EE757
Numerical Techniques in Electromagnetics
Lecture 3

Special Cases in Maxwell's Equations.

- With no sources, Maxwell's Equations are written as

$$(\nabla\times\boldsymbol{E})=-\frac{\partial\boldsymbol{B}}{\partial t}\,, \qquad (\nabla\times\boldsymbol{H})=\boldsymbol{J}_c+\frac{\partial\boldsymbol{D}}{\partial t}$$

- In Cartesian coordinates we have

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial B_x}{\partial t} \qquad \frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=J_x+\frac{\partial D_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{\partial B_y}{\partial t} \qquad \frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=J_y+\frac{\partial D_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\partial B_z}{\partial t} \qquad \frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=J_z+\frac{\partial D_z}{\partial t}$$

Special Cases (Cont'd)

- For a 1D case, the fields depend only on one coordinate (e.g. a uniform plane wave traveling in the x direction)

- For example, if $\partial/\partial y = \partial/\partial z = 0$

$$-\frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \quad , \quad \frac{\partial H_y}{\partial x} = J_z + \frac{\partial D_z}{\partial t}$$ (propagation in –ve x)

or

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \quad , \quad -\frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t}$$ (propagation in +ve x)

Notice that these two systems are decoupled

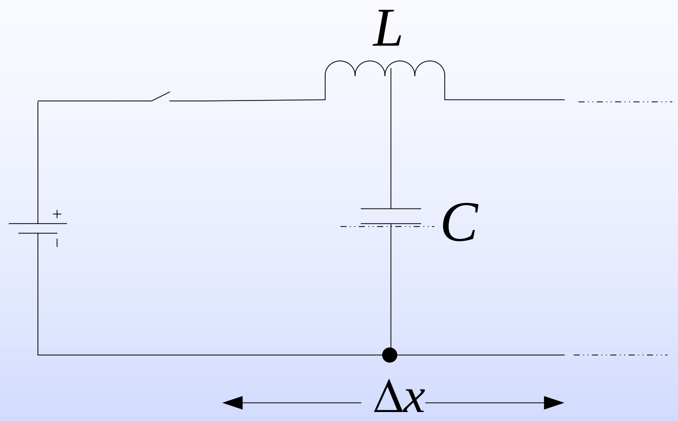
Special Cases (Cont'd)

- For the 2D problems, the fields depend only on two coordinates (e.g. TE10 mode in waveguides)
- For example, if $\partial/\partial y=0$

TE_y	TM_y
$-\dfrac{\partial H_y}{\partial z}=J_x+\dfrac{\partial D_x}{\partial t}$	$-\dfrac{\partial E_y}{\partial z}=-\dfrac{\partial B_x}{\partial t}$
$\dfrac{\partial E_x}{\partial z}-\dfrac{\partial E_z}{\partial x}=-\dfrac{\partial B_y}{\partial t}$	$\dfrac{\partial H_x}{\partial z}-\dfrac{\partial H_z}{\partial x}=J_y+\dfrac{\partial D_y}{\partial t}$
$\dfrac{\partial H_y}{\partial x}=J_z+\dfrac{\partial D_z}{\partial t}$	$\dfrac{\partial E_y}{\partial x}=-\dfrac{\partial B_z}{\partial t}$

Notice that the two systems are decoupled

Transient Response of a Transmission Line



- C_d capacitance per unit length F/m

L_d inductance per unit length H/m

u velocity of propagation

- After time Δt, the disturbance traveled a distance Δx. A charge $\Delta Q = C_d\, \Delta x\, V_s$ is transferred. It follows that

$$i = \Delta Q/\Delta t \quad \Longrightarrow \quad i = C_d\, V_s\, (\Delta x/\, \Delta t) = C_d\, V_s\, u$$

Transient Response of a TL (Cont'd)

- The flow of current establishes a flux Φ associated with line inductance through $\Phi = L_d \Delta x\, i = L_d \Delta x\, C_d V_s u$
- Using Faraday's law we have

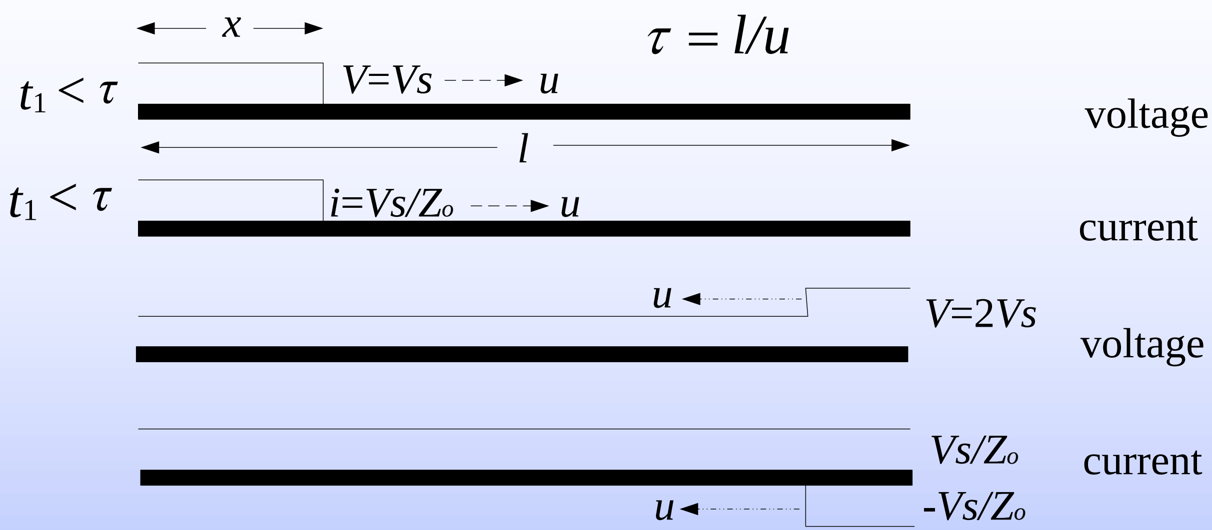
$V_s = \Delta\Phi / \Delta t = L_d C_d V_s u^2 \Longrightarrow L_d C_d u^2 = 1.0$

$$\boxed{u = \frac{1}{\sqrt{C_d L_d}} \quad \text{m/s}}$$

- It follows that $i = C_d V_s u = V_s \sqrt{\dfrac{C_d}{L_d}}$

$$Z_o = \frac{\text{incident voltage}}{\text{incident current}} = \frac{V_s}{i} = \sqrt{\frac{L_d}{C_d}}\ \Omega$$

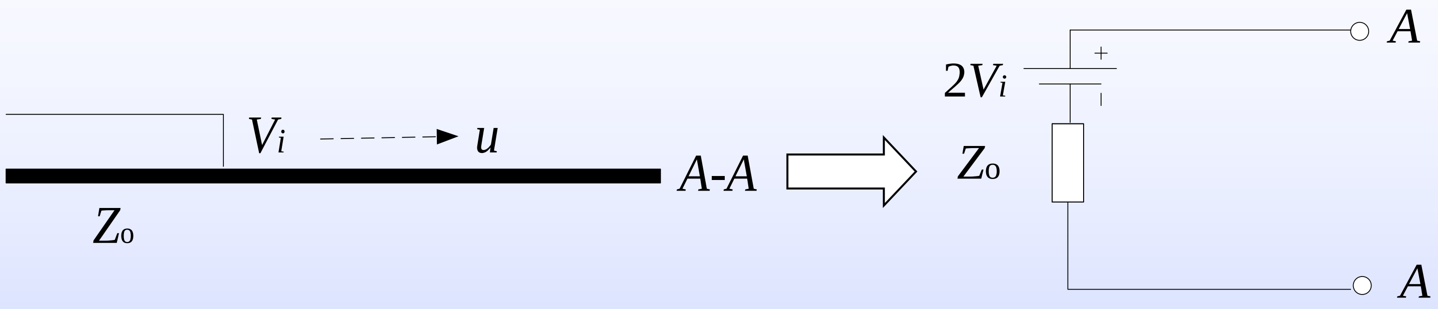
Open-Ended Transmission Line



- After time τ, the incident voltage and current waves reach the open end. A reflected current wave of amplitude $I^r = -V_s / Z_o$ is initiated to make the total current at the open end zero

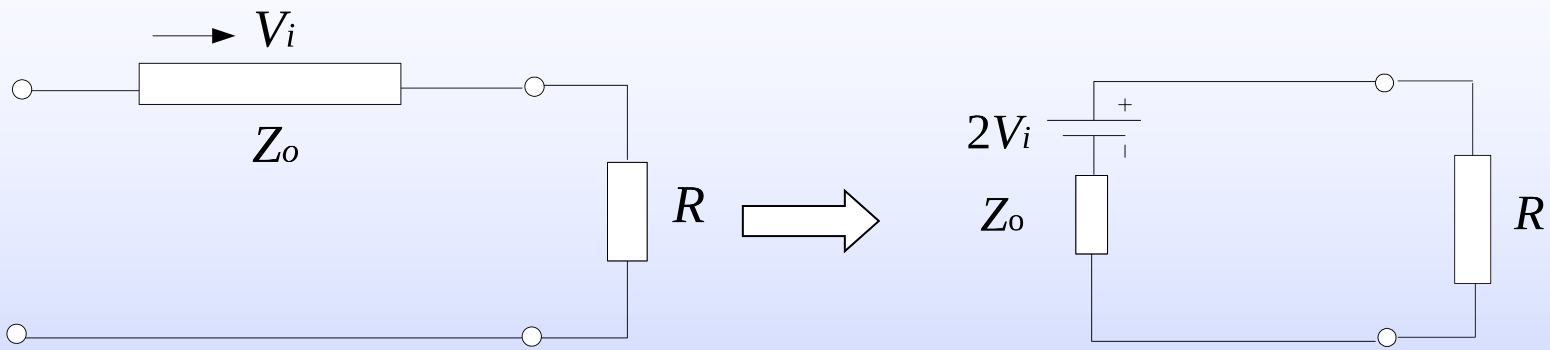
- It follows that for an open-ended transmission line we have $V^i = V^s$, $I^i = V_s / Z_o$ and $V^r = V^s$, $I^r = -V_s / Z_o$

Thevenin Equivalent of a Transmission Line



- The open circuit voltage is $2V^{i}$ (open-ended line)
- The equivalent resistance is Z_o
- This Thevenin equivalent is valid only for limited time period where no reflections takes place on the line (same V^{i})

Example on Utilizing Thevenin's Equivalent of a TL



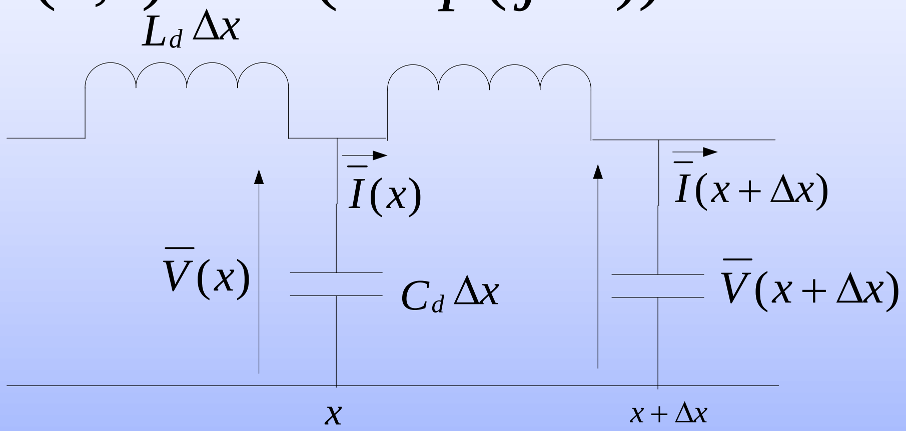
- Using Kirchtoff's voltage law we have $V = 2V^i \dfrac{R}{R+Z_o}$
- But $V = V^i + V^r$ $\Rightarrow$ $V^r = V - V^i$

$$V^r = 2V^i \frac{R}{R+Z_o} - V^i \Rightarrow V^r = \left(\frac{R - Z_o}{R+Z_o}\right) V^i = \Gamma V^i$$

- For open circuit termination $R=\infty$, $\Gamma=1$, $V^r = V^i$
- For short circuit termination $R=0$, $\Gamma=-1$, $V^r = -V^i$

Sinusoidal Steady-State Response of a TL

- Remember that for a steady-state sinusoidal analysis we need only the amplitude and phase at each point on the line $V(x,t) = \mathrm{Re}(\bar{V} exp(j\omega t))$



- Applying KVL, we get $\bar{V}(x) - \bar{V}(x+\Delta x) = j\omega L_d \Delta x \bar{I}(x)$

Leading to $-\dfrac{\partial \bar{V}(x)}{\partial x} = j\omega L_d \bar{I}(x)$

- Applying KCL we get $\bar{I}(x) - \bar{I}(x+\Delta x) = j\omega C_d \Delta x \bar{V}(x+\Delta x)$

Leading to $-\dfrac{\partial \bar{I}(x)}{\partial x} = j\omega C_d \bar{V}(x)$

Sinusoidal Steady State of a TL (Cont'd)

- Differentiating to eliminate the current term we get

$$\frac{\partial^2 \overline{V}(x)}{\partial x^2} = -\omega^2 L_d C_d \overline{V}(x) = -\beta^2 \overline{V}(x)$$

$$\Downarrow$$

$$\overline{V}(x) = V^i exp(-j\beta x) + V^r exp(j\beta x)$$

- Similarly $\overline{I}(x) = I^i exp(-j\beta x) + I^r exp(j\beta x)$
- Using the telegrapher's differential equations we can show that $I^i = V^i / Z_o$ and $I^r = -V^r / Z_o$
- It follows that $\overline{V}(x) = V^i exp(-j\beta x) + V^r exp(j\beta x)$

  and $\overline{I}(x) = (V^i / Z_o) exp(-j\beta x) - (V^r / Z_o) exp(j\beta x)$

ABCD Matrix of a TL Section

- At x=0, we have $\bar{V}(0) = V^i + V^r$ and $\bar{I}(0) = (V^i / Z_o) - (V^r / Z_o)$
- Expressing the solutions in terms of the amplitudes at x=0, we get

$$\begin{bmatrix} \bar{V}(x) \\ \bar{I}(x) \end{bmatrix} = \begin{bmatrix} \cos(\beta x) & -jZ_o \sin(\beta x) \\ -j\sin(\beta x)/Z_o & \cos(\beta x) \end{bmatrix} \begin{bmatrix} \bar{V}(0) \\ \bar{I}(0) \end{bmatrix}$$

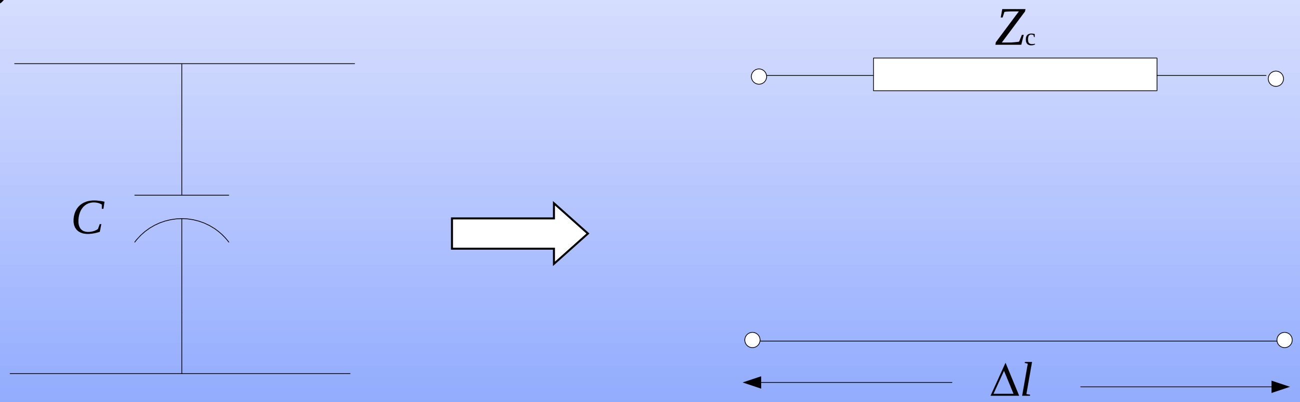
$$\Downarrow$$

$$\begin{bmatrix} \bar{V}(0) \\ \bar{I}(0) \end{bmatrix} = \begin{bmatrix} \cos(\beta x) & jZ_o \sin(\beta x) \\ j\sin(\beta x)/Z_o & \cos(\beta x) \end{bmatrix} \begin{bmatrix} \bar{V}(x) \\ \bar{I}(x) \end{bmatrix}$$

ABCD matrix of a TL section with length x

Discrete Models of Lumped Elements

- The lumped element is replaced by a section of a transmission line
- Using these models the voltages and currents are obtained only at discrete instants of time



- $C_d \Delta l = C$, C_d is the capacitance per unit length
- $u = \Delta l / \Delta t = 1/\sqrt{L_d C_d}$ $\Rightarrow$ $L_d = \left(\dfrac{\Delta t}{\Delta l}\right)^2 \dfrac{1}{C_d}$

Discrete Models of Lumped Elements (Cont'd)

- $Z_c = \sqrt{\dfrac{L_d}{C_d}} = \dfrac{\Delta t}{C_d \Delta l} = \dfrac{\Delta t}{C}$
- L_e = equivalent (parasitic) inductance = $L_d \Delta l = \dfrac{(\Delta t)^2}{C}$
- Δt is selected small enough such that the parasitic inductance is small

Modeling Steps:

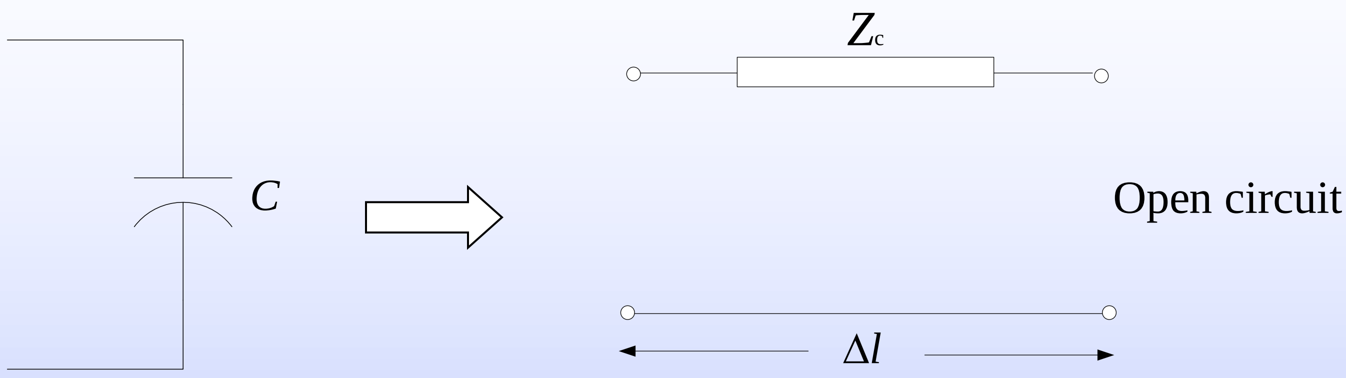
1.Select Δt, Δl 2. evaluate C_d, L_d and Z_c

3. Assume certain incident voltage at the kth time step

4. Use Thevenin's equivalent to get the reflected voltages

5. Obtain incident impulses at the (k+1) time step

Discrete Models of Lumped Elements (Cont'd)



- Using a similar derivation $C_d\,\Delta l = C$

$u = \Delta l/(\Delta t/2) = 1/\sqrt{L_d\, C_d}$, notice that Δt is the round-trip time

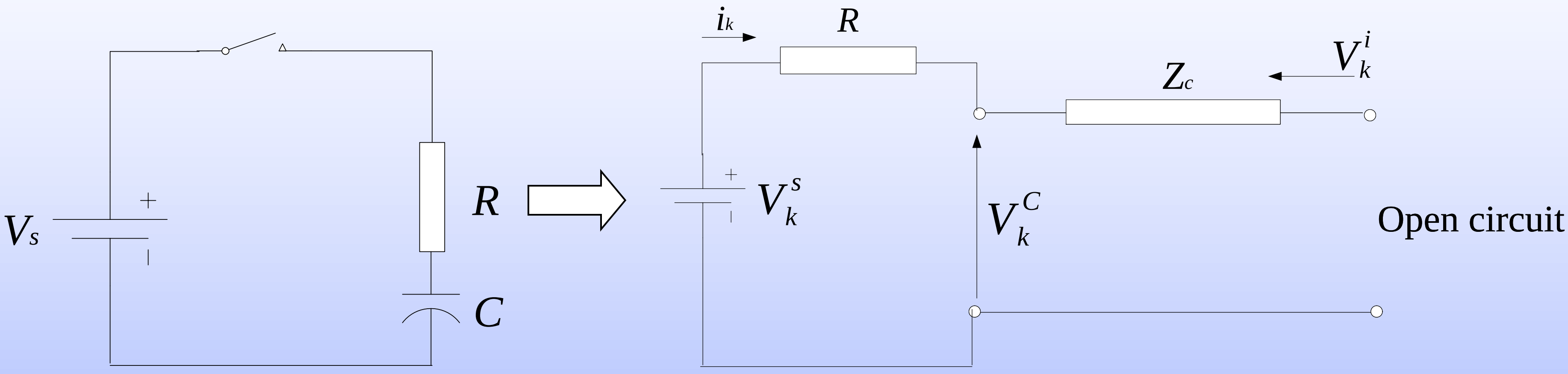
$$L_d = \left(\frac{\Delta t}{2\Delta l}\right)^2 \frac{1}{C_d} = \frac{\Delta t^2}{4C\Delta l}$$

$$Z_c = \sqrt{\frac{L_d}{C_d}} = \frac{\Delta t}{2C}$$

$$L_e = L_d\,\Delta l = \frac{(\Delta t)^2}{4C}$$ = error inductance

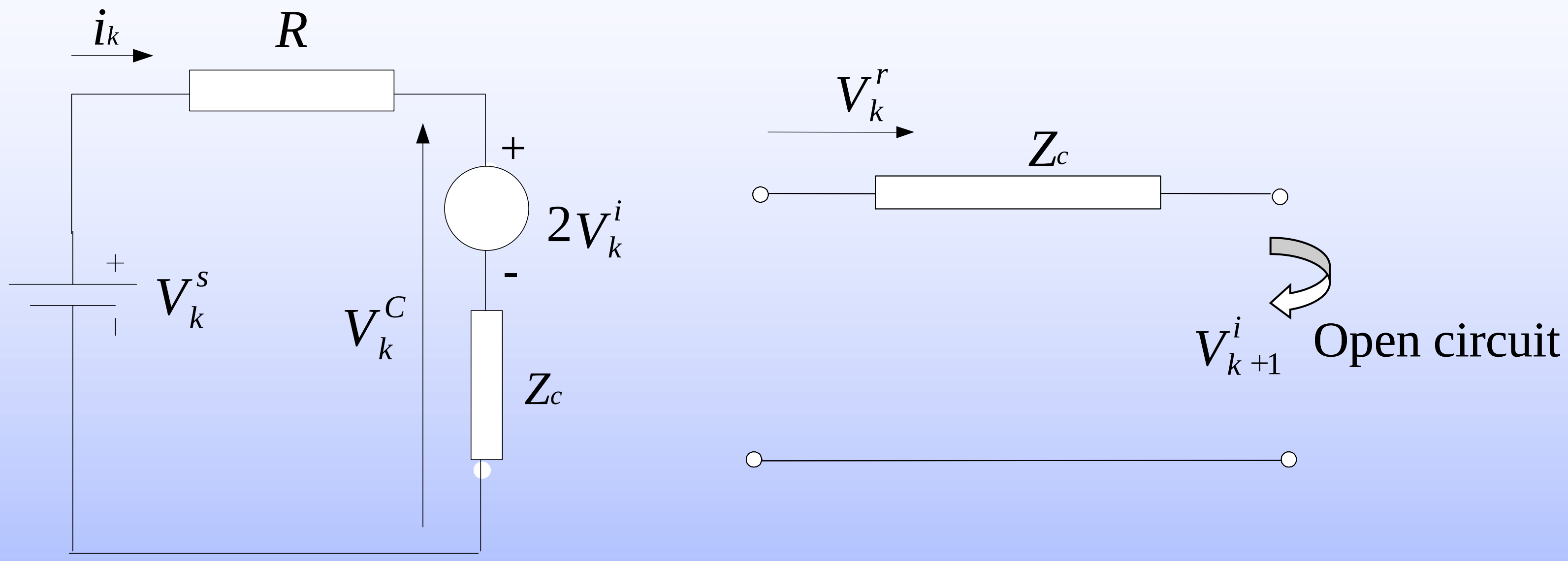
Example

- Evaluate the transient response of the shown circuit

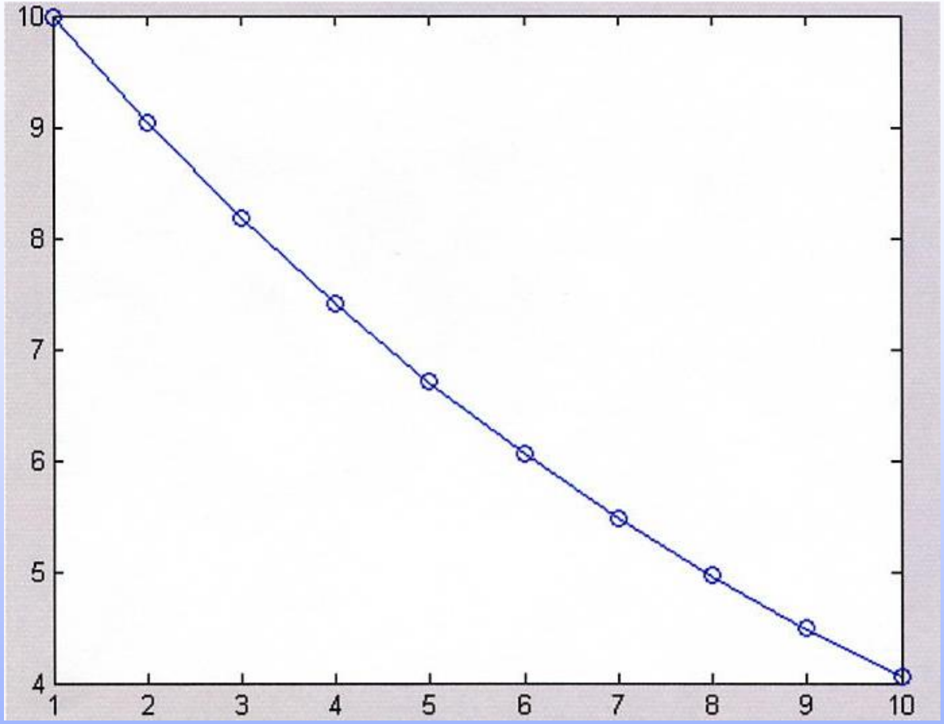


- Evaluate the current $i_k = (V_k^S - 2V_k^i)/(Z_c + R)$
- Evaluate the capacitor voltage $V_k^C = 2V_k^i + Z_c i_k$
- Scattering: $V_k^r = V_k^C - V_k^i$
- Connection: $V_{k+1}^i = V_k^r$

Example (Cont'd)



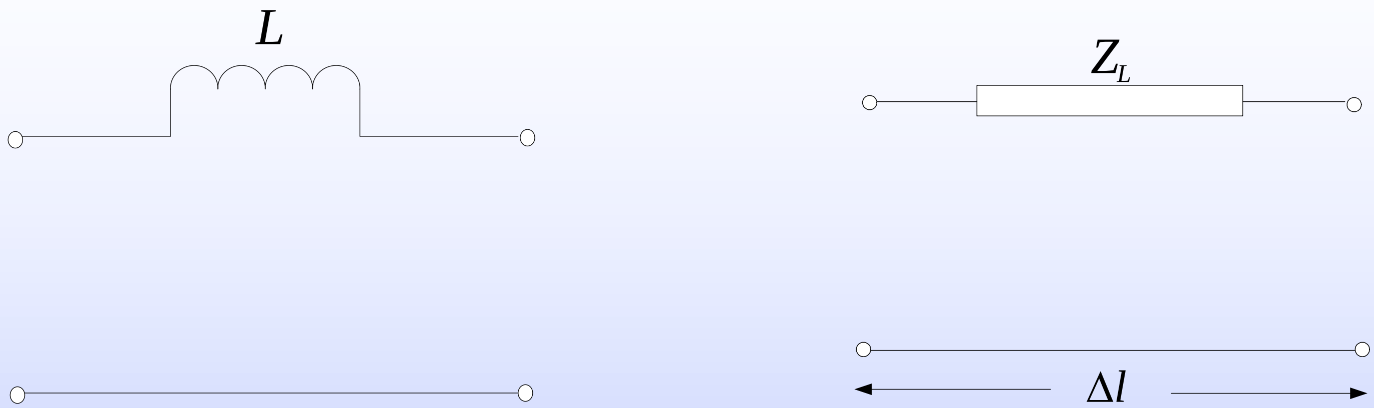
Example (Cont'd)




```
for k=1:10                 %repeat for only time steps
   Ik=(Vs-2*Vi)/(R+Zc);   %evaluate current
   CurrentValuesTLM(k,1)=Ik;  %store current value
   CurrentValuesAnalytic(k,1)=(Vs/R)*exp(-(k-1)*dt/(R*C));
   Vk=2*Vi+Ik*Zc;         %get capacitor voltage
   Vr=Vk-Vi;              %evaluate reflected voltage (scattering)
   Vi=Vr;                 %evaluate new incident voltage (connection)
end
```

Try It!

Link Model of an Inductor



- Using similar derivation to that of the capacitor we have

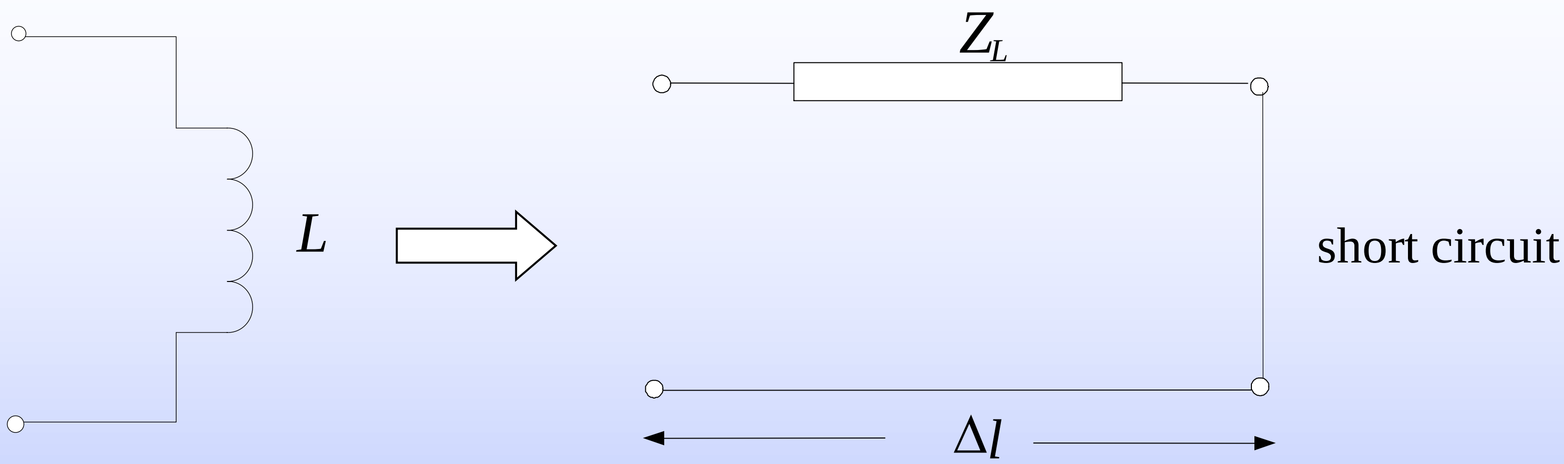
$L_d \Delta l = L$

$$u = \Delta l / \Delta t = 1/\sqrt{L_d C_d} \Longrightarrow C_d = \left(\frac{\Delta t}{\Delta l}\right)^2 \frac{1}{L_d}$$

$$Z_L = \sqrt{\frac{L_d}{C_d}} = \frac{L}{\Delta t}$$

Error capacitance $C_e = C_d \Delta l = (\Delta t)^2 / L$

Stub Model of an Inductor



- Using similar derivation we have

$$L_d \Delta l = L$$

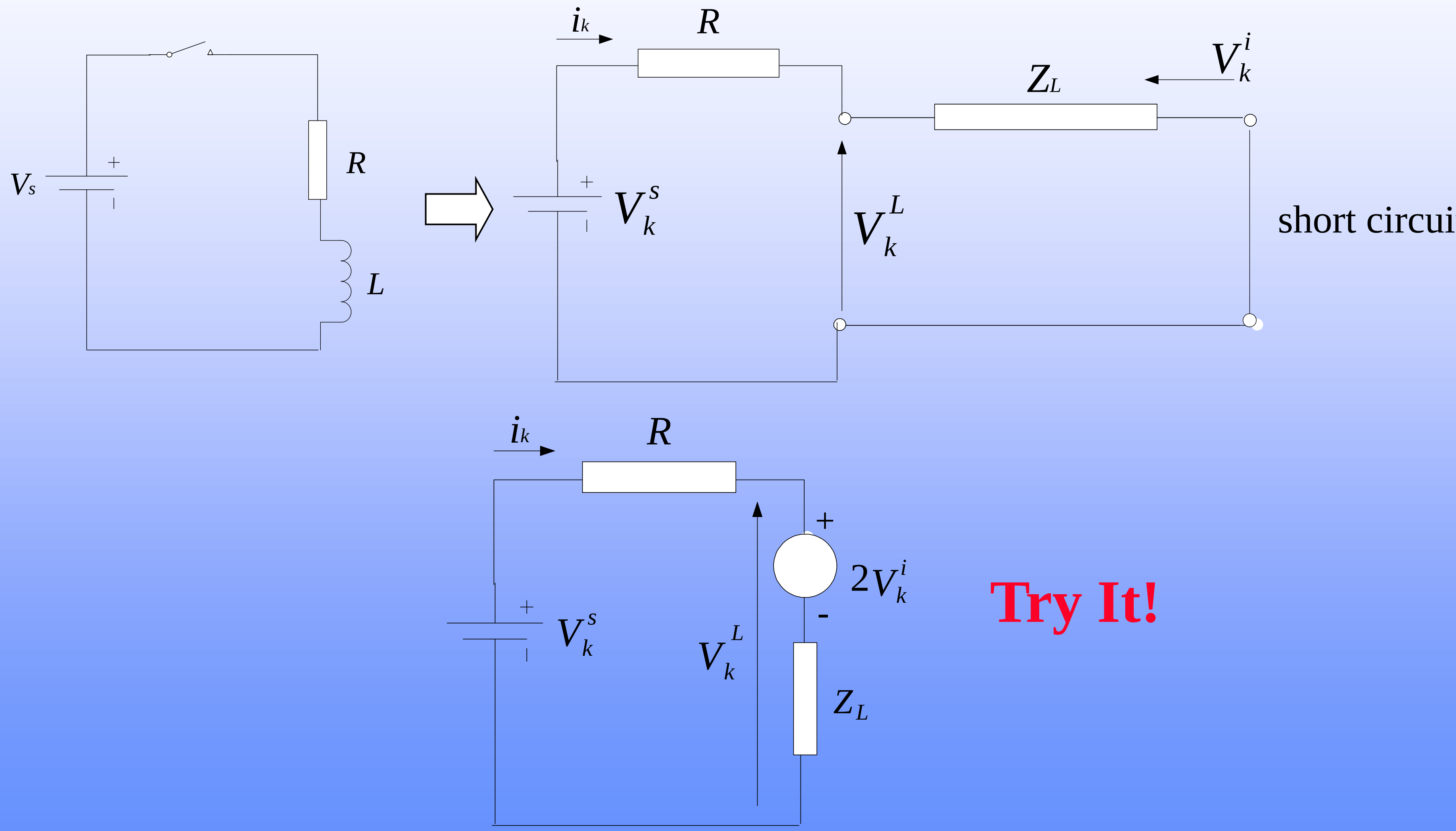
$$u = \Delta l / (\Delta t / 2) = 1 / \sqrt{L_d C_d} \Rightarrow C_d = \left(\frac{\Delta t}{\Delta l}\right)^2 \frac{1}{4L_d}$$

$$Z_L = \sqrt{\frac{L_d}{C_d}} = \frac{2L}{\Delta t}$$

Error capacitance $C_e = C_d \Delta l = (\Delta t)^2 / (4L)$

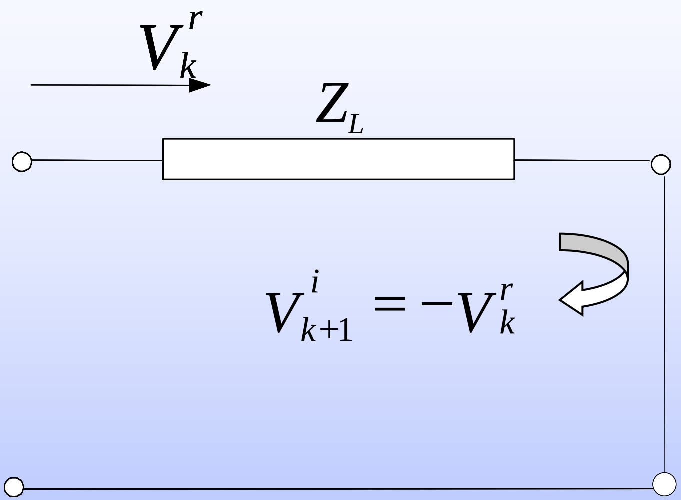
Example

- Evaluate the transient response of the shown circuit



Try It!

Example (Cont'd)



- Evaluate the current $i_k = (V_k^S - 2V_k^i)/(Z_L + R)$
- Evaluate the inductor voltage $V_k^C = 2V_k^i + Z_L i_k$
- Scattering: $V_k^r = V_k^C - V_k^i$
- Connection: $V_{k+1}^i = -V_k^r$