EE757
Numerical Techniques in Electromagnetics
Lecture 1

Notations

$\boldsymbol{E}$: electric field intensity (V/M)

$\boldsymbol{H}$: magnetic field intensity (A/M)

$\boldsymbol{D}$: electric flux density (C/ M^2)

$\boldsymbol{B}$: magnetic flux density (Weber/M^2)

$\boldsymbol{J}_i$: impressed electric current density (A/M^2)

$\boldsymbol{J}_c$: conduction electric current density (A/M^2)

$\boldsymbol{J}_d$: displacement electric current density (A/M^2)

$\boldsymbol{\mu}_i$: impressed magnetic current density (V/M^2)

$\boldsymbol{\mu}_d$: displacement magnetic current density (A/M^2)

Notations (Cont'd)

q_{ev}: volumetric electric charge density (C/M^3)

q_{mv} : volumetric magnetic charge density (Weber/M^3)

Historical Background

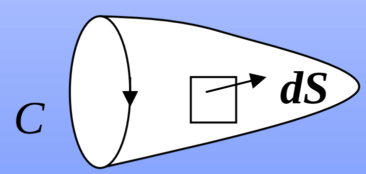
- Ancient civilizations knew the effect of magnetic materials
- Gauss's law for electric fields

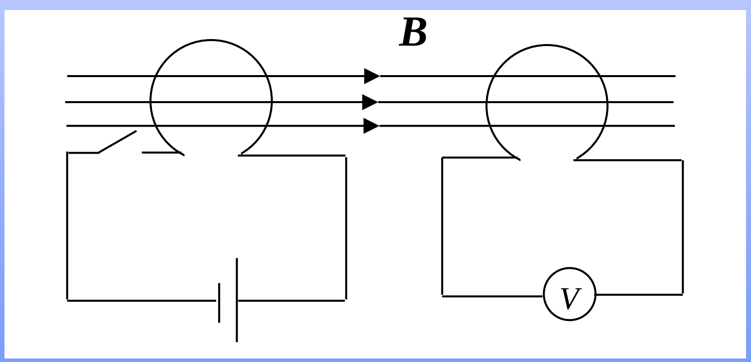
$$\oiint_S \boldsymbol{D}.\boldsymbol{dS} = \iiint_V q_{ev}\, dV = Q_{ev}$$

- Gauss's law for magnetic fields

$$\oiint_S \boldsymbol{B}.\boldsymbol{dS} = 0$$

- Faraday's law





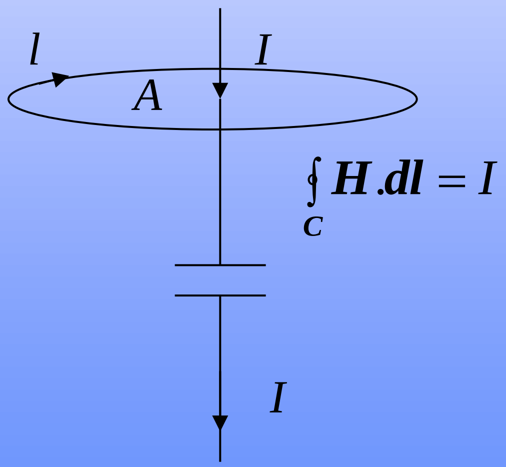
$$\oint_C \boldsymbol{E}.\boldsymbol{dl} = -\frac{\partial}{\partial t}\iint_S \boldsymbol{B}.\boldsymbol{dS}$$

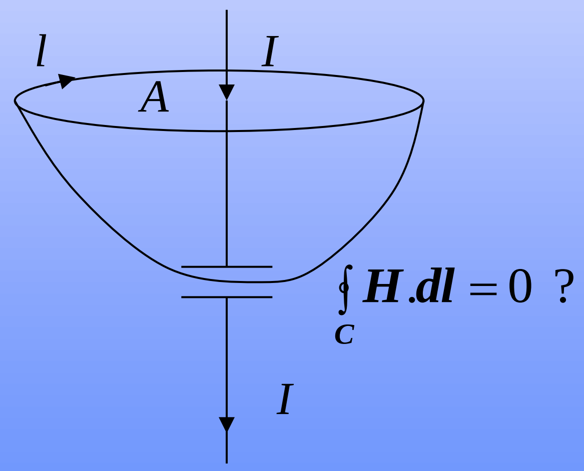
Historical Background (Cont'd)

- Ampere's law

$$\oint_C \boldsymbol{H}.\boldsymbol{dl} = \iint_S \boldsymbol{J}.\boldsymbol{dS} = \text{total current due to flow of charges}$$

- Ampere's law in its original form could not be considered general





Historical Background (Cont'd)

- Ampere's law is modified by introducing the displacement current

$$\oint_C \boldsymbol{H}.\boldsymbol{dl} = \iint_S \boldsymbol{J}.\boldsymbol{dS} + \frac{\partial}{\partial t}\iint_S \boldsymbol{D}.\boldsymbol{dS}$$

- It follows that the 4 main laws are

$$\oiint_S \boldsymbol{D}.\boldsymbol{dS} = \iiint_V q_{ev}\, dV = Q_{ev}$$

$$\oiint_S \boldsymbol{B}.\boldsymbol{dS} = 0$$

$$\oint_C \boldsymbol{E}.\boldsymbol{dl} = -\frac{\partial}{\partial t}\iint_S \boldsymbol{B}.\boldsymbol{dS}$$

$$\oint_C \boldsymbol{H}.\boldsymbol{dl} = \iint_S \boldsymbol{J}.\boldsymbol{dS} + \frac{\partial}{\partial t}\iint_S \boldsymbol{D}.\boldsymbol{dS}$$

Maxwell's Equations (the integral form)

- Maxwell's equations are made symmetric by the introduction of fictitious magnetic charges and currents

$$\oiint_S \boldsymbol{D}.\boldsymbol{dS} = \iiint_V q_{ev}\, dV = Q_{ev} \quad \Rightarrow \quad \oiint_S \boldsymbol{B}.\boldsymbol{dS} = \iiint_V q_{mv}\, dV = Q_{mv}$$

$$\oint_C \boldsymbol{H}.\boldsymbol{dl} = \iint_S \boldsymbol{J}.\boldsymbol{dS} + \frac{\partial}{\partial t}\iint_S \boldsymbol{D}.\boldsymbol{dS} \quad \Rightarrow \quad \oint_C \boldsymbol{E}.\boldsymbol{dl} = -\iint_S \boldsymbol{\mu}.\boldsymbol{dS} - \frac{\partial}{\partial t}\iint_S \boldsymbol{B}.\boldsymbol{dS}$$

$$\boldsymbol{J} = \boldsymbol{J}_i + \boldsymbol{J}_c$$

$$\boldsymbol{\mu} = \boldsymbol{\mu}_i + \boldsymbol{\mu}_c$$

The Divergence Theorem

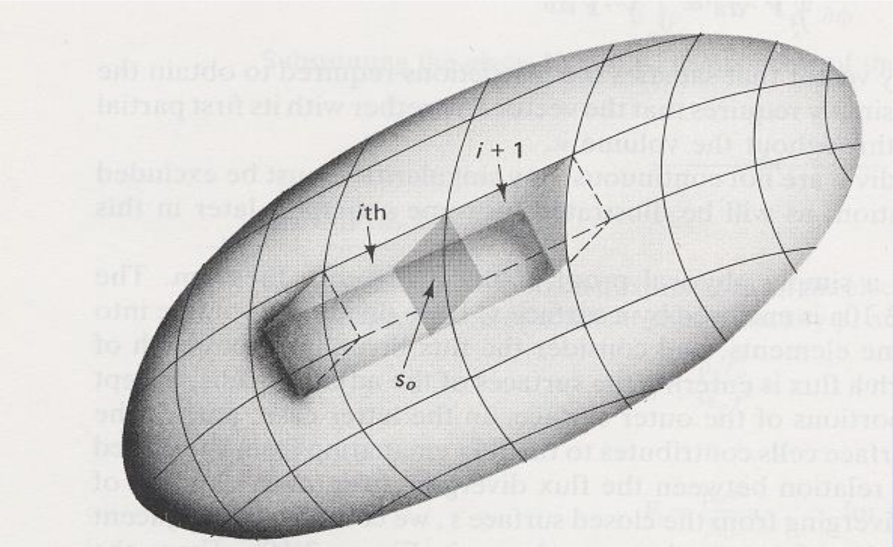
- The divergence of a vector at a point is defined as

$$\text{div}\,\boldsymbol{F} = \lim_{\Delta \boldsymbol{V} \to 0} \frac{\displaystyle\oiint_{\Delta S} \boldsymbol{F.dS}}{\Delta V} = \nabla.\boldsymbol{F}$$

- The divergence of a vector is a scalar value that is position dependent
- Divergence Theorem converts a closed surface integral to a volume integral over the enclosed volume

$$\oiint_{S} \boldsymbol{F.dS} = \iiint_{V} \nabla.\boldsymbol{F}\; dV$$

The Divergence Theorem (Cont'd)



Iskandar 1992

- For the ith element we have $\oiint_{\Delta S_i} \boldsymbol{F.dS} = \text{div}\,\boldsymbol{F}\ \Delta V_i$
- Summing over the N volumetric elements we get

$$\sum_{i=1}^{N} \oiint_{\Delta S_i} \boldsymbol{F.dS} = \sum_{i=1}^{N} \text{div}\,\boldsymbol{F}\ \Delta V_i$$

- Notice that the flux cancels out between adjacent elements leaving only external surface flux

The Divergence Theorem (Cont'd)

- As $N\rightarrow\infty$, we get $\oiint_S \boldsymbol{F.dS} = \iiint_V \nabla.\boldsymbol{F}\ dV$
- As an application of the divergence theorem, we have

$$\oiint_S \boldsymbol{D.dS} = \iiint_V q_{ev}\, dV \quad \Longrightarrow \quad \iiint_V \nabla.\boldsymbol{D}\, dV = \iiint_V q_{ev}\, dV$$

$\nabla.\boldsymbol{D} = q_{ev}$ Gauss's law in differential form

- Similarly, for the magnetic field we have

$$\oiint_S \boldsymbol{B.dS} = \iiint_V q_{mv}\, dV \quad \Longrightarrow \quad \iiint_V \nabla.\boldsymbol{B}\, dV = \iiint_V q_{mv}\, dV$$

$$\nabla.\boldsymbol{B} = q_{mv}$$

Stokes' Theorem

- The curl is a measure of the rotation of a vector

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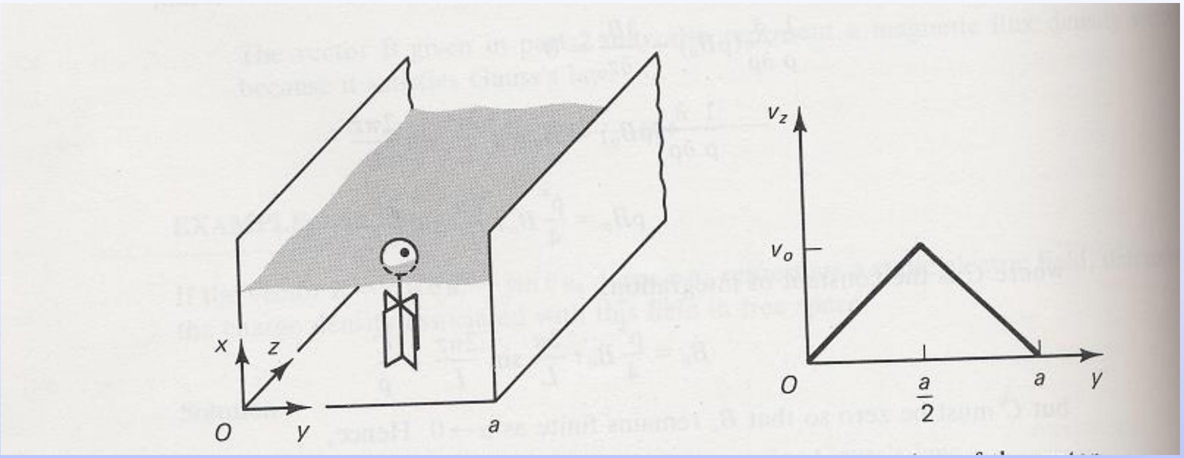
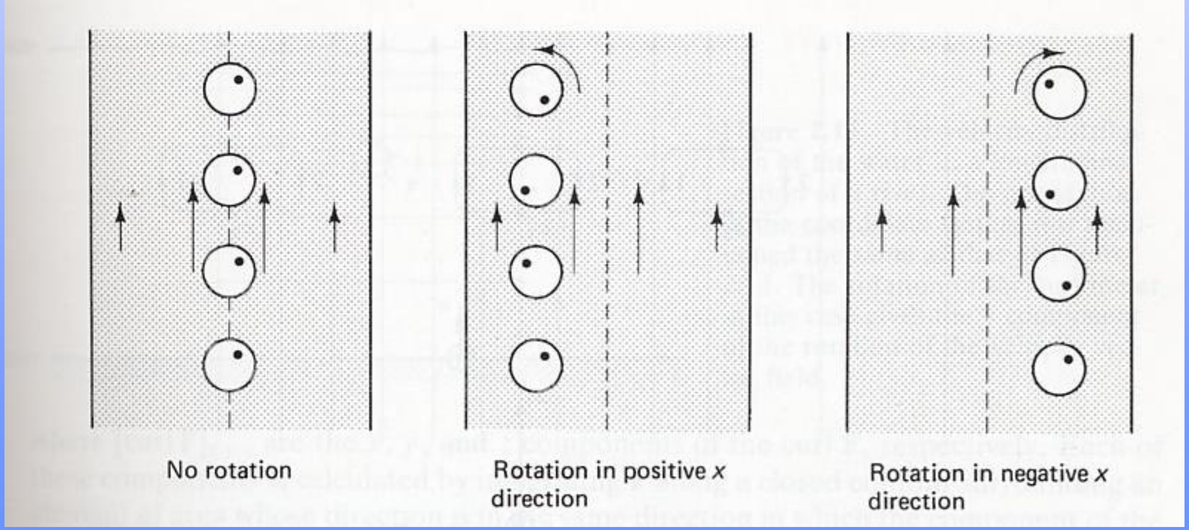
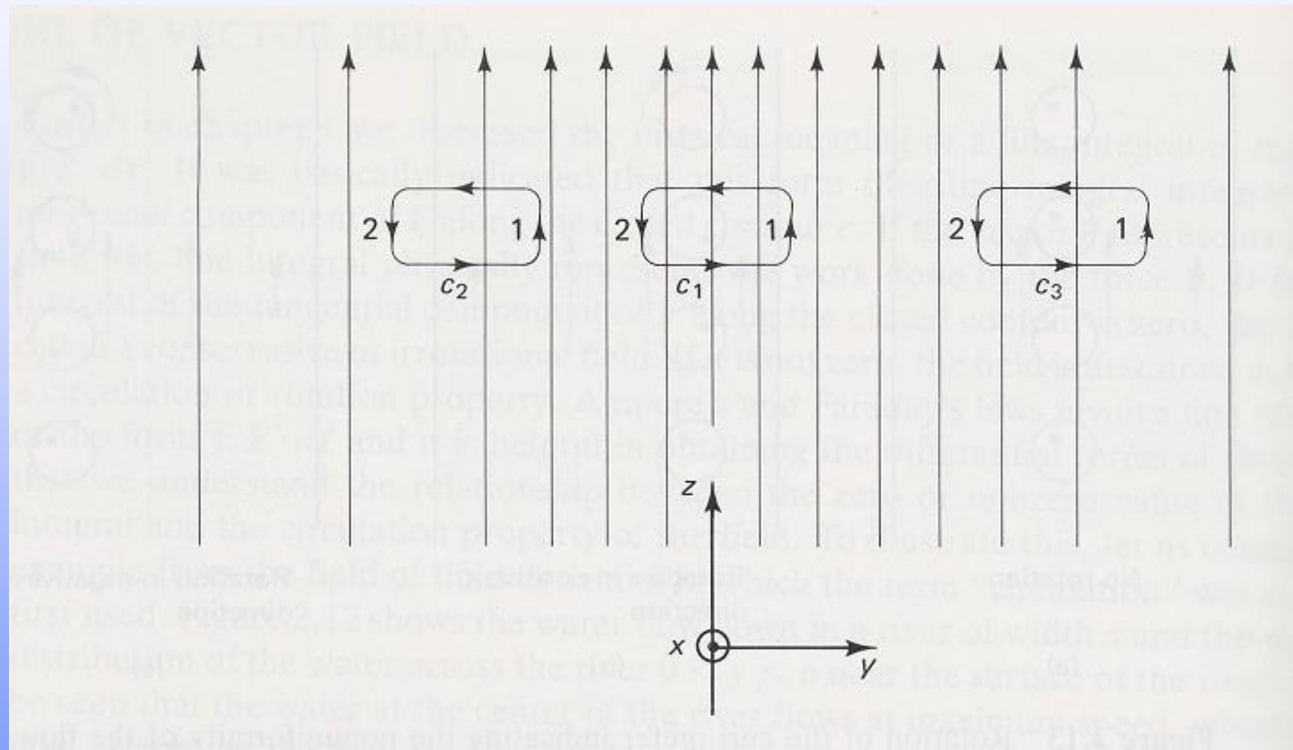




Stokes' Theorem (Cont'd)

- The curl can be measured through a line integral

Iskandar 1992

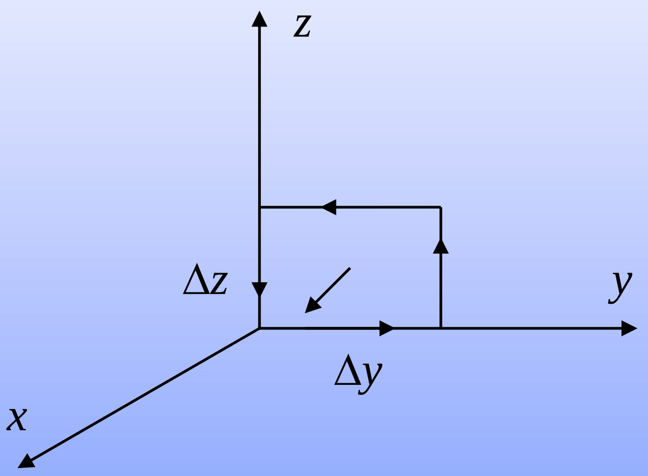


$$\oint \boldsymbol{F}.\boldsymbol{dl} = +\text{ve} \qquad \oint \boldsymbol{F}.\boldsymbol{dl} = 0 \qquad \oint \boldsymbol{F}.\boldsymbol{dl} = -\text{ve}$$

Stokes' Theorem (Cont'd)

- The curl is in general a vector

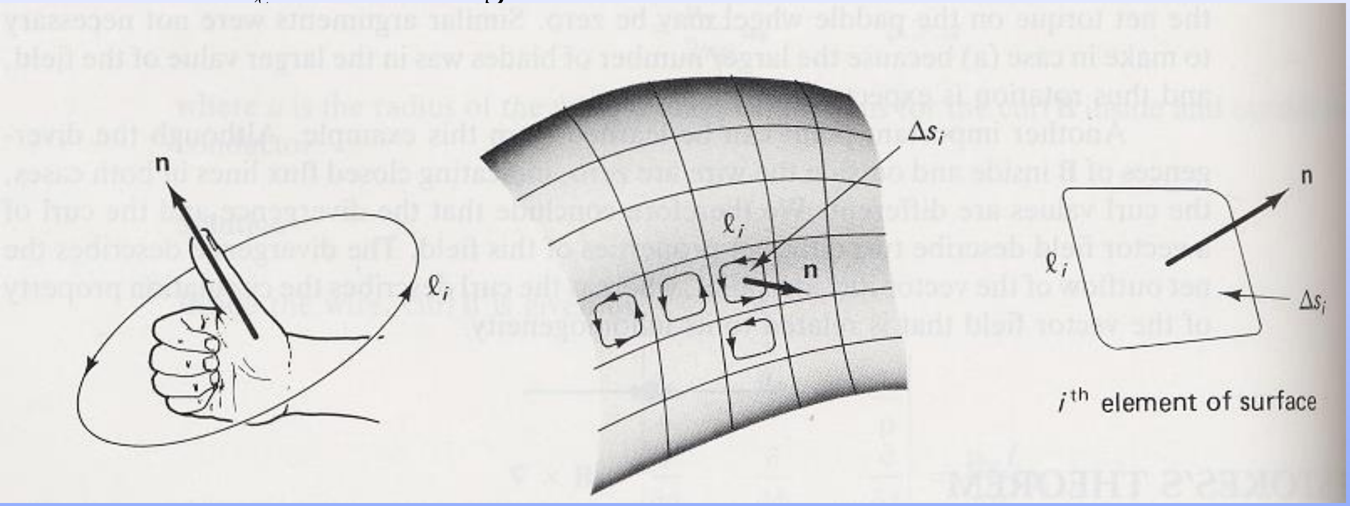
curl $\boldsymbol{F}$= [curl $\boldsymbol{F}]_x$ $\boldsymbol{a}_x$+ [curl $\boldsymbol{F}]_y$ $\boldsymbol{a}_y$+[curl $\boldsymbol{F}]_z$ $\boldsymbol{a}_z$



$$[\nabla\times\boldsymbol{F}]_x = \lim_{\Delta y,\Delta z\to 0}\frac{\oint_{C_1}\boldsymbol{F.dl}}{\Delta y\Delta z}$$

Stokes' Theorem (Cont'd)

- This theorem relates the value of a line integral over a closed contour to a surface integral over the enclosed surface $\oint_{\ell} \boldsymbol{F}.\boldsymbol{dl} = \iint_{S} (\nabla \times \boldsymbol{F}).\boldsymbol{dS}$



- For the ith element we have $\lim_{\Delta S_i \to 0} \dfrac{\oint_{\ell_i} \boldsymbol{F}.\boldsymbol{dl}}{\Delta S_i} = \text{curl}\, \boldsymbol{F}.\boldsymbol{n}$

or $\oint_{\ell_i} \boldsymbol{F}.\boldsymbol{dl} = \text{curl}\, \boldsymbol{F}.\Delta \boldsymbol{S}_i$

Stokes' Theorem (Cont'd)

- Summing for all elements we get

$$\sum_{i=1}^{N} \oint_{\ell_i} \boldsymbol{F}.\boldsymbol{dl} = \sum_{i=1}^{N} (\nabla \times \boldsymbol{F}).\Delta \boldsymbol{S}_i$$

- Notice that the internal line integrals cancel out and only integration over the external contour remains
- It follows that as $\Delta S_i \rightarrow 0$, we get

$$\oint_{\ell} \boldsymbol{F}.\boldsymbol{dl} = \iint_{S} (\nabla \times \boldsymbol{F}).\boldsymbol{dS}$$

Applications of Stokes' Theorem

- Starting with the Maxwell's integral equation

$$\oint_C \boldsymbol{E}.\boldsymbol{dl} = -\iint_S \boldsymbol{\mu}.\boldsymbol{dS} - \frac{\partial}{\partial t}\iint_S \boldsymbol{B}.\boldsymbol{dS}$$

⇓ apply Stokes' Theorem

$$\iint_S (\nabla \times \boldsymbol{E}).\boldsymbol{dS} = -\iint_S \boldsymbol{\mu}.\boldsymbol{dS} - \frac{\partial}{\partial t}\iint_S \boldsymbol{B}.\boldsymbol{dS}$$

⇓

$$(\nabla \times \boldsymbol{E}) = -\boldsymbol{\mu} - \frac{\partial \boldsymbol{B}}{\partial t}$$

- Similarly, starting with $\oint_C \boldsymbol{H}.\boldsymbol{dl} = \iint_S \boldsymbol{J}.\boldsymbol{dS} + \frac{\partial}{\partial t}\iint_S \boldsymbol{D}.\boldsymbol{dS}$

We get $(\nabla \times \boldsymbol{H}) = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$

Maxwell's Equations

Integral form

$$\oiint_S \boldsymbol{D}.\boldsymbol{dS} = \iiint_V q_{ev}\, dV = Q_{ev}$$

$$\oiint_S \boldsymbol{B}.\boldsymbol{dS} = \iiint_V q_{mv}\, dV = Q_{mv}$$

$$\oint_C \boldsymbol{E}.\boldsymbol{dl} = -\iint_S \boldsymbol{\mu}.\boldsymbol{dS} - \frac{\partial}{\partial t}\iint_S \boldsymbol{B}.\boldsymbol{dS}$$

$$\oint_C \boldsymbol{H}.\boldsymbol{dl} = \iint_S \boldsymbol{J}.\boldsymbol{dS} + \frac{\partial}{\partial t}\iint_S \boldsymbol{D}.\boldsymbol{dS}$$

Differential form

$$\nabla.\boldsymbol{D} = q_{ev}$$

$$\nabla.\boldsymbol{B} = q_{mv}$$

$$(\nabla \times \boldsymbol{E}) = -\boldsymbol{\mu} - \frac{\partial \boldsymbol{B}}{\partial t}$$

$$(\nabla \times \boldsymbol{H}) = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

The Constitutive Relations

- A material is characterized by its constitutive parameters ε, μ and σ
- For example, $\boldsymbol{D}(t) = \int_{-\infty}^{\infty} \varepsilon(t-\tau)\boldsymbol{E}(\tau)\,\mathrm{d}\tau$
- For frequency independent permittivity and for frequency-domain analysis we have $\boldsymbol{D}=\varepsilon\boldsymbol{E}$
- For free space we have $\varepsilon_o=10^{-9}/(36\pi)$ F/M
- Similarly, $\boldsymbol{B}=\mu*\boldsymbol{H}$ ⟹ $\boldsymbol{B}=\mu\boldsymbol{H}$ (frequency independent or single frequency analysis)
- For free space we have $\mu_o=4\pi\times10^{-7}$ H/M

The Constitutive Parameters (Cont'd)

- Also, $\boldsymbol{J}=\sigma^*\boldsymbol{E}$ ⟹ $\boldsymbol{J}=\sigma\boldsymbol{E}$ (frequency independent or single frequency analysis)
- For free space, $\sigma_o=0$ S