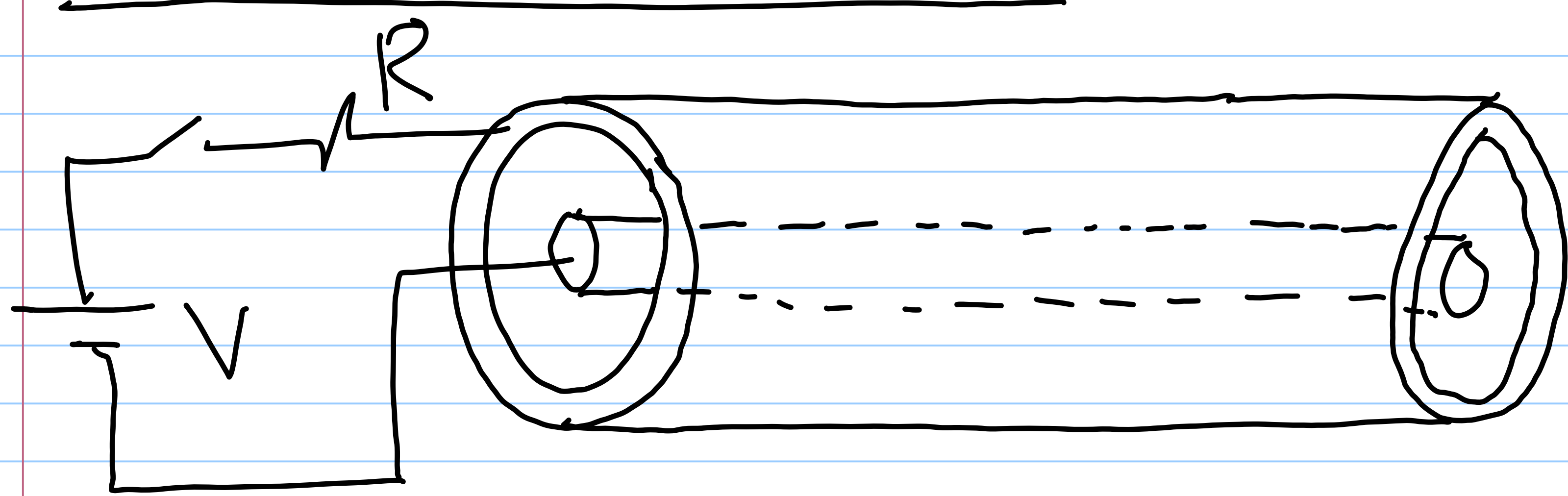
Lecture 14

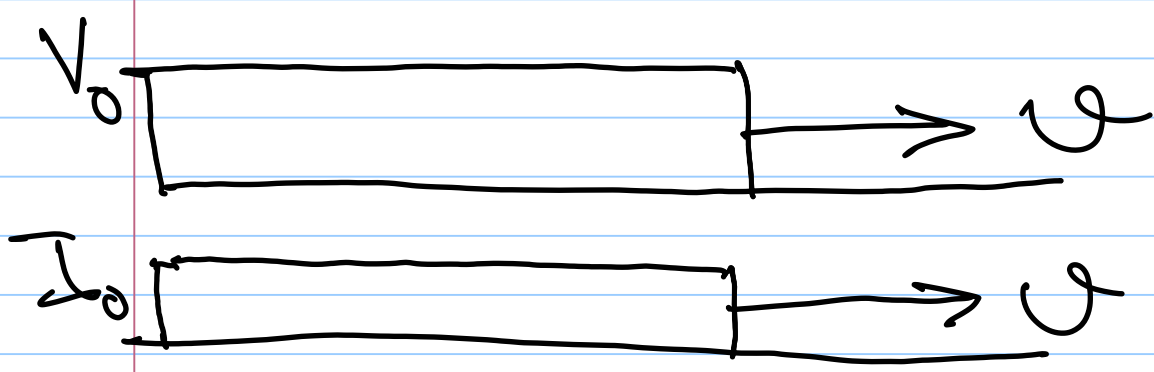
from sections 11.1 - 11.3

Solve 11.4 - 11.16

Transmission Lines

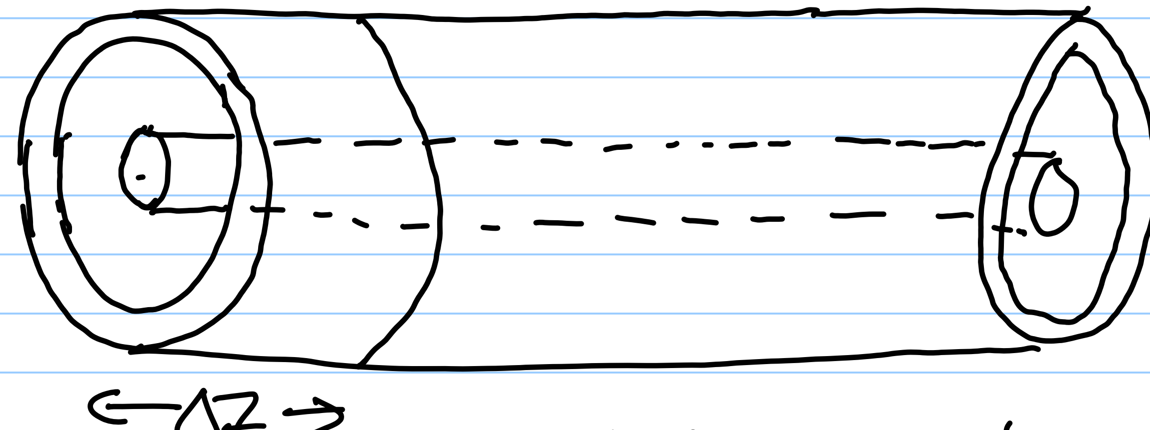


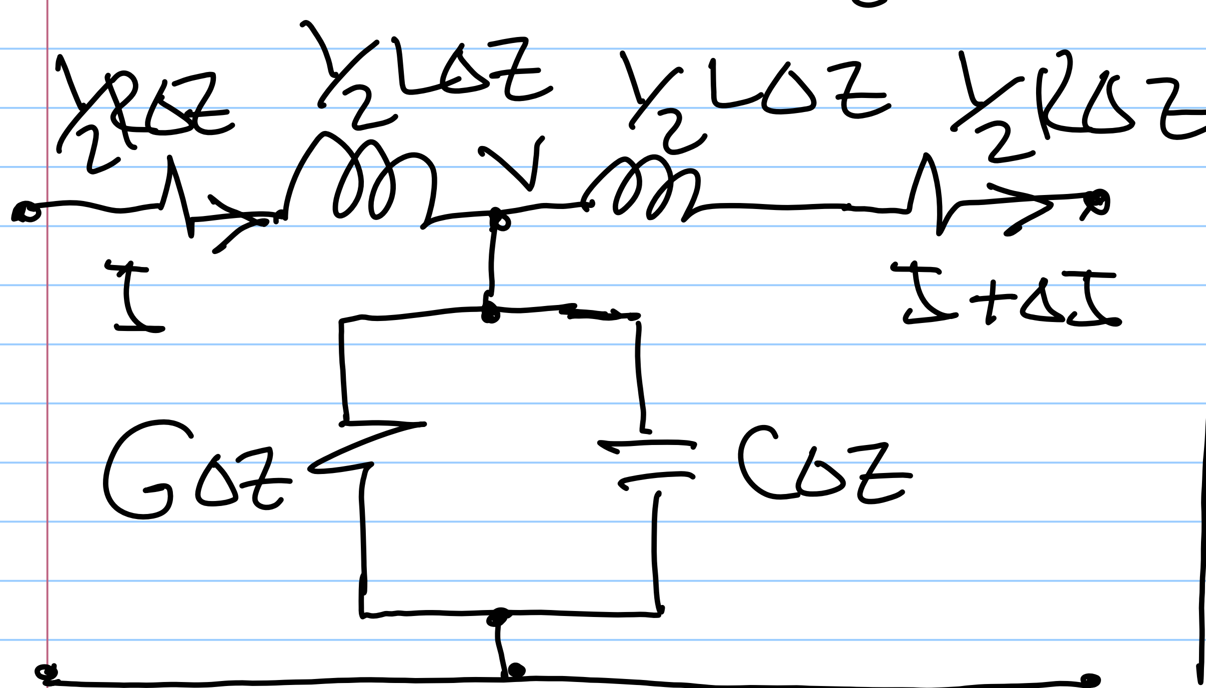
*When switch is closed, voltage and current waves are excited.



$$\frac{V_0}{I_0} = Z_0$$

Equivalent Circuit Model





R: resistance per unit length

G: Conductance per unit length

C: Capacitance per unit length

L: Inductance per unit length

Differential Equations of TLs

$$V = \frac{1}{2} R I \Delta z + \frac{1}{2} L \frac{\partial I}{\partial t} \Delta z + \frac{1}{2} L \left(\frac{\partial}{\partial t} (I + \Delta I) \right) \Delta z$$
$$+ \frac{1}{2} R (I + \Delta I) \Delta z + (V + \Delta V)$$

$$\Rightarrow \frac{\partial V}{\partial z} \simeq \frac{\Delta V}{\Delta z} = -\left(R I + L \frac{\partial I}{\partial t} \right) \leftarrow (1)$$

* By applying KCL, we get

$$I = G \Delta z \left(V + \frac{\Delta V}{2} \right) + C \Delta z \frac{\partial}{\partial t} \left(V + \frac{\Delta V}{2} \right) + (I + \Delta I)$$

$$\Rightarrow \frac{\partial I}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta I}{\Delta z} = -\left(G V + C \frac{\partial V}{\partial t} \right) \leftarrow (2)$$

Differential Equations (Cont'd)

* Differentiating J w.r.t. z and eliminating $\frac{\partial I}{\partial z}$ and $\frac{\partial^2 I}{\partial z \partial t}$, we get

$$\frac{\partial^2 V}{\partial z^2} = LC \frac{\partial^2 V}{\partial t^2} + (LG + RC) \frac{\partial V}{\partial t} + RGV$$

* Similarly, we can obtain for the current I

$$\frac{\partial^2 I}{\partial z^2} = LC \frac{\partial^2 I}{\partial t^2} + (LG + RC) \frac{\partial I}{\partial t} + RGI$$

The Lossless Case

* For the lossless case, we have
  $R = G = 0$

$$\Rightarrow \frac{\partial^2 V}{\partial z^2} = LC \frac{\partial^2 V}{\partial t^2} \text{ and } \frac{\partial^2 I}{\partial z^2} = LC \frac{\partial^2 I}{\partial t^2}$$

* The solution of this 1D wave equation is

$$V(z,t) = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right)$$

with $v = \frac{1}{\sqrt{LC}}$

The Lossless Case (Cont'd)

$$V(z,t) = V^+ + V^-$$

$$I(z,t) = I^+ + I^-$$

V^+, I^+, V^-, and I^- are not independent

$$\frac{\partial I}{\partial t} = -\frac{1}{L}\frac{\partial V}{\partial z} = -\frac{1}{L}\frac{\partial}{\partial z}\left(f_1\left(t-\frac{z}{u}\right) + f_2\left(t+\frac{z}{u}\right)\right)$$

$$\Rightarrow \frac{\partial I}{\partial t} = -\frac{1}{Lu}\left(-f_1'\left(t-\frac{z}{u}\right) - f_2'\left(t+\frac{z}{u}\right)\right)$$

$$\Rightarrow I = \frac{1}{Lu}\left(f_1\left(t-\frac{z}{u}\right) - f_2\left(t+\frac{z}{u}\right)\right)$$

$$\Rightarrow I^+ = \frac{V^+}{Lu} = \frac{V^+}{\left(L \cdot \frac{1}{\sqrt{LC}}\right)} = \frac{V^+}{\sqrt{\frac{L}{C}}} = \frac{V^+}{Z_0}$$

Characteristic Impedance

$Z_0 = \sqrt{\frac{L}{C}}$ = characteristic impedance of TL in Ω

also $= I^- = -\frac{V^-}{Z_0}$

$V = V^+ + V^- \Rightarrow I = \frac{V^+}{Z_0} - \frac{V^-}{Z_0}$

The Sinusoidal Case

* If we take $f_1(\xi) = \cos(\omega\xi + \phi)$
a possible solution of the Telegrapher's equation is $V(z,t) = \cos\left(\omega\left(t - \frac{z}{v}\right) + \phi\right)$

$$\Rightarrow V_+(z,t) = \cos(\omega t - \beta z + \phi), \quad \beta = \frac{\omega}{v}$$

* The backward wave is given by

$$V_-(z,t) = \cos(\omega t + \beta z + \phi)$$

* The corresponding phasors are

$$\tilde{V} = \tilde{V}_+ + \tilde{V}_- + V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

The Lossy Case

* This case is easier to address in the frequency domain
* The voltage satisfies

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2} + (LG+RC)\frac{\partial V}{\partial t} + RGV$$

Using phasor for $\left(\frac{\partial}{\partial t} = j\omega,\ \frac{\partial^2}{\partial t^2} = -\omega^2\right)$

$$\frac{\partial^2 \tilde{V}}{\partial z^2} = -\omega^2 LC\tilde{V} + j\omega(LG+RC)\tilde{V} + RG\tilde{V}$$

$$\Rightarrow \frac{\partial^2 \tilde{V}}{\partial z^2} = \underbrace{(R+j\omega L)}_{Z}\underbrace{(G+j\omega C)}_{Y}\tilde{V}$$

The Lossy Case (Cont'd)

$$\frac{\partial^2 \tilde{V}}{\partial z^2} = ZY\tilde{V} = \gamma^2 \tilde{V}$$

$$\Rightarrow \gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

* The general solution is given by

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$\Rightarrow \tilde{V}(z) = |V_0^+| e^{-\alpha z} e^{-j\beta z + j\phi^+} + |V_0^-| e^{\alpha z} e^{j\beta z + j\phi^-}$$

$$\Rightarrow V(z,t) = |V_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \phi^+) + |V_0^-| e^{\alpha z} \cos(\omega t + \beta z + \phi^-)$$

The Current Phasor

$$\frac{\partial V}{\partial z} = -\left(RI + L\frac{dI}{dt}\right) \Rightarrow \frac{\partial \tilde{V}}{\partial z} = -\underbrace{(R + j\omega L)}_{Z}\tilde{I}$$

$$\Rightarrow -\gamma V_0^+ e^{-\gamma z} + \gamma V_0^- e^{\gamma z} = -(R + j\omega L)\tilde{I}$$

$$\tilde{I} = \underbrace{\frac{\gamma}{Z} V_0^+}_{I_0^+} e^{-\gamma z} - \underbrace{\frac{\gamma}{Z} V_0^-}_{I_0^-} e^{\gamma z}$$

$$I_0^+ = \frac{V_0^+}{Z_0}, \quad Z_0 = \frac{Z}{\gamma} = \frac{Z}{\sqrt{ZY}} = \sqrt{\frac{Z}{Y}}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$ | Similarly $I_0^- = \dfrac{V_0^-}{Z_0}$