Chapter 4 (Solutions of selected problems).

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4.1-1
$$F(\omega)=\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt=\int_{-\infty}^{+\infty} f(t)\,Cos(\omega t)\,dt-j\int_{-\infty}^{+\infty} f(t)\,Sin(\omega t)\,dt$$

If $f(t)$ is an even function of t, $f(t)\,Sin\,\omega t$ is an odd function of t, and the second integral vanishes. Moreover, $f(t)\,Cos(\omega t)$ is an even function of t, and the first integral is twice the integral over the interval 0 to ∞. Thus when $f(t)$ is even:

$$F(\omega)=2\int_0^{\infty} f(t)\,Cos(\omega t)dt. \qquad (1)$$

Similar argument shows that when $f(t)$ is odd:

$$F(\omega)=-2j\int_0^{\infty} f(t)\,Sin(\omega t)\,dt. \qquad (2)$$

If $f(t)$ is also real (in addition to being even), the integral (1) is real. Moreover from (1):

$$F(-\omega)=2\int_0^{\infty} f(t)\,Cos(\omega t)\,dt=F(\omega)$$

Hence $F(\omega)$ is real and even function of ω. Similar arguments can be used to prove the rest of the properties.

4.1-2
$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty} |F(\omega)|e^{j\angle F(\omega)}\cdot e^{j\omega t}d\omega$$

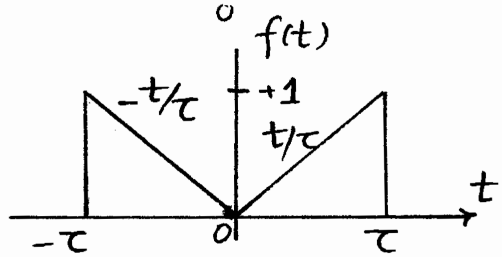
$$=\frac{1}{2\pi}\Big[\underbrace{\int_{-\infty}^{+\infty}|F(\omega)|\,Cos[\omega t+\angle F(\omega)]d\omega}_{\text{an even function which is twice the integral over 0 to }\infty}+\underbrace{j\int_{-\infty}^{+\infty}|F(\omega)|\,Sin[\omega t+\angle F(\omega)]d\omega}_{\text{an odd function which vanishes}}\Big]$$

$$\longrightarrow f(t)=\frac{1}{\pi}\int_0^{\infty}|F(\omega)|\,Cos[\omega t+\angle F(\omega)]\,d\omega.$$

4.1-4 a) $F(\omega)=\int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt = \int_0^T e^{-at} \cdot e^{-j\omega t}\, dt = \int_0^T e^{-(a+j\omega)t}\, dt$

$$= -\frac{1}{(a+j\omega)} e^{-(a+j\omega)t}\Big|_{t=0}^{t=T} = \frac{1-e^{-(a+j\omega)T}}{a+j\omega}$$

4.1-5 (b) $F(\omega)=\int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-\tau}^{0} -\frac{t}{\tau} e^{-j\omega t}\, dt + \int_0^{\tau} \frac{t}{\tau} e^{-j\omega t}\, dt$



$\int_{-\tau}^{0} -\frac{t}{\tau} e^{-j\omega t}\, dt$ using by-part integration as:

$t = u \rightarrow dt = du$

$e^{-j\omega t} dt = dv \rightarrow v = -\frac{1}{j\omega} e^{-j\omega t}$

$$\rightarrow \int_{-\tau}^{0} -\frac{t}{\tau} e^{-j\omega t}\, dt = -\frac{1}{\tau}\left[-\frac{t}{j\omega} e^{-j\omega t}\Big|_{-\tau}^{0} - \int_{-\tau}^{0} \left(-\frac{1}{j\omega}\right) e^{-j\omega t}\, dt \right]$$

$$= -\frac{1}{\tau}\left[-\frac{\tau}{j\omega} e^{j\omega\tau} + \frac{1}{\omega^2} e^{-j\omega t}\Big|_{-\tau}^{0} \right]$$

$$= -\frac{1}{\tau}\left[-\frac{\tau}{j\omega} e^{j\omega\tau} + \frac{1}{\omega^2}\left(1 - e^{j\omega\tau}\right) \right] = \frac{1}{j\omega} e^{j\omega\tau} - \frac{1}{\omega^2\tau}\left(1 - e^{j\omega\tau}\right)$$

Similarly: $\int_0^{\tau} \frac{t}{\tau} e^{-j\omega t}\, dt = -\frac{1}{j\omega} e^{-j\omega\tau} + \frac{1}{\omega^2\tau}\left(e^{-j\omega\tau} - 1\right)$

$$\Rightarrow F(\omega) = \frac{1}{j\omega} e^{+j\omega\tau} - \frac{1}{\omega^2\tau}\left(1 - e^{j\omega\tau}\right) - \frac{1}{j\omega} e^{-j\omega\tau} + \frac{1}{\omega^2\tau}\left(e^{-j\omega\tau} - 1\right)$$

$$= \frac{1}{j\omega}\left(e^{j\omega\tau} - e^{-j\omega\tau}\right) + \frac{1}{\omega^2\tau}\left(e^{j\omega\tau} + e^{-j\omega\tau} - 2\right)$$

$$= \frac{1}{\cancel{j}\omega}\left(\cancel{j}2 \sin\omega\tau\right) + \frac{1}{\omega^2\tau}\left(2\cos\omega\tau - 2\right).$$

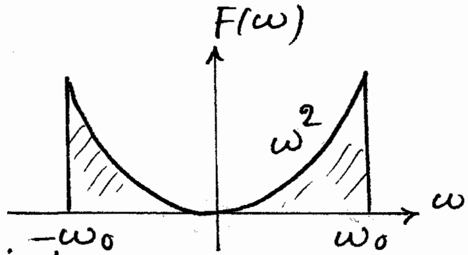
$$= \frac{2}{\omega} \sin(\omega\tau) + \frac{1}{\omega^2\tau}(2\cos\omega\tau - 2)$$

$$\longrightarrow F(\omega) = \frac{2}{\omega^2\tau}(\cos\omega\tau + \omega\tau\sin\omega\tau - 1)$$

▨

4.1-6.

(a)



$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{+\omega_0} \omega^2 e^{j\omega t} d\omega$$

Using part-by-part integration to solve $I = \int_{-\omega_0}^{\omega_0} \omega^2 e^{j\omega t} d\omega$:

$u = \omega^2 \longrightarrow du = 2\omega d\omega$

$e^{j\omega t} d\omega = dv \rightarrow v = \frac{1}{jt} e^{j\omega t}$

$$\rightarrow I = \int_{-\omega_0}^{+\omega_0} \omega^2 e^{j\omega t} d\omega = \left[\frac{\omega^2}{jt} e^{j\omega t} \Big|_{-\omega_0}^{\omega_0} - \int_{-\omega_0}^{+\omega_0} \frac{1}{jt} e^{j\omega t} 2\omega d\omega \right]$$

$$= \left[\frac{1}{jt} \underbrace{\left(\omega_0^2 e^{j\omega_0 t} - \omega_0^2 e^{-j\omega_0 t} \right)}_{\omega_0^2 (e^{j\omega_0 t} - e^{-j\omega_0 t}) = \omega_0^2 (j2 \sin\omega_0 t)} - \frac{2}{jt} \int_{-\omega_0}^{+\omega_0} \omega e^{j\omega t} d\omega \right].$$

The second term, $I_1 = \int_{-\omega_0}^{+\omega_0} \omega e^{j\omega t} d\omega$, can be solve using by-part integration:

$I_1 = \int_{-\omega_0}^{\omega_0} \omega e^{j\omega t} d\omega$: $\quad u = \omega \rightarrow du = d\omega$

$e^{j\omega t} d\omega = dv \rightarrow v = \frac{1}{jt} e^{j\omega t}$

$$\Rightarrow I_1 = \int_{-\omega_0}^{+\omega_0} \omega e^{j\omega t}\, d\omega = \frac{\omega}{jt} e^{j\omega t}\Big|_{-\omega_0}^{+\omega_0} - \int_{-\omega_0}^{+\omega_0} \frac{1}{jt} e^{j\omega t}\, d\omega$$

$$= \frac{1}{jt}\left(\omega_0 e^{j\omega_0 t} + \omega_0 e^{-j\omega_0 t}\right) - \frac{1}{jt}\left(\frac{1}{jt}\right) e^{j\omega t}\Big|_{-\omega_0}^{+\omega_0}$$

$$= \frac{\omega_0}{jt}(2\cos\omega_0 t) + \frac{1}{t^2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

$$= \frac{\omega_0}{jt}\left(2\cos\omega_0 t\right) + \frac{1}{t^2}\left(j2\sin\omega_0 t\right)$$

Putting I_1 in the expression of I,

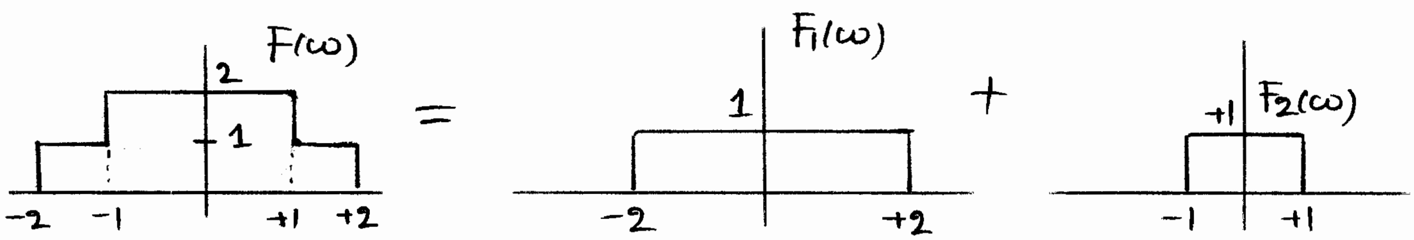
$$\Rightarrow I = \frac{1}{jt}\omega_0^2\left(\cancel{j}2\sin\omega_0 t\right) - \frac{2}{jt}\left[\frac{\omega_0}{jt}2\cos\omega_0 t + \frac{1}{t^2}\left(\cancel{j}2\sin\omega_0 t\right)\right]$$

$$= \frac{2\omega_0^2}{t}\sin(\omega_0 t) + \frac{4\omega_0}{t^2}\cos(\omega_0 t) - \frac{4}{t^3}\sin\omega_0 t$$

$$= \frac{2(\omega_0^2 t^2 - 2)\sin(\omega_0 t) + 4\omega_0 t\cos\omega_0 t}{t^3}$$

$$f(t) = \frac{1}{2\pi} I \longrightarrow f(t) = \frac{(\omega_0^2 t^2 - 2)\sin(\omega_0 t) + 2\omega_0 t\cos\omega_0 t}{\pi t^3}$$

(b) The derivation can be simplified by observing that $F(\omega)$ can be expressed as a sum of two gate functions $F_1(\omega)$ and $F_2(\omega)$ as shown below:



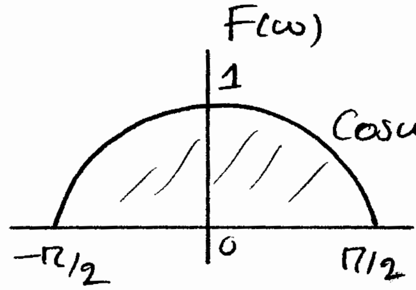
Therefore:

$$f(t)=\frac{1}{2\pi}\int_{-2}^{+2}\left[F_1(\omega)+F_2(\omega)\right]e^{j\omega t}d\omega=\frac{1}{2\pi}\left\{\int_{-2}^{+2}e^{j\omega t}d\omega+\int_{-1}^{+1}e^{j\omega t}d\omega\right\}$$

$$=\frac{1}{2\pi}\left\{\frac{1}{jt}\left(e^{j2t}-e^{-j2t}\right)+\frac{1}{jt}\left(e^{jt}-e^{-jt}\right)\right\}$$

$$=\frac{1}{2\pi}\left\{\frac{1}{jt}\left(j2\sin 2t\right)+\frac{1}{jt}\left(j2\sin t\right)\right\}=\frac{\sin 2t+\sin t}{\pi t}$$

4.1-7 (a)



$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2}\cos\omega\, e^{j\omega t}d\omega$$

$$=\frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2}\frac{1}{2}\left(e^{j\omega}+e^{-j\omega}\right)e^{j\omega t}d\omega=\frac{1}{4\pi}\left[\int_{-\pi/2}^{+\pi/2}\left(e^{j\omega}\cdot e^{j\omega t}+e^{-j\omega}\cdot e^{j\omega t}\right)d\omega\right]$$

$$=\frac{1}{4\pi}\left[\int_{-\pi/2}^{+\pi/2}e^{j(t+1)\omega}d\omega+\int_{-\pi/2}^{+\pi/2}e^{j\omega(t-1)}d\omega\right]$$

$$=\frac{1}{4\pi}\left[\frac{1}{j(t+1)}e^{j(t+1)\omega}\Big|_{\omega=-\pi/2}^{+\pi/2}+\frac{1}{j(t-1)}e^{j(t-1)\omega}\Big|_{-\pi/2}^{+\pi/2}\right].$$

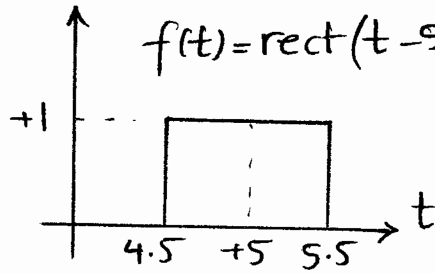
$$=\frac{1}{4\pi}\left[\frac{1}{j(t+1)}\left(e^{j\frac{\pi}{2}(t+1)}-e^{-j\frac{\pi}{2}(t+1)}\right)+\frac{1}{j(t-1)}\left(e^{j\frac{\pi}{2}(t-1)}-e^{-j\frac{\pi}{2}(t-1)}\right)\right]$$

$$=\frac{1}{4\pi}\left[\frac{1}{j(t+1)}\left(j2\sin\left(\frac{\pi}{2}(t+1)\right)+\frac{1}{j(t-1)}\left(j2\sin\left(\frac{\pi}{2}(t-1)\right)\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{t+1}\cos\left(\frac{\pi t}{2}\right) - \frac{1}{t-1}\cos\left(\frac{\pi t}{2}\right)\right]$$

$$= \frac{1}{2\pi}\cos(\pi t/2)\left(\frac{t+1-t+1}{t^2-1}\right) = \frac{1}{\pi(t^2-1)}\cos\left(\frac{\pi t}{2}\right)$$

4.2-2

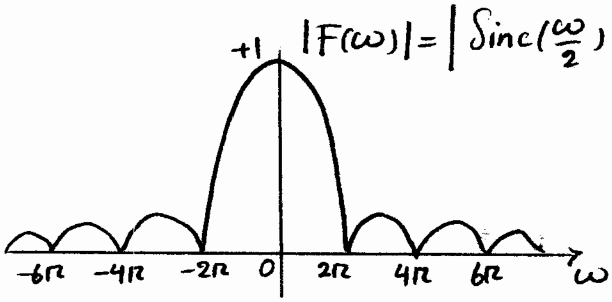


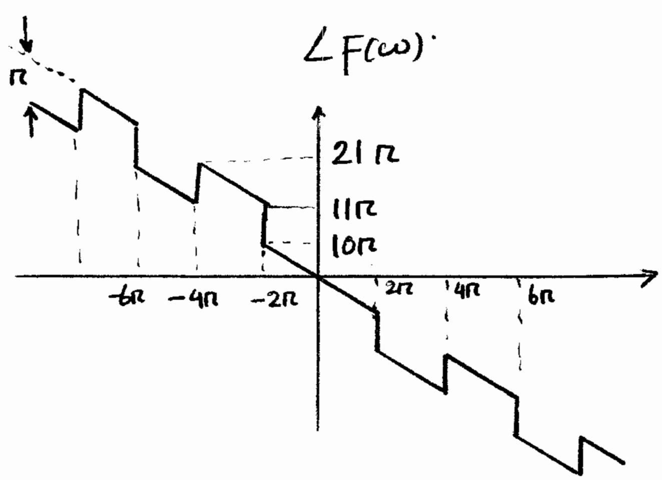
$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = \int_{4.5}^{5.5} e^{-j\omega t}dt = -\frac{1}{j\omega}e^{-j\omega t}\Big|_{t=4.5}^{5.5}$$

$$= -\frac{1}{j\omega}\left(e^{-j5.5\omega} - e^{-j4.5\omega}\right) = \frac{-e^{-j5\omega}}{j\omega}\left(e^{-j0.5\omega} - e^{j0.5\omega}\right)$$

$$= \frac{-e^{-j5\omega}}{j\omega}\left(-j2\sin(0.5\omega)\right) = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)e^{-j5\omega} = \frac{\sin(0.5\omega)}{0.5\omega}e^{-j5\omega}$$

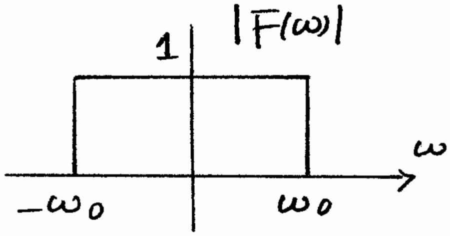
$$\rightarrow F(\omega) = \mathrm{Sinc}(\omega/2)\, e^{-j5\omega}$$

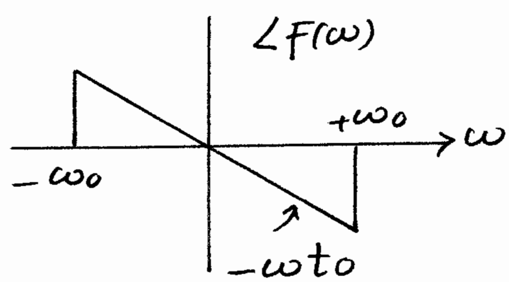




4.2-4

(a)





(b)

$$F(\omega) = |F(\omega)| e^{j\angle F(\omega)} = e^{-j\omega t_0} \qquad -\omega_0 \leq \omega \leq \omega_0$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{+\omega_0} e^{-j\omega t_0} \cdot e^{j\omega t} d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{+\omega_0} e^{-j\omega(t-t_0)} d\omega$$

$$= \frac{1}{2\pi}\left(\frac{1}{-j(t-t_0)}\right) e^{-j\omega(t-t_0)}\Big|_{\omega=-\omega_0}^{\omega_0} = \frac{1}{2\pi}\left(-\frac{1}{j(t-t_0)}\right)\left(e^{-j\omega_0(t-t_0)} - e^{j\omega_0(t-t_0)}\right)$$

$$= \frac{1}{2\pi}\left(-\frac{1}{\cancel{j}(t-t_0)}\right)\left(-\cancel{j}2\sin\omega_0(t-t_0)\right)$$

$$\rightarrow f(t) = \frac{\sin\omega_0(t-t_0)}{\pi(t-t_0)} = \frac{\omega_0}{\pi}\,\mathrm{Sinc}\,\omega_0(t-t_0)$$

4.3-1

(a)

$$\underbrace{u(t)}_{f(t)} \longleftrightarrow \underbrace{\pi\delta(\omega) + \frac{1}{j\omega}}_{F(\omega)}$$

Using duality property:

$$\pi\delta(t) + \frac{1}{jt} \longleftrightarrow 2\pi u(-\omega)$$

$$\text{or } \frac{1}{2}\left[\delta(t) + \frac{1}{j\pi t}\right] \longleftrightarrow u(-\omega)$$

Application of equation (4.35) yeilds:

$$\frac{1}{2}\left[\delta(-t) + \frac{j}{\pi t}\right] \longleftrightarrow u(\omega)$$

but $\delta(t)$ is an even function, that is $\delta(-t) = \delta(t)$ and

$$\frac{1}{2}\left[\delta(t) + \frac{j}{\pi t}\right] \longleftrightarrow u(\omega)$$

(c).

$$\underbrace{\sin(\omega_0 t)}_{f(t)..} \longleftrightarrow \underbrace{j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]}_{F(\omega)}$$

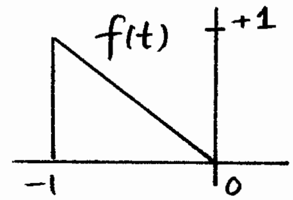
Using duality property:

$$\underbrace{j\pi[\delta(t+\omega_0)-\delta(t-\omega_0)]}_{F(t)} \longleftrightarrow \underbrace{2\pi\sin(-\omega_0\omega)}_{2\pi f(-\omega)} = -2\pi\sin(\omega_0\omega)$$

Setting $\omega_0 = T$ yeilds:

$$\delta(t+T)-\delta(t-T) \longleftrightarrow 2j\sin T\omega.$$

4.3-2



$$\leftrightarrow F(\omega) = \frac{1}{\omega^2}\left(e^{j\omega} - j\omega e^{j\omega} - 1\right)$$

(b) $f_1(t) = f(-t) \longrightarrow F_1(\omega) = F(-\omega) = \frac{1}{\omega^2}\left[e^{-j\omega} + j\omega e^{-j\omega} - 1\right]$

(c) $f_2(t) = f(t-1) + f_1(t-1)$ therefore

$$F_2(\omega) = F(\omega)e^{-j\omega} + F_1(\omega)e^{-j\omega} = [F(\omega)+F(-\omega)]e^{-j\omega}$$

$$= \left(\frac{1}{\omega^2}e^{j\omega} - \frac{j}{\omega}e^{j\omega} - \frac{1}{\omega^2} + \frac{1}{\omega^2}e^{-j\omega} + \frac{j}{\omega}e^{-j\omega} - \frac{1}{\omega^2}\right)e^{-j\omega}$$

$$= \left[\frac{1}{\omega^2}\left(e^{j\omega}+e^{-j\omega}\right) - \frac{j}{\omega}\left(e^{j\omega}-e^{-j\omega}\right) - \frac{2}{\omega^2}\right]e^{-j\omega}$$

$$= \left[\frac{1}{\omega^2}(2\cos\omega) - \frac{j}{\omega}(2j\sin\omega) - \frac{2}{\omega^2}\right]e^{-j\omega}$$

$$= \frac{2e^{-j\omega}}{\omega^2}\left(\cos\omega + \omega\sin\omega - 1\right)$$

(f) $f_5(t)$ can be obtained in three steps: (i) time expanding $f(t)$ by a factor 2. (ii) then delaying it by 2 seconds, (iii) and multiplying it by 1.5 [we may interchange the sequence for steps (i) and (ii)]

The first step yeilds:

$$f\left(\frac{t}{2}\right) \longleftrightarrow 2F(2\omega) = \frac{1}{2\omega^2}\left(e^{j2\omega} - j2\omega e^{j2\omega} - 1\right)$$

Second step of time delay of 2 seconds yeilds:

$$f\left(\frac{t-2}{2}\right) \longleftrightarrow \frac{1}{2\omega^2}\left(e^{j2\omega} - j2\omega e^{j2\omega} - 1\right)e^{-j2\omega}$$

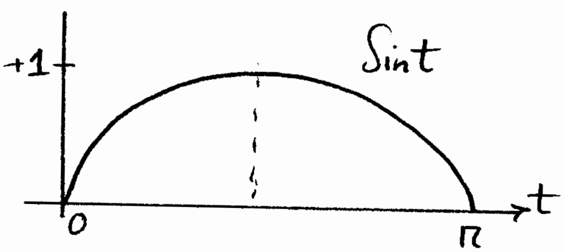
$$= \frac{1}{2\omega^2}\left(1 - j2\omega - e^{-j2\omega}\right)$$

The third step of multiplying the resulting signal by 1.5 yeilds:

$$f_5(t) = 1.5 f\left(\frac{t-2}{2}\right) \longleftrightarrow \frac{3}{4\omega^2}\left(1 - j2\omega - e^{-j2\omega}\right)$$

4.3-3

(b)



$f(t) = \sin t\, u(t) + \sin(t-\pi)\, u(t-\pi)$

Note that $\sin(t-\pi)u(t-\pi)$ is $\sin t\, u(t)$ delayed by π.

$$\sin t\, u(t) \longleftrightarrow \frac{\pi}{j2}\left[\delta(\omega-1) - \delta(\omega+1)\right] + \frac{1}{1-\omega^2} \quad \text{and:}$$

$$\sin(t-\pi)u(t-\pi) \longleftrightarrow \left\{\frac{\pi}{j2}\left[\delta(\omega-1) - \delta(\omega+1)\right] + \frac{1}{1-\omega^2}\right\} e^{-j\pi\omega}$$

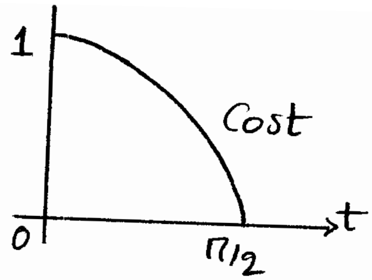
$$\rightarrow F(\omega) = \left\{\frac{\pi}{j2}\left[\delta(\omega-1) - \delta(\omega+1)\right] + \frac{1}{1-\omega^2}\right\}\left(1 + e^{-j\pi\omega}\right)$$

Recall that $f(x)\delta(x-x_0) = f(x_0)\,\delta(x-x_0)$. Therefore:

$$\delta(\omega \pm 1)\left(1 + e^{-j\pi\omega}\right) = 0.$$

$$\longrightarrow F(\omega) = \frac{1}{1-\omega^2}\left(1 + e^{-j\pi\omega}\right)$$

(c)



$$f(t) = \cos t\left[u(t) - u(t-\tfrac{\pi}{2})\right] = \cos t\, u(t) - \cos t\, u(t-\tfrac{\pi}{2})$$

but $\sin(t-\frac{\pi}{2}) = -\cos t$ and the second term can be written as: $\sin(t-\frac{\pi}{2})u(t-\frac{\pi}{2})$. Hence:

$$f(t) = \cos t\, u(t) + \sin(t-\tfrac{\pi}{2})u(t-\tfrac{\pi}{2})$$

$$\rightarrow F(\omega) = \frac{\pi}{2}\left[\delta(\omega-1)+\delta(\omega+1)\right] + \frac{j\omega}{1-\omega^2} + \left\{\frac{\pi}{j2}\left[\delta(\omega-1) - \delta(\omega+1)\right] + \frac{1}{1-\omega^2}\right\} e^{-j\frac{\pi\omega}{2}}$$

Also because $f(x)\,\delta(x-x_0) = f(x_0)\,\delta(x-x_0)$ we have:

$$\delta(\omega\pm 1)\, e^{-j\frac{\pi\omega}{2}} = \delta(\omega\pm 1)\, e^{\pm j\frac{\pi}{2}} = \pm j\,\delta(\omega\pm 1)$$

Therefore:

$$F(\omega) = \frac{j\omega}{1-\omega^2} + \frac{e^{-j\pi\omega/2}}{1-\omega^2} = \frac{1}{1-\omega^2}\left(j\omega + e^{-j\pi\omega/2}\right)$$

■

4.3-5

$$f(t)\sin(\omega_0 t) = f(t)\left(\frac{1}{j2}e^{j\omega_0 t} - \frac{1}{j2}e^{-j\omega_0 t}\right)$$

$$= \frac{1}{2j}\left(f(t)e^{j\omega_0 t} - f(t)e^{-j\omega_0 t}\right)$$

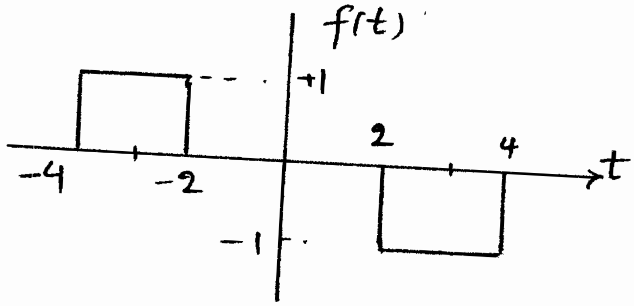
$$\longleftrightarrow \frac{1}{j2}\left[F(\omega-\omega_0) - F(\omega+\omega_0)\right]$$

$$\frac{1}{2j}\left[f(t+T) - f(t-T)\right] \longleftrightarrow \frac{1}{2j}\left[F(\omega)e^{+j\omega T} - F(\omega)e^{-j\omega T}\right]$$

$$= \frac{1}{2j}\left[F(\omega)\left(e^{j\omega T} - e^{-j\omega T}\right)\right]$$

$$= \frac{1}{2j}\left[F(\omega)\,(\cancel{2j}\,\mathrm{Sin}\,\omega T)\right] = F(\omega)\,\mathrm{Sin}\,\omega T.$$

For the following signal:



$$f(t) = \mathrm{rect}\left(\frac{t+3}{2}\right) - \mathrm{rect}\left(\frac{t-3}{2}\right)$$

$$\mathrm{rect}(t) \longleftrightarrow \mathrm{Sinc}(\omega/2)$$

$$\rightarrow \mathrm{rect}\left(\frac{t}{2}\right) \longleftrightarrow 2\,\mathrm{Sinc}(\omega)$$

$$\rightarrow \mathrm{rect}\left(\frac{t+3}{2}\right) - \mathrm{rect}\left(\frac{t-3}{2}\right) \longleftrightarrow 2j\left[2\,\mathrm{Sinc}(\omega)\,\mathrm{Sin}(3\omega)\right]$$

$$= j4\,\mathrm{Sinc}(\omega)\,\mathrm{Sin}(3\omega)$$

4.3-6.

(a) The signal $f(t)$ in this case is a triangle pulse $\Delta(\frac{t}{2\pi})$ multiplied by $\mathrm{Cos}(10t)$:

$$f(t) = \Delta\left(\frac{t}{2\pi}\right)\mathrm{Cos}(10t)$$

From table 4.1 (pair 19): $\Delta(\frac{t}{2\pi}) \longleftrightarrow \pi\,\mathrm{Sinc}^2(\frac{\pi\omega}{2})$. Using modulation property (4.41) it follows that:

$$f(t) = \Delta\left(\frac{t}{2\pi}\right)\mathrm{Cos}(10t) \longleftrightarrow \frac{\pi}{2}\left\{\mathrm{Sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \mathrm{Sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}$$

(b) The signal $f(t)$ here is the same as the signal in Fig. (a) delayed by 2π. From time shifting property, its Fourier transform can be written as:

$$F(\omega)=\frac{\pi}{2}\left\{\text{Sinc}^2\left[\frac{\pi(\omega-10)}{2}\right]+\text{Sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}e^{-j2\pi\omega}$$

The Fourier transform in this case is the same as that in part (a) multiplied by $e^{-j2\pi\omega}$. This multiplying factor represents a linear phase spectrum $-2\pi\omega$. Thus we have an "amplitude spectrum" as well as a linear "phase spectrum" $\angle F(\omega)=-2\pi\omega$.

(c) In this case the signal is identical to that in Fig. b except that the basic pulse is $\text{rect}\left(\frac{t}{2\pi}\right)$ instead of a triangle pulse $\Delta\left(\frac{t}{2\pi}\right)$.
Now:

$$\text{rect}\left(\frac{t}{2\pi}\right)\longleftrightarrow 2\pi\,\text{Sinc}(\pi\omega)$$

Using the same argument as for part (b), we obtain:

$$F(\omega)=\pi\left\{\text{Sinc}[\pi(\omega+10)]+\text{Sinc}[\pi(\omega-10)]\right\}e^{-j2\pi\omega}$$

4.3-8

(a) $$u(t)\longleftrightarrow \pi\delta(\omega)+\frac{1}{j\omega}$$

$$e^{\lambda t}u(t)\longleftrightarrow\frac{1}{j\omega-\lambda}$$

if $f(t)=e^{\lambda t}u(t)*u(t)$ then

$$F(\omega)=\left(\frac{1}{j\omega-\lambda}\right)\left(\pi\delta(\omega)+\frac{1}{j\omega}\right)$$

$$=\frac{\pi\delta(\omega)}{j\omega-\lambda}+\left[\frac{1}{j\omega(j\omega-\lambda)}\right]=-\frac{\pi}{\lambda}\delta(\omega)+\left[\frac{-\frac{1}{\lambda}}{j\omega}+\frac{\frac{1}{\lambda}}{j\omega-\lambda}\right]$$

$$=\frac{1}{\lambda}\left[\frac{1}{j\omega-\lambda}-\left(\pi\delta(\omega)+\frac{1}{j\omega}\right)\right]$$

Taking the inverse Fourier transform of this equations yeilds:

$f(t)=\frac{1}{\lambda}\left(e^{\lambda t}-1\right)u(t).$

(c) $e^{\lambda_1 t}u(t) \longleftrightarrow \frac{1}{j\omega-\lambda_1}$; $e^{\lambda_2 t}u(-t) \longleftrightarrow -\frac{1}{j\omega-\lambda_2}$

If $f(t)=e^{\lambda_1 t}u(t) * e^{\lambda_2 t}u(-t)$ then:

$$F(\omega)=\frac{-1}{(j\omega-\lambda_1)(j\omega-\lambda_2)}=\frac{\frac{1}{\lambda_2-\lambda_1}}{j\omega-\lambda_1}-\frac{\frac{1}{\lambda_2-\lambda_1}}{j\omega-\lambda_2}$$

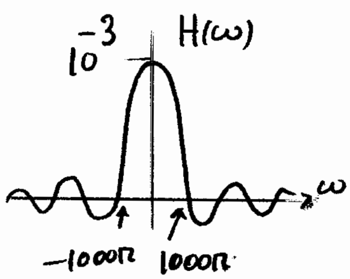
therefore: $f(t)=\frac{1}{\lambda_2-\lambda_1}\left[e^{\lambda_1 t}u(t)+e^{\lambda_2 t}u(-t)\right].$

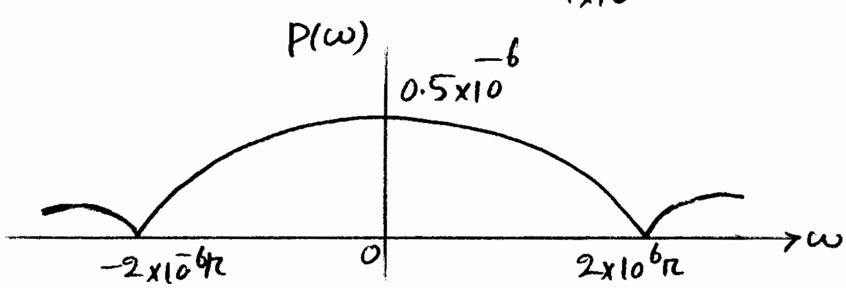
Note that because $\lambda_2>0$ the inverse transform of $-\frac{1}{j\omega-\lambda_2}$ is $e^{\lambda_2 t}u(-t)$ and not $-e^{\lambda_2 t}u(t)$. The Fourier transform of the latter does not exist because $\lambda_2>0$.

4.4-4

$$h(t)=\text{rect}\left(\frac{t}{10^{-3}}\right) \longleftrightarrow H(\omega)=10^{-3}\,\text{Sinc}\left(\frac{\omega}{2000}\right)$$

$$p(t)=\Delta\left(\frac{t}{10^{-6}}\right) \longleftrightarrow P(\omega)=0.5\times10^{-6}\,\text{Sinc}^2\left(\frac{\omega}{4\times10^{6}}\right)$$





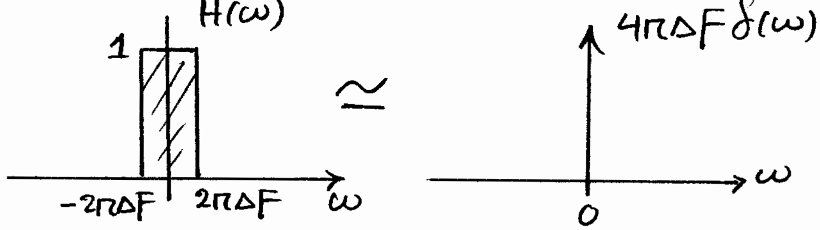
From the Figures, it is clear that $H(\omega)$ is much narrower than $P(\omega)$ and we may consider $P(\omega)$ to be nearly constant of value $P(0)=10^{-6}/2$ over the entire band of $H(\omega)$. Hence:

$$Y(\omega)=P(\omega)H(\omega)\approx P(0)H(\omega)=0.5\times10^{-6}H(\omega)\rightarrow y(t)=0.5\times10^{-6}h(t)$$

Recall that $h(t)$ is the unit impulse response of the system. Therefore the output $y(t)$ is equal to the system response to an input $0.5 \times 10^{-6} \delta(\omega) = A\delta(\omega)$.

4.6-3

If $f^2(t) \longleftrightarrow A(\omega)$ then the output $Y(\omega) = A(\omega) H(\omega)$ where $H(\omega)$ is the lowpass filter transfer function. As $\Delta F \to 0$, we may express $H(\omega)$ as an impulse function of area $4\pi\Delta f$:



$$\longrightarrow H(\omega) \approx [4\pi\Delta F]\,\delta(\omega) \quad \text{and} \quad Y(\omega) \approx [4\pi A(\omega)\Delta F]\,\delta(\omega)$$
$$= [4\pi A(0)\Delta F]\,\delta(\omega)$$

Here we used the property $f(x)\,\delta(x) = f(0)\,\delta(x)$. This yeilds:

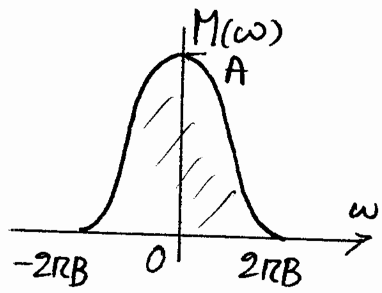
$$y(t) = 2A(0)\Delta F.$$

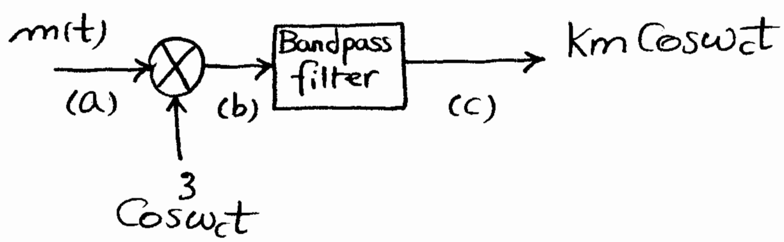
Now, because $f^2(t) \longleftrightarrow A(\omega)$, we have:

$$A(\omega) = \int_{-\infty}^{+\infty} f^2(t)\, e^{-j\omega t}\, dt \longrightarrow A(0) = \int_{-\infty}^{+\infty} f^2(t)\, e^{0}\, dt = \int_{-\infty}^{+\infty} f^2(t)\, dt = E_f.$$

$$\Rightarrow y(t) = 2E_f\Delta f.$$

4.7-2





(a) The signal at point b is:

$$f(t) = m(t)\cos^3\omega_c t$$

Using the following trigonometric equality:

$$\cos 3\omega_c t = 4\cos^3\omega_c t - 3\cos\omega_c t \longrightarrow \cos^3\omega_c t = \frac{3}{4}\cos\omega_c t + \frac{1}{4}\cos 3\omega_c t$$

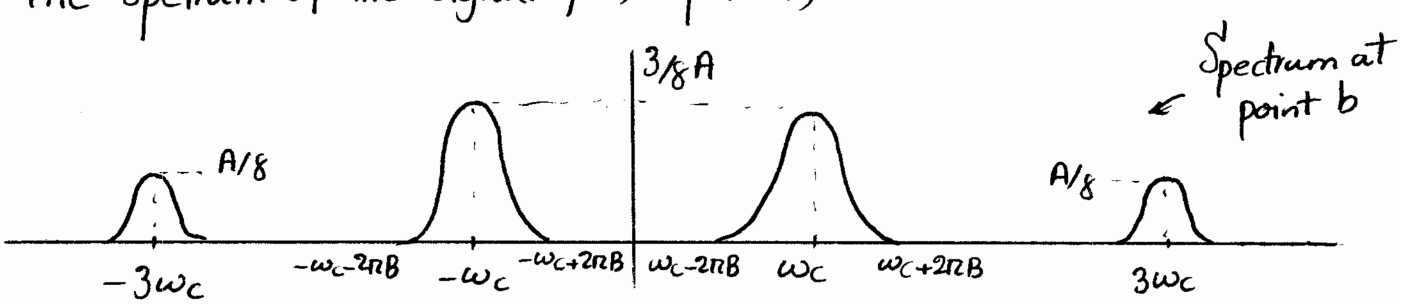
$$\rightarrow f(t) = m(t)\left(\frac{3}{4}\cos\omega_c t + \frac{1}{4}\cos 3\omega_c t\right)$$

$$\frac{3}{4}m(t)\cos\omega_c t + \frac{1}{4}m(t)\cos(3\omega_c t)$$

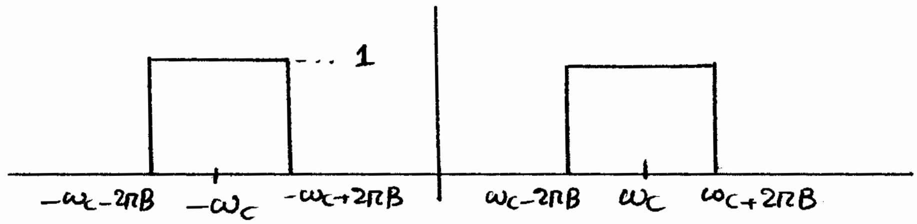
Using the "modulation" property of Fourier transform, the frequency Spectrum of f(t) can be obtained as:

$$F(\omega) = \underbrace{\frac{3}{8}\left[M(\omega-\omega_c)+M(\omega+\omega_c)\right]}_{\text{desired modulated signal spectrum centered at } \omega=\pm\omega_c} + \underbrace{\frac{1}{8}\left[M(\omega-3\omega_c)+M(\omega+3\omega_c)\right]}_{\text{undesired signal spectrum centered at } \omega=\pm 3\omega_c}$$

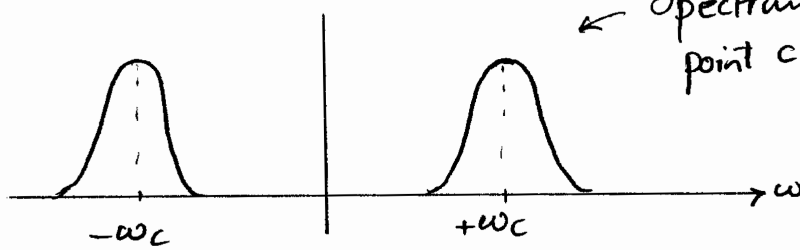
The spetrum of the signal f(t) (point b) is shown below:



For the bandpass filter centered at $\pm\omega_c$ with bandwidth $4\pi B$, the Spectrum is:

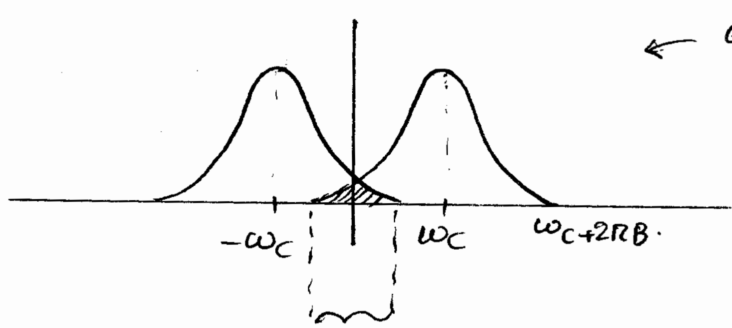


Passing $F(\omega)$ through the shown bandpass filter allows the desired modulated signal to pass, but the undesired signal centered at $\omega=\pm 3\omega_c$ will be suppressed. Hence the output of the bandpass filter is a signal with spectrum $\frac{3}{8}\left[M(\omega-\omega_c)+M(\omega+\omega_c)\right]$.



Spectrum at point c.

c) The minimum usable value of ω_c is $2\pi B$ in order to avoid spectral folding at dc. The following figure illustrates a value of ω_c less than $2\pi B$. Notice to the region where the Spectrum of modulated signal overlapps. Overlapping the spectrum in this case is equal to lossing information which is undesirable.



$\omega_c < 2\pi B$

Overlapping of the spectrum of the modulated signal (lossing information!)

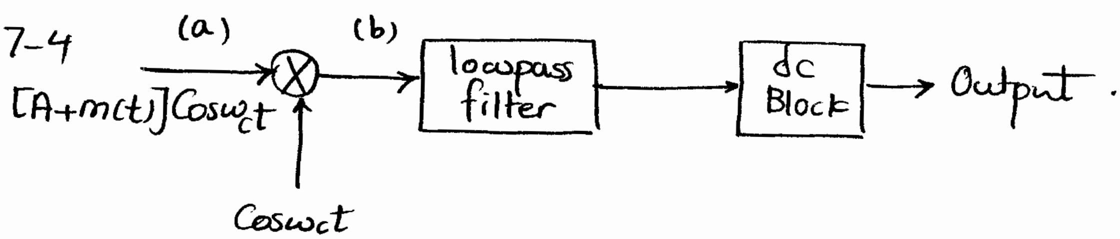
d) $m(t)\cos^2\omega_c t = \frac{m(t)}{2}\left[1+\cos(2\omega_c t)\right].$

$$= \frac{1}{2}m(t) + \frac{1}{2}m(t)\cos(2\omega_c t)$$

This signal at point (b) consists of a baseband signal $\frac{1}{2}m(t)$ and a modulated signal $\frac{1}{2}m(t)\cos(2\omega_c t)$ which has a carrier frequency $2\omega_c t$ not the desired value $\omega_c t$. Both of these two components will be suppressed by the filter. whose center frequency is ω_c. Hence this system will not do the desired job.

(e) You may verify that the identity for $\cos n\omega_c t$ contains a term $\cos(\omega_c t)$ when n is odd. This is not true when n is even. Hence the system works. for a carrier $\cos^n \omega_c t$ only when n is odd.

4.7-4



$f_a(t) = [A+m(t)]\cos\omega_c t$. Hence

$$f_b(t) = [A+m(t)]\cos^2\omega_c t = \frac{1}{2}[A+m(t)] + \frac{1}{2}[A+m(t)]\cos(2\omega_c t)$$

The first term is a lowpass signal centered at $\omega = 0$, The low pass filter allows this term to pass, but suppresses the second term whose spectrum is centered at $\pm 2\omega_c$. Hence the output of the lowpass filter is:

$$y(t) = A + m(t)$$

The dc block simply suppresses the dc component, A. This shows that the system can demodulate AM signal regardless of the value of A. This is synchronous or coherent demodulation.