Lecture 9

From Sections 9.5 - 9.7 of textbook

Solve 9.18, 9.21, 9.23, 9.27, 9.32, 9.37, 9.38, 9.39

Maxwell's Equations

Integral Form	Differential Form
$\oint_L \underline{E} \cdot d\underline{l} = -\frac{\partial}{\partial t} \iint_S \underline{B} \cdot d\underline{s}$	$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$
$\oint_L \underline{H} \cdot d\underline{l} = \iint_S \left(\underline{J} + \frac{\partial \underline{D}}{\partial t} \right) \cdot d\underline{s}$	$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$
$\oiint_S \underline{D} \cdot d\underline{s} = \iiint_V \rho_v \, dv$	$\nabla \cdot \underline{D} = \rho_v$
$\oiint_S \underline{B} \cdot d\underline{s} = 0$	$\nabla \cdot \underline{B} = 0$
$\oiint_S \underline{J} \cdot d\underline{s} = -\frac{\partial}{\partial t} \iiint_V \rho_v \, dv$	$\nabla \cdot \underline{J} = -\frac{\partial \rho_v}{\partial t}$

<u>Constitutive Equations</u>

* The quantities $\underline{E}, \underline{D}, \underline{H}, \underline{B}$, and $\underline{J}$ are related through material properties

* $\underline{D} = \varepsilon \underline{E} = \varepsilon_0 \varepsilon_r \underline{E}$

  $\underline{B} = \mu \underline{H} = \mu_0 \mu_r \underline{H}$

  $\underline{J} = \sigma \underline{E}$

In general $\left(\underline{J}_t = \sigma \underline{E} + \underline{J}_i \right)$

Boundary Conditions

* Same boundary conditions still apply at interfaces of different materials.

$$E_{1t} - E_{2t} = 0 \Rightarrow (\underline{E}_1 - \underline{E}_2) \times \underline{a}_n = 0$$

$$H_{1t} - H_{2t} = K \Rightarrow (\underline{H}_1 - \underline{H}_2) \times \underline{a}_n = \underline{K}$$

$$D_{1n} - D_{2n} = \rho_s \Rightarrow (\underline{D}_1 - \underline{D}_2) \cdot \underline{a}_n = \rho_s$$

$$B_{1n} - B_{2n} = 0 \Rightarrow (\underline{B}_2 - \underline{B}_1) \cdot \underline{a}_n = 0$$

Retarded Potentials

* In the static case we have

$$V = \iiint_V \frac{\rho_v \, dv}{4\pi\varepsilon R}$$

and $\underline{A} = \iiint_V \frac{\mu \underline{J} \, dv}{4\pi R}$

$\underline{E} = -\nabla V$ and $\underline{B} = \nabla \times \underline{A}$

Retarded Potentials (Cont'd)

* Starting with the $\underline{E}$ field curl equation, we have

$$\nabla \times \underline{E} = -\frac{\partial}{\partial t}\underline{B} = -\frac{\partial}{\partial t}(\nabla \times \underline{A})$$

$$\Rightarrow \nabla \times \left(\underline{E} + \frac{\partial \underline{A}}{\partial t}\right) = \underline{0}$$

$$\Rightarrow \underline{E} + \frac{\partial \underline{A}}{\partial t} = -\nabla V \Rightarrow \boxed{\underline{E} = -\nabla V - \frac{\partial \underline{A}}{\partial t}}$$

Retarded Potentials (Cont'd)

* The D.Es governing V and $\underline{A}$ are given by

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \underline{A}) = -\frac{\rho_v}{\varepsilon}$$

and

$$\nabla^2 \underline{A} - \nabla(\nabla \cdot \underline{A}) = -\mu \underline{J} + \mu\varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) + \mu\varepsilon \frac{\partial^2 \underline{A}}{\partial t^2}$$

We choose $\nabla \cdot \underline{A} = -\frac{\partial V}{\partial t}$

Retarded Potential (Cont'd)

$$\left.\begin{aligned}\nabla^2 V - \underline{\mu\varepsilon}\frac{\partial^2 V}{\partial t^2} &= -\frac{\rho_v}{\varepsilon}\\ \nabla^2 \underline{A} - \underline{\mu\varepsilon}\frac{\partial^2 \underline{A}}{\partial t^2} &= -\mu \underline{J}\end{aligned}\right\} \quad u = \frac{1}{\sqrt{\mu\varepsilon}} \;\; m/s$$

these equations have the solution

$$V(x,y,z,t) = \iiint_{v'} \frac{\rho_v(x',y',z',t-\frac{R}{u})}{4\pi\varepsilon R}$$

$$\underline{A}(x,y,z,t) = \iiint_{v'} \frac{\mu \underline{J}(x',y',z',t-\frac{R}{u})}{4\pi R}$$

Time Harmonic Fields

* If the excitation is sinusoidal, then all fields are sinusoidal quantities
* In this case, we can use phasors

$$E_x = E_0 \cos(\omega t + \phi_x)$$

$$= \mathrm{Re}\left(E_0 e^{j(\omega t + \phi_x)}\right)$$

$$E_x = \mathrm{Re}\left(\underbrace{E_0 e^{j\phi_x}}_{\tilde{E}_x} e^{j\omega t}\right)$$

Maxwell's Equations For Harmonic Fields

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \Rightarrow \nabla \times \underline{\tilde{E}} = -j\omega \underline{\tilde{B}}$$

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} \Rightarrow \nabla \times \underline{\tilde{H}} = \underline{\tilde{J}} + j\omega \underline{\tilde{D}}$$

$$\nabla \cdot \underline{D} = \rho_v \Rightarrow \nabla \cdot \underline{\tilde{D}} = \tilde{\rho}_v$$

$$\nabla \cdot \underline{B} = 0 \Rightarrow \nabla \cdot \underline{\tilde{B}} = 0$$

$$\nabla \cdot \underline{J} = -\frac{\partial \rho_v}{\partial t} \Rightarrow \nabla \cdot \underline{\tilde{J}} = -j\omega \tilde{\rho}_v$$

* Same boundary Conditions still apply

EM Power Flow

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \cdot \underline{H}$$

$$\nabla \times \underline{H} = \underline{J}_c + \frac{\partial \underline{D}}{\partial t} \qquad \cdot \underline{E}$$

Subtract

$$\underbrace{\underline{H} \cdot (\nabla \times \underline{E}) - \underline{E} \cdot (\nabla \times \underline{H})}_{\nabla \cdot (\underline{E} \times \underline{H})} = \underline{H} \cdot \left(-\frac{\partial \underline{B}}{\partial t}\right) - \underline{E} \cdot \left(\underline{J}_c + \frac{\partial \underline{D}}{\partial t}\right)$$

$$\nabla \cdot (\underline{E} \times \underline{H}) = -\underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} - \underline{E} \cdot \underline{J}_c$$

EM Power Flow (Cont'd)

* Typically $\frac{\partial \varepsilon}{\partial t} = 0$, $\frac{\partial \mu}{\partial t} = 0$

$$\nabla \cdot (\underline{E} \times \underline{H}) = -\mu \underline{H} \cdot \frac{\partial \underline{H}}{\partial t} - \varepsilon \underline{E} \cdot \frac{\partial \underline{E}}{\partial t} - \underline{E} \cdot (\sigma \underline{E})$$

$$\Rightarrow \nabla \cdot (\underline{E} \times \underline{H}) = -\frac{1}{2}\mu \frac{\partial |\underline{H}|^2}{\partial t} - \frac{1}{2}\varepsilon \frac{\partial |\underline{E}|^2}{\partial t} - \sigma |\underline{E}|^2$$

$w_e = \frac{1}{2}\varepsilon |\underline{E}|^2$ J/m^3 electric energy density

$w_m = \frac{1}{2}\mu |\underline{H}|^2$ J/m^3 magnetic energy density

$p_\sigma = \sigma |\underline{E}|^2$ W/m^3 power density of Losses

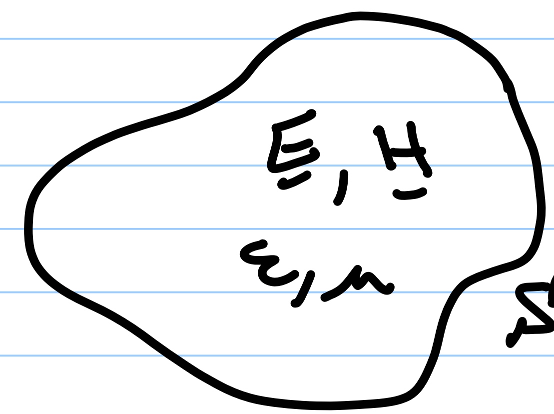
Vector of Poynting

* The vector $\underline{S} = \underline{E} \times \underline{H}$ is called the vector of poynting

* This vector points in the direction of EM power propagation

* Integrating over the volume of interest, we get

Poynting Theorem



$$\iiint_V \nabla \cdot (\underline{E} \times \underline{H}) \, dv$$

$$= -\frac{1}{2} \iiint_V \varepsilon \frac{\partial |\underline{E}|^2}{\partial t} dv - \frac{1}{2} \iiint_V \mu \frac{\partial |\underline{H}|^2}{\partial t} dv$$

$$- \iiint_V \sigma |\underline{E}|^2 dv$$

$$-\iint_S (\underline{E} \times \underline{H}) \cdot d\underline{s} = \frac{1}{2} \iiint_V \varepsilon \frac{\partial |\underline{E}|^2}{\partial t} dv$$

$$+ \frac{1}{2} \iiint_V \mu \frac{\partial |\underline{H}|^2}{\partial t} dv + \iiint_V \sigma |\underline{E}|^2 dv$$

Poynting Theorem (Cont'd)

$$-\iint_S (\underline{E}\times\underline{H})\cdot d\underline{s} = \frac{1}{2}\iiint_V \varepsilon \frac{\partial (\underline{E})^2}{\partial t}\,dv + \frac{1}{2}\iiint_V \mu \frac{\partial (\underline{H})^2}{\partial t}\,dv + \iiint_V \sigma |\underline{E}|^2\,dv$$

Energy Flowing in through S'

= stored Energy + dissipated Energy

Poynting Theorem (Cont'd)

If $\underline{J} = \underline{J}_c + \underline{J}_i$, we get

$$\iiint_V (\underline{E} \cdot \underline{J}_i)\, dv = \iint_S (\underline{E} \times \underline{H}) \cdot d\underline{s} + \frac{1}{2} \iiint_V \varepsilon \frac{\partial (\underline{E})^2}{\partial t}\, dv + \frac{1}{2} \iiint_V \mu \frac{\partial (\underline{H})^2}{\partial t}\, dv + \iiint_V \sigma |\underline{E}|^2\, dv$$

Energy generated = Energy leaving + Energy stored + Energy dissipated

EM energy is Conserved!