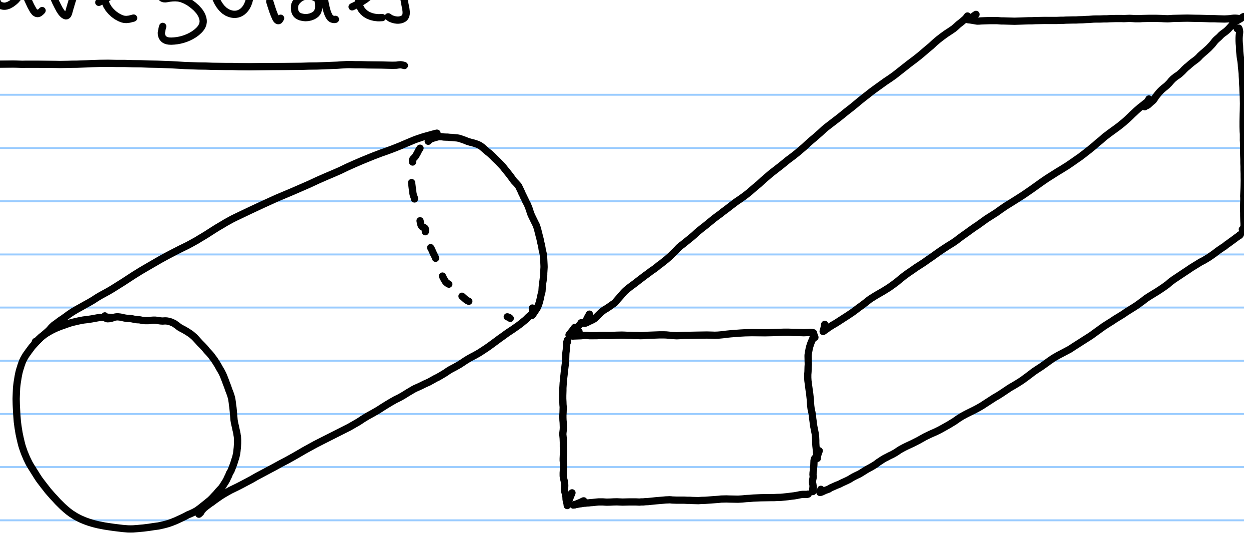
Lecture 17

From sections 12.1 - 12.3

Waveguides



* metalic waveguides are hollow tubes used to guide EM waves.
* They are used mainly in high power applications (e.g. Sattelite equipments)

Governing Equations

* Maxwell's equations for the lossless, sourceless, and phasor case are given by

$$\nabla \times \underline{\tilde{E}} = -j\omega\mu \underline{\tilde{H}} \quad \& \quad \nabla \times \underline{\tilde{H}} = j\omega\varepsilon \underline{\tilde{E}}$$

$$\Rightarrow \nabla \times \nabla \times \underline{\tilde{E}} = -j\omega\mu(j\omega\varepsilon \underline{\tilde{E}})$$

$$\nabla(\nabla \cdot \underline{\tilde{E}}) - \nabla^2 \underline{\tilde{E}} = \omega^2\mu\varepsilon \underline{\tilde{E}} \quad (\nabla \cdot \underline{\tilde{E}} = 0)$$

$$\nabla^2 \underline{\tilde{E}} + K^2 \underline{\tilde{E}} = 0 \quad , \quad K = \omega\sqrt{\mu\varepsilon}$$

Governing Equations (Cont'd)

* Similarly, for the magnetic field phasor, we have $\nabla^2 \underline{\tilde{H}} + k^2 \underline{\tilde{H}} = 0$

* It follows that we have 6 partial differential equations to solve!

* We can only solve for the components in the direction of wave propagation ($\tilde{E}_z$ and $\tilde{H}_z$) and then obtain all other fields in terms of $\tilde{E}_z$ and $\tilde{H}_z$

Governing Equations of Longitudinal Components

$$\nabla^2 \tilde{E}_z + k^2 \tilde{E}_z = 0 \Rightarrow \frac{\partial^2 \tilde{E}_z}{\partial x^2} + \frac{\partial^2 \tilde{E}_z}{\partial y^2} + \frac{\partial^2 \tilde{E}_z}{\partial z^2} + k^2 \tilde{E}_z = 0$$

* Using separation of variables, we have

$$\tilde{E}_z(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -k^2$$

$$\boxed{k^2 = k_x^2 + k_y^2 - \gamma^2}$$

we take $\frac{X''}{X} = -k_x^2$, $\frac{Y''}{Y} = -k_y^2$, $\frac{Z''}{Z} = \gamma^2$

(Why is k_z different?)

Solution of $\tilde{E}_z$

$$\tilde{E}_z = (C_1 \cos K_x x + C_2 \sin K_x x)(C_3 \cos K_y y + C_4 \sin K_y y)$$
$$(\underbrace{C_5 e^{-\gamma z}}_{\text{incident wave}} + \underbrace{C_6 e^{\gamma z}}_{\text{reflected wave}})$$

* Similarly for $\tilde{H}_z$, we have

$$\tilde{H}_z = (B_1 \cos K_x x + B_2 \sin K_x x)(B_3 \cos K_y y + B_4 \sin K_y y)$$
$$(B_5 e^{-\gamma z} + B_6 e^{-\gamma z})$$

* Boundary conditions determine K_x and K_y

Relation to other field Components

* Using Maxwell's Equations, we can show that

$$\tilde{E}_x = -\frac{\gamma}{h^2}\frac{\partial \tilde{E}_z}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial \tilde{H}_z}{\partial y}$$

$$\tilde{E}_y = -\frac{\gamma}{h^2}\frac{\partial \tilde{E}_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial \tilde{H}_z}{\partial x}$$

$$\tilde{H}_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial \tilde{E}_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial \tilde{H}_z}{\partial x}$$

$$\tilde{H}_y = -\frac{j\omega\varepsilon}{h^2}\frac{\partial \tilde{E}_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial \tilde{H}_z}{\partial y}$$

$$h^2 = K_x^2 + K_y^2 = K^2 + \gamma^2$$

Prove It!

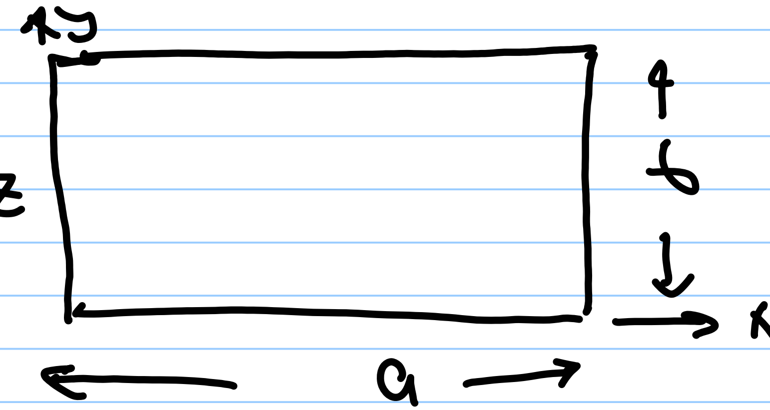
Types of Waveguides Modes

TEM: $\tilde{E}_z = 0$ and $\tilde{H}_z = 0 \Longrightarrow$ all fields are Zero

TE : $\tilde{E}_z = 0$, $\tilde{H}_z \neq 0 \Longrightarrow$ Non Zero lateral fields

TM: $\tilde{H}_z = 0$, $\tilde{E}_z \neq 0 \Longrightarrow$ Non Zero lateral fields

Transverse Magnetic (TM) Modes



$$\tilde{E}_z = (A_1 \cos K_x x + A_2 \sin K_x x)(A_3 \cos K_y y + A_4 \sin K_y y)\, e^{-\gamma z}$$

* we must have the following boundary conditions

$\tilde{E}_z(x=0)=0$, $\tilde{E}_z(x=a)=0$

$\tilde{E}_z(y=0)=0$, $\tilde{E}_z(y=b)=0$

$A_1 = 0$

$A_3 = 0$

$K_x a = m\pi \Rightarrow K_x = \frac{m\pi}{a}$

$K_y b = n\pi \Rightarrow K_y \frac{n\pi}{b}$

TM Case

$$\tilde{E}_z = E_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, e^{-\gamma z} \qquad (m \neq 0 \;\&\; n \neq 0)$$

For other field components, we get

$$\tilde{E}_x = -\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$$

$$\tilde{E}_y = -\frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$$

$$\tilde{H}_x = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$$

$$\tilde{H}_y = -\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\gamma z}$$

TM case (cont'd)

* 'Each value of m and n presents a mode of propagation
* remember that $K^2 = K_x^2 + K_y^2 - \gamma^2$

$$\Rightarrow \quad \gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon} = \alpha + j\beta$$

* Below a certain cut-off frequency ω, γ is real $\Longrightarrow$ Mode not propagating

TM Case (Cont'd)

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$f_c = \frac{u}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$m = 1, 2, \ldots$
$n = 1, 2, \ldots$

$$\beta = \sqrt{K^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \omega\sqrt{\mu\varepsilon}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\eta_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$