Chapter 9. (Solution of selected problems).

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9.1-4.

$$y[k+2]+3y[k+1]+2y[k]=f[k+2]+3f[k+1]+3f[k].$$

$$\begin{cases} y[-1]=3 \\ y[-2]=2 \\ f[k]=(3)^k u[k]. \end{cases}$$

The equation can be written as:

$$y[k+2]=-3y[k+1]-2y[k]+f[k+2]+3f[k+1]+3f[k].$$

$k=-2$: $\quad y[0]=-3(3)-2(2)+1+3(0)+3(0)=-12$

$k=-1$: $\quad y[1]=-3(-12)-2(3)+3+3(1)+3(0)=36.$

$k=0$: $\quad y[2]=-3(36)-2(-12)+9+3(3)+3(1)=-63.$

9.2-3.

$$y[k+2]-2y[k+1]+2y[k]=0 \qquad y[-1]=1\ ;\ y[-2]=0$$

$$\rightarrow (E^2-2E+2)y[k]=0$$

characteristic equation is: $\gamma^2-2\gamma+2=0 \rightarrow (\gamma-1-j)(\gamma-1+j)=0$

$$\rightarrow \begin{cases} \gamma_1=1+j=\sqrt{2}\,e^{j(\pi/4)}. \\ \gamma_2=1-j=\sqrt{2}\,e^{-j(\pi/4)}. \end{cases}$$

$$\Rightarrow y[k]=C(\sqrt{2})^k\cos\left(\frac{\pi}{4}k+\theta\right)$$

$$k=-1 \longrightarrow 1=\frac{c}{\sqrt{2}}\cos\left(-\frac{\pi}{4}+\theta\right)=\frac{c}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right).$$

$$k=-2 \longrightarrow 0=\frac{c}{2}\cos\left(-\frac{\pi}{2}+\theta\right)=\frac{c}{2}\sin\theta$$

$$\begin{cases} c\cos\theta=2 \longrightarrow C=2 \\ c\sin\theta=0 \longrightarrow \theta=0 \end{cases} \Rightarrow y[k]=2(\sqrt{2})^k\cos\left(\frac{\pi}{4}k\right).$$

9.3-4. $y[k]-6y[k-1]+25y[k-2]=2f[k]-4f[k-1]$.

replace: $k \to k+2$:

$$y[k+2]-6y[k+1]+25y[k]=2f[k+2]-4\ [k+1]$$

$$\Rightarrow (E^2-6E+25)y[k]=(2E^2-4E)f[k]$$

Characteristic equation: $(\gamma^2-6\gamma+25)=0 \longrightarrow \begin{cases}\gamma_1=5e^{j0.923}\\ \gamma_2=5e^{-j0.923}\end{cases}$

$$\Rightarrow h[k]=c(5)^k\cos(0.923k+\theta)u[k]. \quad (I)$$

we need two values of $h[k]$ to determine c and θ. This is done by solving iteratively:

$$h[k]-6h[k-1]+25h[k-2]=2\delta[k]-4\delta[k-1].$$

$k=0$: $h[0]-6(0)+25(0)=2(1)-4(0) \longrightarrow h(0)=2$

$k=1$: $h[1]-6(2)+25(0)=2(0)-4 \longrightarrow h(1)=8$.

Setting $k=0,1$ in (I) and substituting $h[0]=2$, $h[1]=8$ yeilds:

$$\begin{cases}2=c\cos\theta.\\ 8=5c\cos(0.923+\theta)=3.017c\cos\theta-3.987c\sin\theta.\end{cases}$$

Solving for $\underline{c}$ and $\underline{\theta}$ we get: $c=2.061$, $\theta=-0.244$ rad.

$$\Rightarrow h[k]=2.061(5)^k\cos(0.923k-0.244)u[k].$$

9.4-2

$$y[k]=e^{-k}u[k]*\left\{\tfrac{1}{2}\delta[k]-\tfrac{1}{2}(-2)^k\right\}u[k]$$

$$=\tfrac{1}{2}e^{-k}u[k]*\delta[k]-\tfrac{1}{2}e^{-k}u[k]*(-2)^ku[k]$$

$$=\tfrac{1}{2}e^{-k}u[k]-\tfrac{1}{2}\sum_{m=-\infty}^{+\infty}e^{-m}u[m](-2)^{k-m}u[k-m]$$

$$= \frac{1}{2} e^{-k} u[k] - \frac{1}{2} \sum_{m=0}^{k} e^{-m} (-2)^{k} (-2)^{m}$$

$$= \frac{1}{2} e^{-k} u[k] - \frac{1}{2} (-2)^{k} \sum_{m=0}^{k} \left(\frac{1}{-2e}\right)^{m}$$

Using equation (B-7-4) of textbook $\left(\sum_{m=0}^{k} \left(\frac{a}{b}\right)^{m} = \frac{a^{k+1} - b^{k+1}}{b^{k}(a-b)} \quad a \neq b \right)$ and select $a = 1$; $b = -2e$; we have:

$$\rightarrow \frac{1}{2} e^{-k} u[k] - \frac{1}{2} (-2)^{k} \left[\frac{1 - (-2e)^{k+1}}{(-2e)^{k}(1+2e)} \right] \underset{u[k]}{\wedge} = \frac{1}{2} e^{-k} u[k] - \frac{1}{2} (e)^{-k} \frac{\left[1 - (-2e)^{k+1}\right]}{(1+2e)} u[k]$$

$$= \frac{1}{2} e^{-k} u[k] - \frac{1}{2} \frac{e^{-k} - (-2e)^{k+1} \cdot e^{-k}}{(2e+1)} u[k].$$

$$= \frac{1}{2} e^{-k} u[k] - \frac{1}{2} \frac{e^{-k} - (-2)^{k+1} \cdot e}{(2e+1)} u[k] = \frac{1}{2} e^{-k} u[k] - \frac{e\left[e^{-(k+1)} - (-2)^{k+1}\right]}{2(2e+1)} u[k]$$

$$\Rightarrow y[k] = \left\{ \frac{1}{2} e^{-k} - \frac{e}{2(2e+1)} \left[e^{-(k+1)} - (-2)^{k+1} \right] \right\} u[k].$$

9.4-5.

$$y[k] = (3)^{k} \cos\left(\frac{\pi}{3} k - 0.5\right) u[k] * (2)^{k} u[k]$$

Using Table 9.1 (page 590).

$$R = \left[(3)^{2} + (2)^{2} - 2(3)(2)(0.5) \right]^{1/2} = \sqrt{7}$$

$$\phi = \tan^{-1}\left[\frac{3\sqrt{3}/2}{1.5-2} \right] = 1.761 \text{ rad}.$$

$$y[k] = \frac{1}{\sqrt{7}} \left\{ (3)^{k+1} \cos\left[\frac{\pi}{3}(k+1) - 2.261\right] - (2)^{k+1} \cos(2.261) \right\} u[k].$$

$$= \frac{1}{\sqrt{7}} \left\{ (3)^{k+1} \cos\left[\frac{\pi}{3}(k+1) - 2.261\right] + 0.637 (2)^{k+1} \right\} u[k].$$

$$= \left\{ \frac{1}{2} e^{-k} - \frac{e}{2(2e+1)} \left[e^{-(k+1)} - (-2)^{k+1} \right] u[k] \right.$$

9.4-9

Second method:

In this approach, the initial condition is $y[0] = -M$ and the input is $f[k] = Pu[k-1]$ because the monthly payment of P starts at $k = 1$. The characteristic root is a, and the zero-input response is:

$$y_0[k] = c a^k u[k]$$

Setting $k = 0$; and substituting $y_0[0] = -M$. yeilds $c = -M$ and

$$y_0[k] = -M a^k u[k]$$

The zero_state response $y[k]$ is:

$$y[k] = h[k] * f[k] = h[k] * Pu[k-1] = Pa^k u[k] * u[k-1]$$

Here we use shift property of convolution. If we let

$$x[k] = a^k u[k] * u[k] = \frac{a^{k+1} - 1}{a-1} u[k]$$

The shift property yields

$$Pa^k u[k] * u[k-1] = x[k-1] = P\left[\frac{a^k - 1}{a-1}\right] u[k-1]$$

The total balance is $y_0[k] + y[k]$

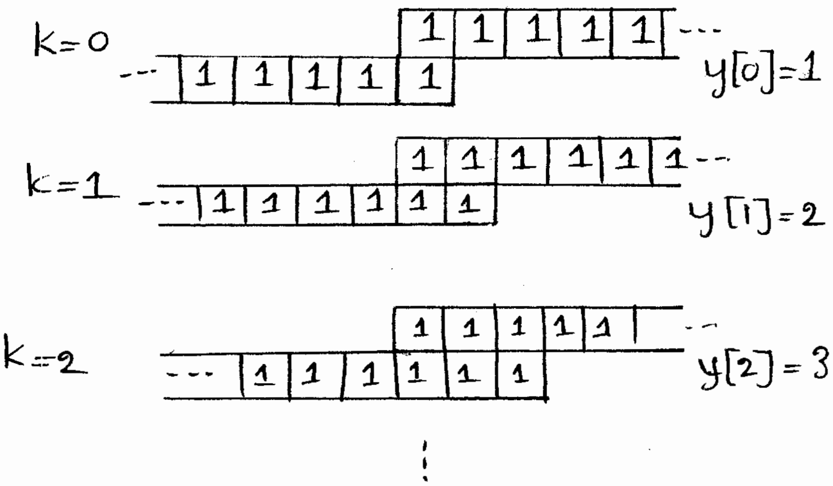
$$y_0[k] + y[k] = -Ma^k u[k] + P\left[\frac{a^k - 1}{a-1}\right] u[k-1]$$

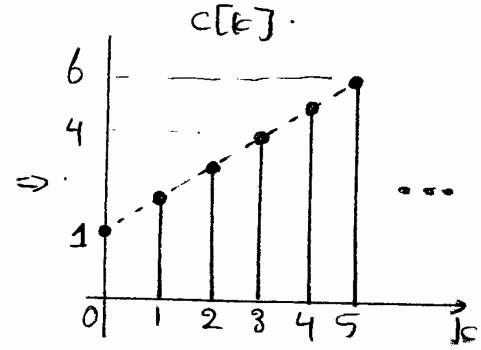
For $k > 1$, $u[k] = u[k-1] = 1$. Thus:

$$\text{Loan balance} = -Ma^k + P\left[\frac{a^k - 1}{a-1}\right] \quad k > 1.$$

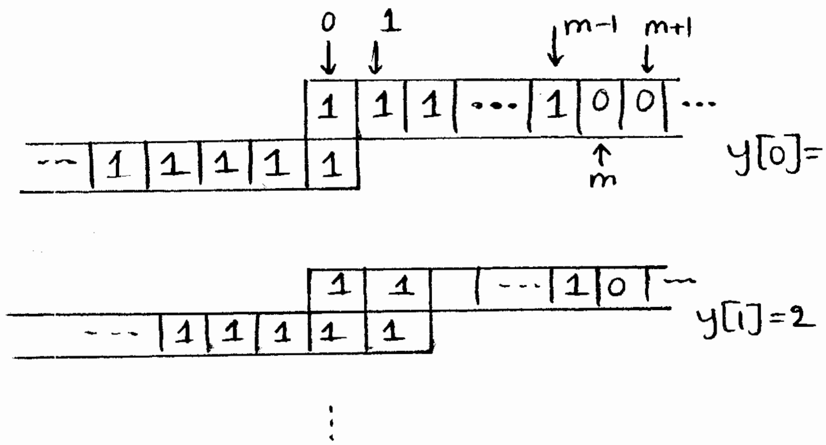
9.4-11.

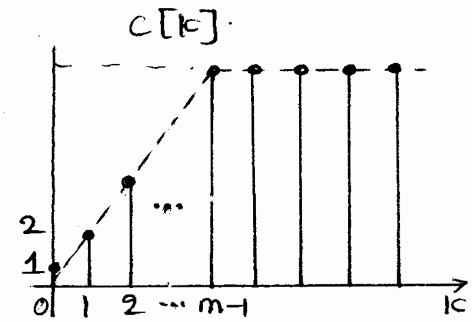
a)





b).





9.5-3

$$y[k+2]+3y[k+1]+2y[k]=f[k+2]+3f[k+1]+3f[k]$$

$$\rightarrow (E^2+3E+2)y[k]=(E^2+3E+3)f[k]$$

The characteristic equation is $\gamma^2+3\gamma+2=(\gamma+1)(\gamma+2)=0$, hence:

$$y_n[k]=B_1(-1)^k+B_2(-2)^k$$

For $f[k]=3^k$

$$y_\phi[k]=H[3]3^k=\frac{(3)^2+3(3)+3}{(3)^2+(3)^3+2}\,3^k=\left(\frac{21}{20}\right)3^k$$

The total response:

$$y[k] = B_1(-1)^k + B_2(-2)^k + \left(\tfrac{21}{20}\right)3^k \quad k \geq 0$$

a) Setting $k=0,1$ and substituting $y[0]=1$, $y[1]=3$ yeilds;

$$\left.\begin{aligned} 1 &= B_1 + B_2 + \tfrac{21}{20} \\ 3 &= -B_1 - 2B_2 + \tfrac{63}{20} \end{aligned}\right\} \longrightarrow \begin{aligned} B_1 &= -1/4 \\ B_2 &= 1/5 \end{aligned}$$

b) We solve system equation iteratively to find $y[0]$ and $y[1]$. We are given $y[-1]=y[-2]=1$. System equation is:

$$y[k+2] + 3y[k+1] + 2y[k] = f[k+2] + 3f[k+1] + 3f[k]$$

$$k=-2 \longrightarrow y[0] + 3(1) + 2(1) = (3)^0 + (3)^0 + (3)^0 \longrightarrow y[0] = -4.$$

$$k=-1 \longrightarrow y[1] + 3(-4) + 2(1) = (3)^1 + 3(3)^0 + 3(0) \longrightarrow y[1] = 16$$

Also

$$y[k] = B_1(-1)^k + B_2(-2)^k + \tfrac{21}{20}(3)^k \quad k \geq 0$$

Setting $k=1,2$ and substituting $y[0]=-4$, $y[1]=16$ yeilds:

$$\left.\begin{aligned} -4 &= B_1 + B_2 + \tfrac{21}{20} \\ 16 &= -B_1 - 2B_2 + \tfrac{63}{20} \end{aligned}\right\} \longrightarrow \begin{aligned} B_1 &= 11/4 \\ B_2 &= -39/5 \end{aligned}$$

$$\Rightarrow y[k] = \tfrac{11}{4}(-1)^k - \tfrac{39}{5}(-2)^k + \tfrac{21}{20}(3)^k \quad k \geq 0$$

9.5-6

$$y[k+2] - y[k+1] + 0.16y[k] = f[k+1]$$

We solve this equation iteratively for $f[k] = \cos\left(\frac{\pi k}{2} + \frac{\pi}{3}\right)$, $y[-1]=y[-2]=0$ to find $y[0]$ and $y[1]$. Remember also that $f[k]=0$ for $k<0$.

$$k=-2 \rightarrow y[0] - 0 + 0.16(0) = 0 \longrightarrow y[0] = 0$$

$$k=-1 \longrightarrow y[1] - 0 + 0.16(0) = \cos\tfrac{\pi}{3} = 0.5 \longrightarrow y[1] = 0.5$$

For the input $f[k] = \cos\left(\frac{\pi k}{2} + \frac{\pi}{3}\right)$

$$y_\phi[k] = c\cos\left(\frac{\pi k}{2} + \frac{\pi}{3} + \phi\right)$$

But $y_\phi[k]$ satisfies the system equation. that is:

$$y_\phi[k+2] - y_\phi[k+1] + 0.16 y_\phi[k] = f[k+1]$$

or $c\cos\left[\frac{\pi}{2}(k+2) + \frac{\pi}{3} + \phi\right] - c\cos\left[\frac{\pi}{2}(k+1) + \frac{\pi}{3} + \phi\right] + 0.16c\cos\left(\frac{\pi}{2}k + \frac{\pi}{3} + \phi\right)$

$$= \cos\left[\frac{\pi}{2}(k+1) + \frac{\pi}{3}\right]$$

After simplifying the above expression:

$$1.306c\cos\left(\frac{\pi k}{2} + \frac{\pi}{3} + \phi - 2.27\right) = \cos\left(\frac{\pi k}{2} + \frac{\pi}{3} + \frac{\pi}{2}\right)$$

Therefore:

$$1.306\,c = 1 \longrightarrow c = 0.765$$

$$\phi - 2.27 = \frac{\pi}{2} \longrightarrow \phi = 3.84 = -2.44 \text{ rad}.$$

Therefore

$$y_\phi[k] = 0.765\cos\left(\frac{\pi k}{2} + \frac{\pi}{3} - 2.44\right)$$

$$= 0.765\cos\left(\frac{\pi k}{2} - 1.393\right).$$

$$y[k] = B_1(0.2)^k + B_2(0.8)^k + 0.765\cos\left(\frac{\pi k}{2} - 1.393\right)$$

Setting $k=0,1$ and substituting $y[0]=0$, $y[1]=0.5$ yeilds:

$$\left.\begin{aligned} 0 &= B_1 + B_2 + 0.1354 \\ 0.5 &= 0.2B_1 + 0.8B_2 + 0.753 \end{aligned}\right\} \longrightarrow \begin{aligned} B_1 &= 0.241 \\ B_2 &= -0.377 \end{aligned}$$

$$\longrightarrow y[k] = 0.241(0.2)^k - 0.377(0.8)^k + 0.765\cos\left(\frac{\pi k}{2} - 1.393\right).$$

9.6-1 (a) $\gamma^2 + 0.6\gamma - 1.6 = (\gamma - 0.2)(\gamma + 0.8)$
Roots are $\gamma = 0.2$ and $\gamma = -0.8$. Both are inside the unit circle.
The system is asymptotically stable.

(b). $(\gamma^2+1)(\gamma^2+\gamma+1) = (\gamma-j)(\gamma+j)(\gamma+\frac{1}{2}-\frac{j\sqrt{3}}{2})(\gamma+\frac{1}{2}+j\frac{\sqrt{3}}{2})$

Roots are $\pm j, -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} = e^{\pm j\pi/3}$

All the roots are simple and on unit circle. The system is marginally stable.

(c) $(\gamma-1)^2(\gamma+\frac{1}{2})$

Roots are 1 (repeated twice) and -0.5. Repeated root on unit circle. The system is unstable.

(d) $\gamma^2+2\gamma+0.96 = (\gamma+0.8)(\gamma+1.2)$

Roots are -0.8 and -1.2. One root (-1.2) is outside the unit circle. The system is unstable.

(e) $(\gamma^2-1)(\gamma^2+1) = (\gamma+1)(\gamma-1)(\gamma+j)(\gamma-j)$

Roots are $\pm 1, \pm j$. All the roots are simple and on unit circle. The system is marginally stable.