EE750
Advanced Engineering Electromagnetics
Lecture 13

Differential Equations Vs. Integral Equations

- Integral equations may take several forms, e.g.,

$$f(x) = \int_a^b K(x,t)\varphi(t)\,\mathrm{d}t$$

$$f(x) = \varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)\,\mathrm{d}t$$

- Most differential equations can be expressed as integral equations, e.g.,

$$d^2\varphi/d\,x^2 = F(x,\varphi), \quad a \le x \le b$$

$$\Downarrow$$

$$d\,\varphi/d\,x \quad = \int_a^x F(t,\varphi(t))\,dt + C_1 \Rightarrow C_1 = \varphi'(a)$$

$$\Downarrow$$

$$\varphi(x) = \int_a^x (x-t)F(t,\varphi(t))\,dt + C_1 x + C_2 \Rightarrow C_2 = \varphi(a) - a\varphi'(a)$$

Green's Functions

- Green's functions offer a systematic way of converting a Differential Equation (DE) to an Integral Equation (IE)
- A Green's function is the solution of the DE corresponding to an impulsive (unit) excitation
- Consider the differential equation $L\Phi = g$, where L is a differential operator, Φ is the unknown field and g is the known given excitation
- For this problem, the Green's function $G(\boldsymbol{r},\boldsymbol{r}')$ is the solution of the DE $LG = \delta(\boldsymbol{r}')$ subject to the same boundary conditions
- For an arbitrary excitation we have $\Phi = \int_{\text{excitation volume}} g(\boldsymbol{r}')G(\boldsymbol{r},\boldsymbol{r}')dv'$

Green’s Functions: Examples

- Obtain the Green’s function for the DE $(\partial^2/\partial x^2 + \partial^2/\partial y^2)\Phi = g$ subject to $\Phi = f$ on the boundary B

- The Green’s function is the solution of

$$\nabla^2 G(x, y, x', y') = \delta(x - x')\delta(y - y')$$

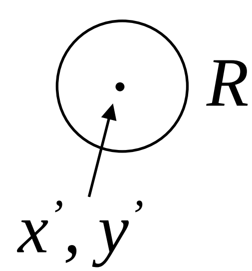
- G can be decomposed into a particular integral and a homogeneous solution $G=F+U$ with F and U satisfying

$$\nabla^2 F = \delta(x - x')\delta(y - y'), \quad \nabla^2 \boldsymbol{U} = 0$$

- Switching to polar form we get $\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial F}{\partial \rho}\right) = 0, \forall x \neq x', y \neq y'$

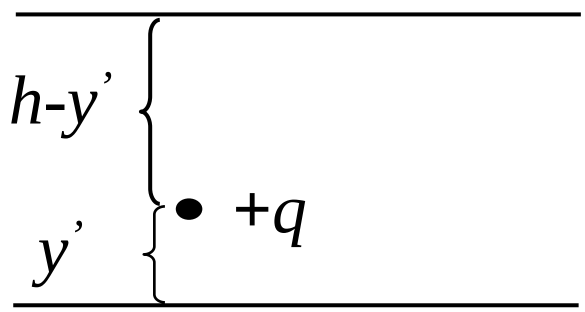
$$\Longrightarrow F = A \ln \rho + C_1$$

- A is obtained using $\lim\limits_{R \to 0} \oint \dfrac{\partial F}{\partial \rho} dl = 1 \quad \Longrightarrow \quad 2\pi A = 1$



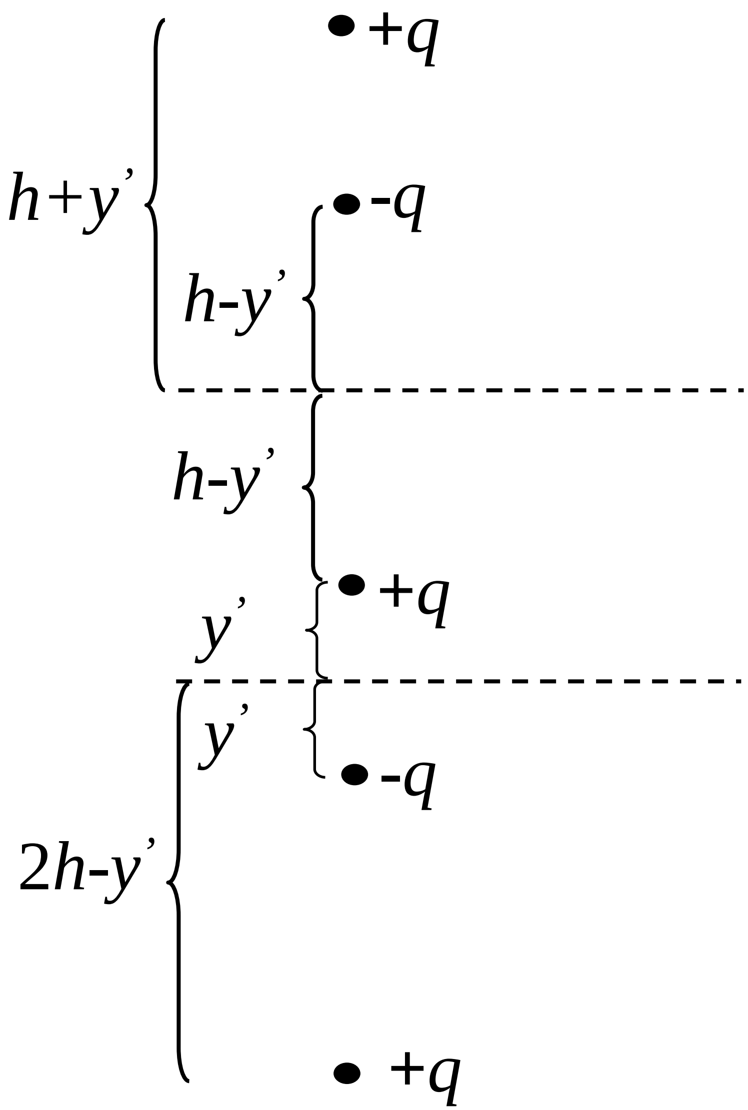
Green's Function: Examples (Cont'd)

- The method of images can also be applied to obtain an infinite series expansion of Green's functions
- Consider the case of a line charge between two conducting planes
- $G(x,y,x',y')$ represents the potential at (x, y) due to a line charge of value 1.0 c/m located at (x', y')



Original problem

Green's Function: Examples (Cont'd)



An infinite number of charges is required to maintain the same boundary conditions

Green's Function: Examples (Cont'd)

• The potential caused by a 1 c/m line charge in an unbounded medium is given by

$$\boldsymbol{V}(\rho)=\frac{1}{4\pi\varepsilon}\ln\rho^2$$

• Using the figure, we conclude that the Green's function is given by the infinite series

$$G(x,y,x',y')=\frac{1}{4\pi\varepsilon}\left(\begin{array}{l}\ln\left[(x-x')^2+(y-y')^2\right]-\ln\left[(x-x')^2+(y+y')^2\right]+\\ \sum_{n=1}^{\infty}(-1)^n\left[\begin{array}{l}\ln\left[(x-x')^2+(y+y'-2nh)^2\right]-\ln\left[(x-x')^2+(y-y'-2nh)^2\right]\\ \ln\left[(x-x')^2+(y+y'+2nh)^2\right]-\ln\left[(x-x')^2+(y-y'+2nh)^2\right]\end{array}\right]\end{array}\right)$$

• Special mathematical techniques are usually utilized to sum such a slowly convergent series

Green's Function: Examples (Cont'd)

- The Green's function can also be expanded in terms of the eigenfunctions of the homogeneous problem
- As an example consider the wave equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k^2 \psi = 0, \text{ Subject to } \frac{\partial \psi}{\partial n} = 0 \text{ or } \psi = 0 \text{ on } B$$

- Let the eigenvalues and eigenfunctions be k_j and ψ_j

$$\nabla^2 \psi_j + k_j^2 \psi_j = 0$$

- The set ψ_j is an orthonormal set, i.e.,

$$\int_S \psi_j^* \psi_i \, dxdy = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Green's Function: Examples (Cont'd)

- We then expand the Green's function in terms of the eigenfunctions $G(x,y,x',y') = \sum_{j=1}^{\infty} a_j \psi_j(x,y)$
- But as the Green's function satisfy

$$\left(\nabla^2 + k^2\right) G(x,y,x',y') = \delta(x-x')\delta(y-y')$$

$\Downarrow$ Substitute for G

$$\sum_{j=1}^{\infty} a_j (k^2 - k_j^2)\psi_j = \delta(x-x')\delta(y-y')$$

$\Downarrow$ Multiply by ψ_i^* and integrate

$$\sum_{j=1}^{\infty} a_j (k^2 - k_j^2)\iint_S \psi_i^* \psi_j \, ds = \psi_i^*(x',y')$$

$\Downarrow$

$$a_i = \frac{\psi_i^*(x',y')}{\left(k^2 - k_i^2\right)}$$

Green's Function: Examples (Cont'd)

- Using Green's functions, construct the solution for the Poisson's equation $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = f(x, y),$

  Subject to $V(0, y)=V(a, y)=V(x, o)=V(x, b)$

  Show that $\psi_{mn} = \dfrac{2}{\sqrt{ab}} \sin(\dfrac{m\pi x}{a}) \sin(\dfrac{n\pi y}{b})$

$$\lambda_{mn} = -\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right), \quad A_{mn} = \frac{-2}{\sqrt{ab}} \frac{\sin\left(\dfrac{m\pi x'}{a}\right)\sin\left(\dfrac{n\pi y'}{b}\right)}{\left(\dfrac{m^2\pi^2}{a^2} + \dfrac{n^2\pi^2}{b^2}\right)}$$

$$V(x, y) = \int_0^a \int_0^b G(x, y, x', y') f(x', y') dx' dy'$$

Dyadic Green's Functions

- Dyadic Green's functions are used to express the situation where a source in one direction gives rise to fields in different directions

- In general, a dyadic Green's function will have 9 components

$$\boldsymbol{G}(x,y,z,x',y',z') = G_{xx}\,\boldsymbol{ii} + G_{xy}\,\boldsymbol{ij} + G_{xz}\,\boldsymbol{ik} +$$
$$G_{yx}\,\boldsymbol{ji} + G_{yy}\,\boldsymbol{jj} + G_{yz}\,\boldsymbol{jk} + G_{zx}\,\boldsymbol{ki} + G_{zy}\,\boldsymbol{kj} + G_{zz}\,\boldsymbol{kk}$$

- For a unit source in the x direction $\boldsymbol{J} = \boldsymbol{i}\delta(x-x')\delta(y-y')\delta(z-z')$

  we obtain the field $\boldsymbol{E}=\boldsymbol{G}.\boldsymbol{J}$ = $G_{xx}\,\boldsymbol{i} + G_{yx}\,\boldsymbol{j} + G_{zx}\,\boldsymbol{k}$

- For a general source (arbitrary distribution and orientations)

$$\boldsymbol{E}(x,y,z) = \iiint_{V'} \boldsymbol{G}(x,y,z,x',y',z').\boldsymbol{J}(x',y',z')dv'$$

Inner Products

- The inner product of two functions is a scalar that must satisfy the following conditions:

$$< f, g > = < g, f > \qquad \text{commutative}$$

$$< \alpha f + \beta g, h > = \alpha < f, h > + \beta < g, h > \qquad \text{distributive}$$

$$< f, f^* > > 0 \quad \text{if} \quad f \neq 0$$

$$< f, f^* > = 0 \quad \text{iff} \quad f = 0$$

- Example: $< f(x), g(x) > = \int_0^1 f(x) g(x) dx$

Adjoint Operators

- For an operator L, we sometimes define an adjoint operator L^a defined by $<Lf, g> = <f, L^a g>$
- For the DE $-d^2 f/dx^2 = g(x)$, $f(0)=f(1)=0 \Rightarrow L = -d^2/dx^2$
- We utilize the inner product $<f(x), g(x)> = \int_0^1 f(x)g(x)dx$

$$<Lf, g> = \int_0^1 -\frac{d^2 f}{dx^2} g(x)dx \Rightarrow -\frac{df}{dx}g + f\frac{dg}{dx}\Big|_0^1 + \int_0^1 f\left(-\frac{d^2 g}{dx^2}\right)dx$$

if $g(0)=g(1)=0$, we have $<Lf, g> = \int_0^1 -\frac{d^2 g}{dx^2} f\, dx = <f, Lg>$

$L=L^a$

Method of Moments (MoM)

- MoM aims at obtaining a solution to the inhomogeneous equation $Lf = g$, where L is a known linear operator, g is a known excitation and f is unknown

- Let f be expanded in a series of known basis functions f_1, f_2, …, f_N $\Rightarrow$ $f = \sum_n \alpha_n f_n$

- Substituting in the equation we get

$$L(\sum_n \alpha_n f_n) = g \Rightarrow \sum_n \alpha_n L(f_n) = g \quad \text{(One equation in } N \text{ unknowns)}$$

- We define a set of N weighting functions w_1, w_2, …, w_N

MoM (Cont'd)

- Taking the inner product of both sides with the *m*th weighting function we obtain

  $$\sum_n \alpha_n < w_m, L(f_n) > = < w_m, g >, \quad m = 1, 2, \cdots, N$$

  (N equations in N unknowns)

- In matrix form we can write $[l_{mn}][\alpha_n] = [g_m]$

$$[l_{mn}] = \begin{bmatrix} < w_1, L f_1 > & < w_1, L f_2 > & \cdots & < w_1, L f_N > \\ < w_2, L f_1 > & < w_2, L f_2 > & \cdots & < w_2, L f_N > \\ \vdots & \vdots & \vdots & \vdots \\ < w_N, L f_1 > & < w_N, L f_2 > & \cdots & < w_N, L f_N > \end{bmatrix}$$

MoM (Cont'd)

$$[\alpha_n]=\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\end{bmatrix},\qquad [g_m]=\begin{bmatrix}<w_1,g>\\ <w_2,g>\\ \vdots\\ <w_N,g>\end{bmatrix}$$

- The unknown coefficients are thus given by $[\alpha_n]=[l_{mn}]^{-1}[g_m]$
- The unknown function f can now be expressed in the compact form

$$f=\sum_n \alpha_n f_n=\begin{bmatrix}f_1 & f_2 & \cdots & f_N\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_N\end{bmatrix}=[\tilde{f}_n][\alpha_n]=[\tilde{f}_n][l_{mn}]^{-1}[g_m]$$

MoM Example

- Solve $d^2f/dx^2=1+4x^2$, $f(0)=f(1)=0$ using MoM
- We choose the basis functions as $f_n=x-x^{n+1}$, $n=1, 2, \ldots, N$

  f is thus approximated by $f=\sum_{n=1}^{N}\alpha_n(x-x^{n+1})$
- Also we choose $w_n=f_n$, $n=1, 2, \ldots N$ (Galerkin's approach)
- our inner product is $<f,g>=\int_0^1 f(x)g(x)dx$
- We have $Lf_n=d^2f_n/dx^2=n(n+1)x^{n-1}$
- Show that $l_{mn}=<w_m, Lf_n>=mn/(m+n+1)$

  $g_m=<w_m,g>=m(3m+8)/(2(m+2)(m+4))$

MoM Example (Cont'd)

- For N=1, we have l_{11}=1/3, g_1=11/30 $\Rightarrow$ α_1=11/10
- For N=2, we have

$$\begin{bmatrix} 1/3 & 1/2 \\ 1/2 & 4/5 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 11/30 \\ 7/12 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 2/3 \end{bmatrix}$$

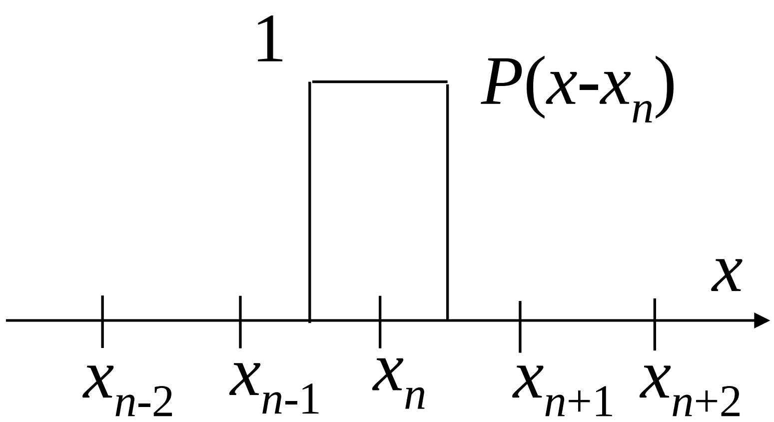
- For N=3, we have

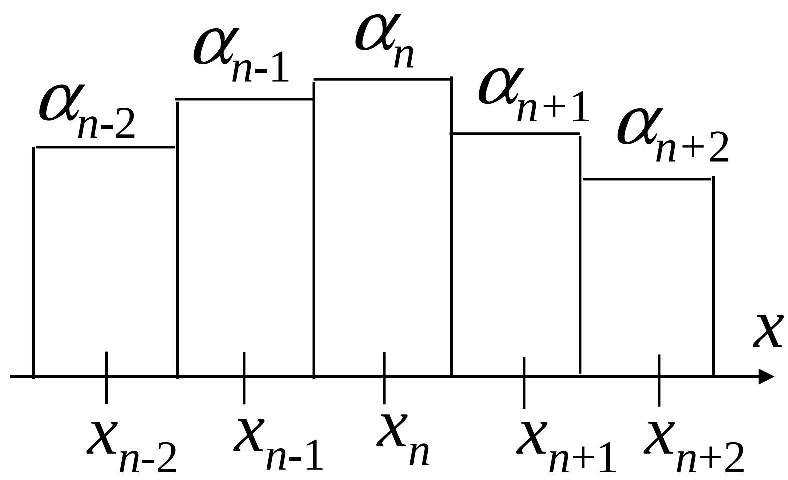
$$\begin{bmatrix} 1/3 & 1/2 & 3/5 \\ 1/2 & 4/5 & 1 \\ 3/5 & 1 & 9/7 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 11/30 \\ 7/12 \\ 51/70 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/3 \end{bmatrix}$$

- Exact solution is obtained for N=3!

Types of Basis Functions

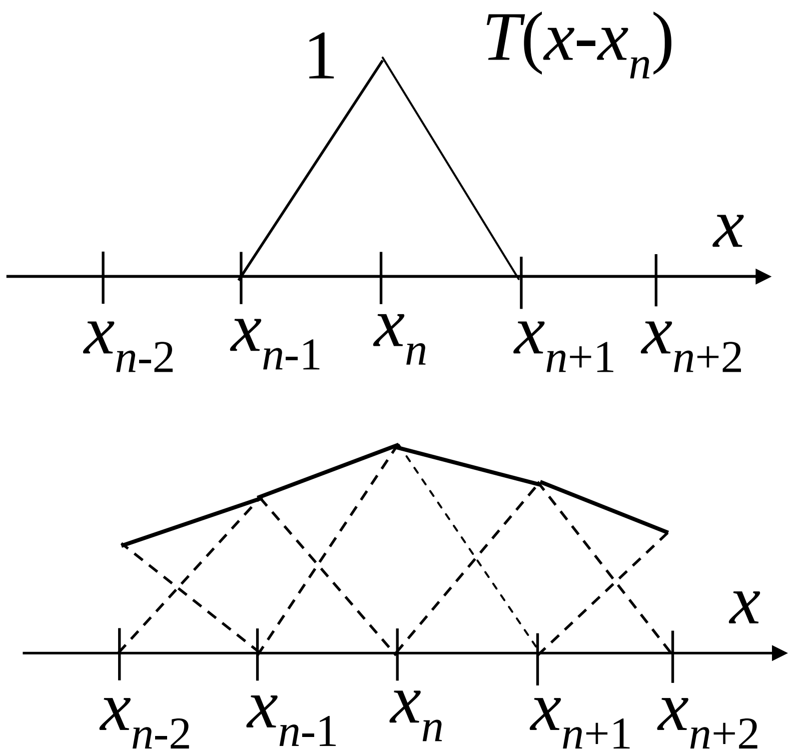
- Entire domain basis functions f_n are defined for the entire domain of the function f
- Subsectional basis functions are defined only over a subsection of the domain of the function f





Pulse functions in 1D

Types of Basis Functions (Cont'd)



Triangular functions in 1D

Types of Weighting Functions

- Recall that

$$\sum_n \alpha_n < w_m, L(f_n) > = < w_m, g >, \quad m = 1, 2, \cdots, N$$

- If we choose $w_n = f_n$, $n = 1, 2, \ldots, N$ (Galerkin matching)

$$\sum_n \alpha_n < f_m, L(f_n) > = < f_m, g >, \quad m = 1, 2, \cdots, N$$

- If we choose $w_n = \delta(\boldsymbol{r} - \boldsymbol{r}_n)$, $n = 1, 2, \ldots, N$ (Point matching)

$$\sum_n \alpha_n < \delta(\boldsymbol{r} - \boldsymbol{r}_m), L(f_n) > = < \delta(\boldsymbol{r} - \boldsymbol{r}_m), g >, \quad m = 1, 2, \cdots, N$$

$$\Downarrow$$

$$\sum_n \alpha_n L(f_n(\boldsymbol{r}_m)) = g(\boldsymbol{r}_m), \quad m = 1, 2, \cdots, N$$

- The two sides of the system equation are matched at a number of space points