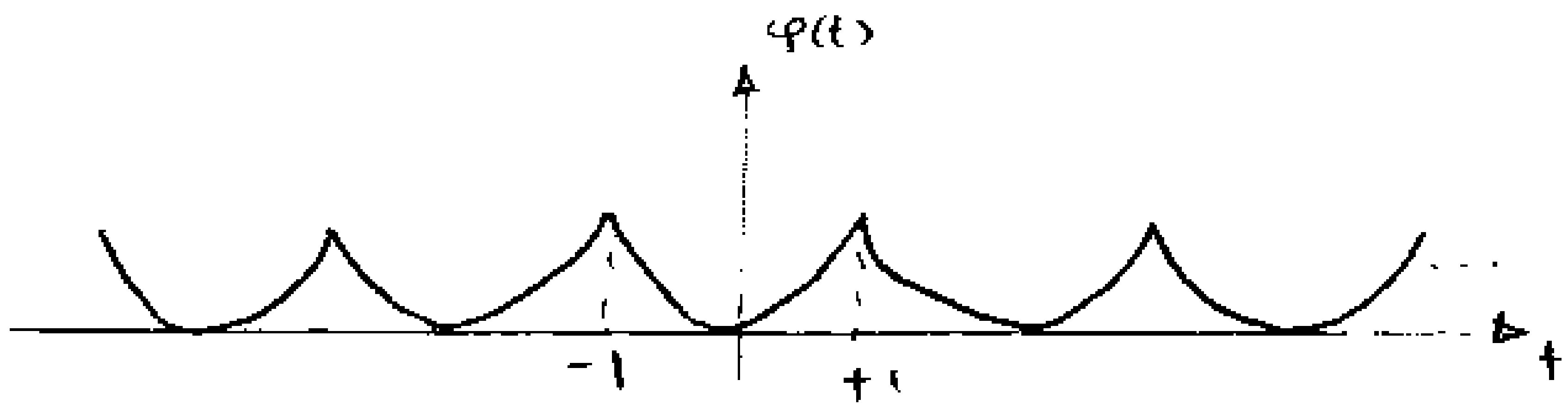
$$= 2 \times \left[t^2\left(\frac{1}{n\pi}\sin n\pi t\right) + \frac{2t}{(n\pi)^2}\cos n\pi t - \frac{2}{(n\pi)^3}\sin n\pi t \right|_0^1$$

$$= \frac{4(-1)^n}{\pi^2 n^2}$$

$$b_n = \frac{2}{T_0}\int_{T_0} f(t)\sin n\omega_0 t\,dt = \frac{2}{2}\int_{-1}^{1} \underbrace{t^2 \sin n\pi t\,dt}_{\text{odd function}} = 0$$

Therefore $f(t) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi t \qquad -1 \le t \le 1$





3.4-3)

(d) $T_0 = \pi$, $\omega_0 = 2$ and $f(t) = \frac{4}{\pi}t \quad -\frac{\pi}{4} \le t \le \frac{\pi}{4}$

$$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\,dt = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \underbrace{\frac{4}{\pi}t}_{\text{odd function of } t}\,dt = 0$$

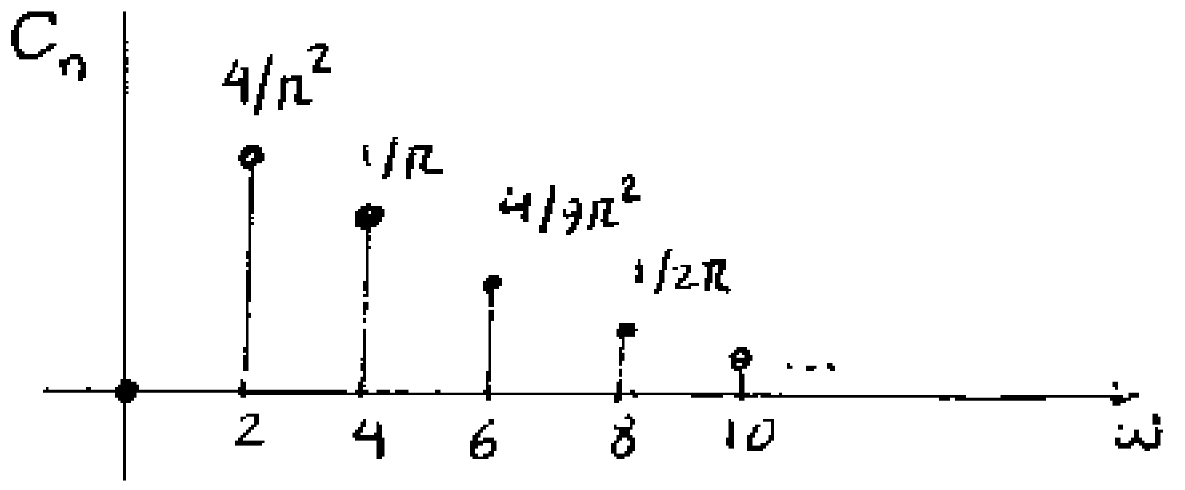
$a_n = 0$ because of odd symmetry

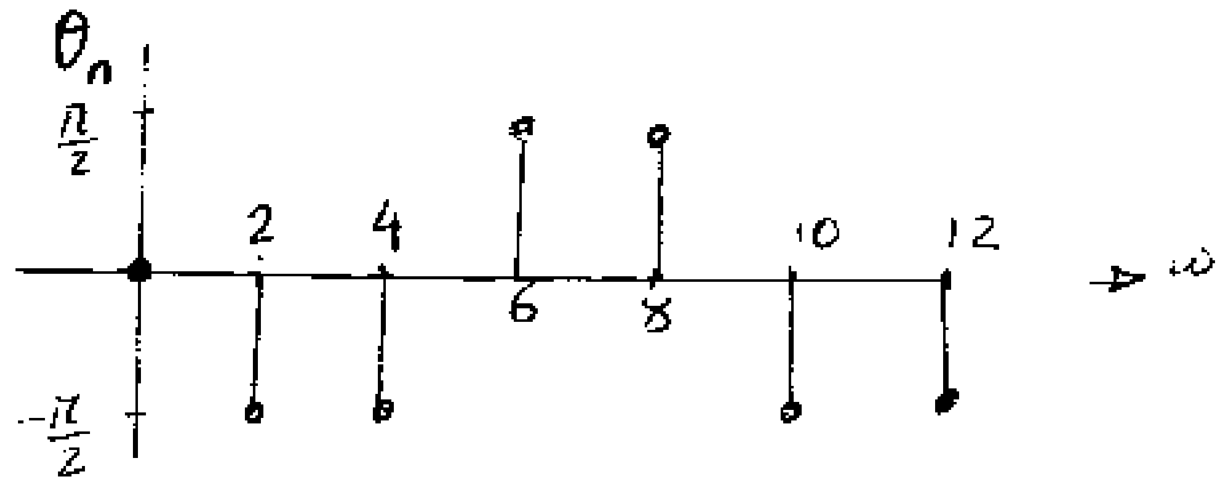
$$b_n = \frac{4}{\pi}\int_0^{\pi/4} \frac{4}{\pi}t \sin 2nt\,dt = \frac{2}{n\pi}\left(\frac{2}{\pi n}\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right)$$

$$\Rightarrow f(t) = \frac{4}{\pi^2}\sin 2t + \frac{1}{\pi}\sin 4t - \frac{4}{9\pi^2}\sin 6t - \frac{1}{2\pi}\sin 8t + \cdots$$

$$= \frac{4}{\pi^2}\cos\left(2t - \frac{\pi}{2}\right) + \frac{1}{\pi}\cos\left(4t - \frac{\pi}{2}\right) + \frac{4}{9\pi^2}\cos\left(6t + \frac{\pi}{2}\right)$$
$$+ \frac{1}{\pi}\cos\left(8t + \frac{\pi}{2}\right)$$

(3)





(f) $T_0 = 6$, $\omega_0 = \pi/3$, $a_0 = 0.5$ (by inspection).

$b_n = 0$ even symmetry

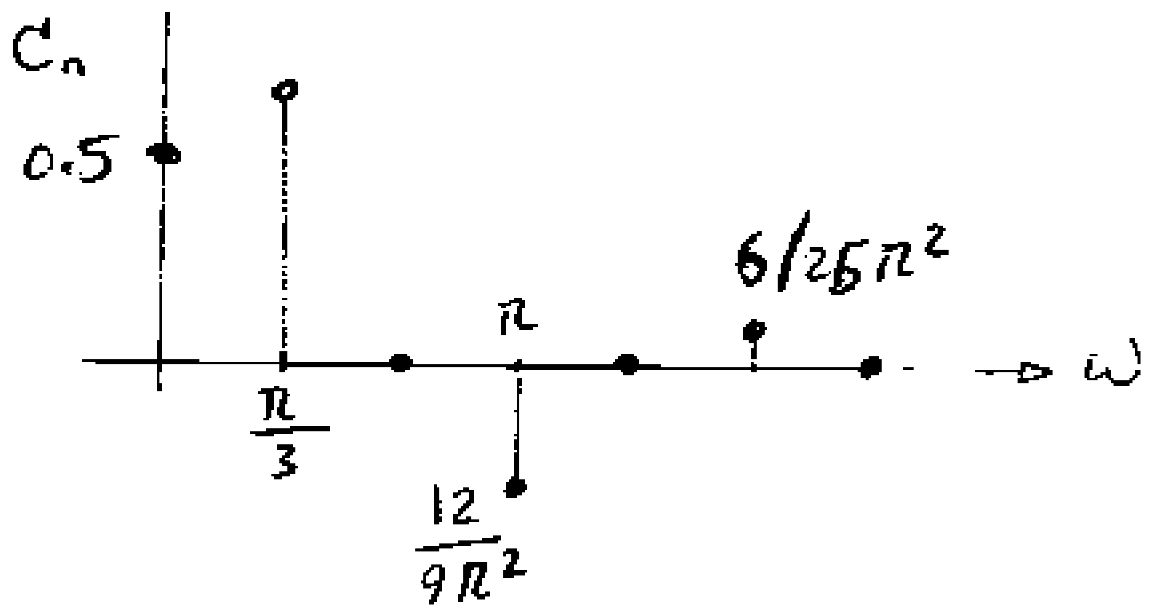
$$a_n = \frac{4}{6}\int_0^3 f(t)\cos\frac{n\pi}{3}\,dt$$

$$= \frac{2}{3}\left[\int_0^1 \cos\frac{n\pi}{3}\,dt + \int_1^2 (2-t)\cos\frac{n\pi}{3}t\,dt\right]$$

$$= \frac{6}{n^2\pi^2}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right]$$

$$\Rightarrow f(t) = 0.5 + \frac{6}{\pi^2}\left(\cos\frac{\pi}{3}t - \frac{2}{9}\cos\pi t + \frac{1}{25}\cos\frac{5\pi}{3}t + \frac{1}{49}\cos\frac{7\pi}{3}t + \dots\right)$$

observe that even harmonics vanish. The reason is that if the dc (0.5) is subtracted from $f(t)$, the resulting function has half-wave symmetry



3.4-5) a) We have $T_0 = \pi/2$, $\omega_0 = \frac{2\pi}{T_0} = 4$.
Therefore

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos 4nt + b_n\sin 4nt$$

Where

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2} e^{-t}\,dt = 0.504$$

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} e^{-t}\cos 4nt\,dt = 0.504\left(\frac{2}{1+16n^2}\right)$$

$$b_n = \frac{4}{\pi}\int_0^{\pi/2} e^{-t}\sin 4nt\,dt = 0.504\left(\frac{8n}{1+16n^2}\right)$$

Therefore $\qquad C_0 = a_0 = 0.504\ ,\ C_n = \sqrt{a_n^2 + b_n^2} = 0.504\left(\frac{2}{\sqrt{1+16n^2}}\right)$

$$\theta_n = -\tan^{-1} 4n$$

(b) This Fourier series is identical to that in Eq. (3.56a) with t replaced by 2t

(c) If $f(t) = C_0 + \sum C_n \cos(n\omega_0 t + \theta_n)$, then

$$f(at) = C_0 + \sum C_n \cos(n(a\omega_0)t + \theta_n)$$

Thus, time scaling by a factor a merely scales the fundumental frequency by the same factor a. Everything else remains unchanged. If we time compress (or time expand) a periodic signal by a factor a, its fundumental frequency increases by the same factor a (or decreases by the same factor a). Comparison of the results in part (a) with those in Example 3.3 confirms this conclusion. This result applies equally well.

3.4-7) a) For half wave symmetry

$$f(t) = -f(t \pm T_0/2)$$

and

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos n\omega_0 t\,dt = \frac{2}{T_0}\int_0^{T_0/2} f(t)\cos n\omega_0 t + \int_{T_0/2}^{T_0} f(t)\cos n\omega_0 t\,dt$$

Let $x = t - T_0/2$ in the second integral. This gives

$$a_n = \frac{2}{T_0}\left[\int_0^{T_0/2} f(t)\cos n\omega_0 t + \int_0^{T_0/2} f(x + T_0/2)\cos n\omega_0\left(x + \frac{T_0}{2}\right)dx\right]$$

$$= \frac{2}{T_0}\left[\int_0^{T_0/2} f(t)\cos n\omega_0 t + \int_0^{T_0/2} -f(t)\left[-\cos n\omega_0 t\right]dx\right.$$

$$= \frac{4}{T_0}\left[\int_0^{T_0/2} f(t)\cos n\omega_0 t\,dt\right.$$

In a similar way we can show

$$b_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\sin n\omega_0 t\,dt$$

(b) (i) $T_0 = 8,\ \omega_0 = \frac{\pi}{4},\quad a_0 = 0$ (by inspection)

Half wave symmetry. Hence

$$a_n = \frac{4}{8}\left[\int_0^4 f(t)\cos\frac{n\pi}{4}t\,dt\right] = \frac{1}{2}\left[\int_0^2 \frac{t}{2}\cos\frac{n\pi}{4}t\,dt\right]$$

$$= \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + \frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \qquad (n \text{ odd})$$

$$= \frac{4}{n^2\pi^2}\left(\frac{n\pi}{2}\sin\frac{n\pi}{2} - 1\right) \qquad (n \text{ odd}$$

Therefore

$$a_n = \begin{cases} \dfrac{4}{n^2\pi^2}\left(\dfrac{n\pi}{2} - 1\right) & n = 1, 5, 9, 13, \ldots \\[2ex] \dfrac{-4}{n^2\pi^2}\left(\dfrac{n\pi}{2} + 1\right) & n = 3, 7, 11, 15, \ldots \end{cases}$$

Similarly

$$b_n = \frac{1}{2}\int_0^2 \frac{t}{2}\sin\frac{n\pi}{4}t\,dt = \frac{4}{n^2\pi^2}\left(\sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right)$$

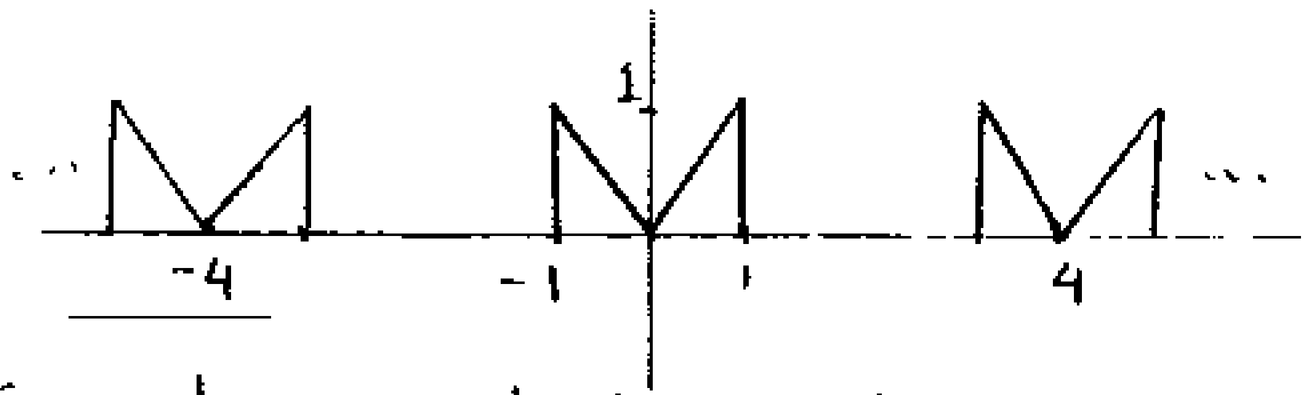
$$= \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \qquad (n \text{ odd})$$

and

$$f(t) = \sum_{n=1,3,5,\ldots}^{\infty} a_n\cos\frac{n\pi}{4}t + b_n\sin\frac{n\pi}{4}t$$

3.4-8)

Here, we need only cosine terms and $\omega_0 = \pi/2$. Hence, we must construct a pulse that it is an even function of t, has a value t over the interval $0 \le t \le 1$, and repeats every 4 seconds as shown in the fig.



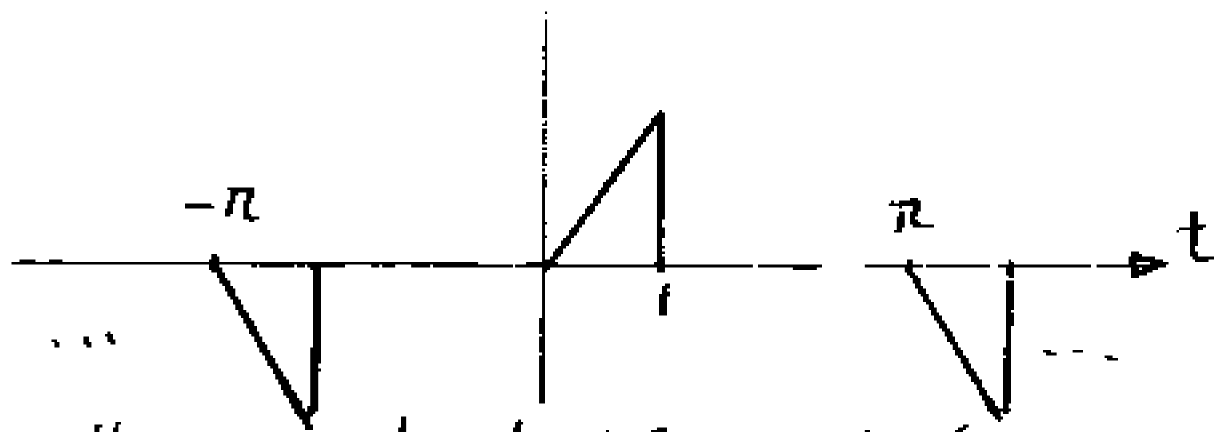
We selected the pulse width $W = 2$ seconds. But it can be anywhere from 2 to 4, and still satisfy these conditions. Each value of W results in different series. Yet all of them converge to t over 0 to 1, and satisfy the other requirements. Clearly, there are infinite number of Fourier series that will satisfy the given requirements. The present choice yields

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{2}\right)t$$

By inspection, we find $a_0 = 1/4$. Because of symmetry $b_n = 0$ and

$$a_n = \frac{4}{4}\int_0^1 t \cos\frac{n\pi}{2}t\,dt = \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right]$$

(f) Here, we need both sine and cosine terms with $\omega_0 = 1$ and odd harmonics only. Hence, we must construct a pulse such that it has half-wave symmetry, but neither odd nor even symmetry, has a value t over the interval $0 \le t \le 1$, and repeats every 2π seconds as shown in the fig..



observe that the first half cycle (from 0 to π) and second half cycle (from π to 2π) are negatives of each other as required in half-wave symmetry. By inspection, $a_0 = 0$.

This yields $$f(t) = \sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} a_n \cos nt + b_n \sin nt$$

Because of half-wave symmetry (see Prob. 3.4-7)

$$a_n = \frac{4}{2\pi}\int_0^1 t \cos nt\,dt = \frac{2}{n^2\pi}\left(\cos n + n\sin n - 1\right)$$

$$b_n = \frac{4}{2\pi}\int_0^1 t \sin nt\,dt = \frac{2}{n^2\pi}\left(\sin n - n\cos n\right) \qquad n \text{ odd}.$$

3.5-4) a)

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$\hat{f}(t) = f(t-T) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 (t-T)} = \sum_{n=-\infty}^{\infty} \left(D_n e^{-jn\omega_0 T}\right) e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \hat{D}_n e^{jn\omega_0 t}$$

where $\hat{D}_n = D_n e^{-jn\omega_0 t}$ so that $|\hat{D}_n| = |D_n|$

$$\angle \hat{D}_n = \angle D_n - jn\omega_0 T$$

(b) $$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$\hat{f}(t) = f(at) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 (at)}$$

3.6-1) Period $T_0 = \pi$, and $\omega_0 = 2$, and

$H(j\omega) = \dfrac{j\omega}{(-\omega^2+3)+j2\omega}$ and from eq. (3.74) $D_n = \dfrac{0.504}{1+j4n}$

Therefore, $y(t) = \sum_{n=-\infty}^{\infty} D_n H(jn\omega_0) e^{jn\omega_0 t}$

$$= \sum_{n=-\infty}^{\infty} \frac{j1.08n}{(1+j4n)(-\omega^2+3+j2\omega)} e^{j2nt}$$