EE750
Advanced Engineering Electromagnetics
Lecture 17

2D FEM

- We consider a 2D differential equation of the form

$$-\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial \varphi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial \varphi}{\partial y}\right) + \beta\varphi = f, \ (x, y) \in \Omega$$

subject to $\varphi = p \quad \text{on } \Gamma_1$

$$\left(\alpha_x \frac{\partial \varphi}{\partial x}\boldsymbol{i} + \alpha_y \frac{\partial \varphi}{\partial y}\boldsymbol{j}\right).\boldsymbol{n} + \gamma\varphi = q \quad \text{on } \Gamma_2$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ is the contour enclosing the domain Ω and $\boldsymbol{n}$ is the unit outward normal

- Notice that the boundary conditions may be a Dirichlet, Neuman or mixed Dirichlet and Neuman.

- α_x, α_y and β are functions associated with the physical parameters and f is the excitation

2D FEM (Cont’d)

- The functional associated with this problem is

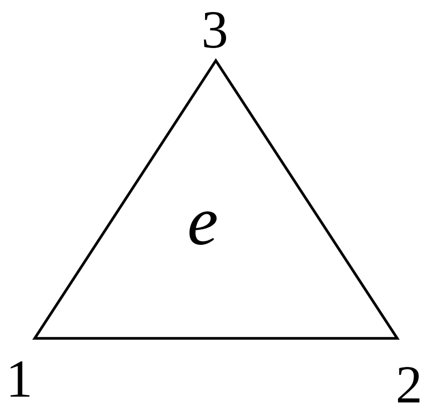
$$F(\varphi)=0.5\iint_{\Omega}\left[\alpha_x\left(\frac{\partial\varphi}{\partial x}\right)^2+\alpha_y\left(\frac{\partial\varphi}{\partial y}\right)^2+\beta\,\varphi^2\right]d\Omega$$

$$-\iint_{\Omega} f\varphi\, d\Omega+\int_{\Gamma 2}\left[\frac{\gamma}{2}\varphi^2-q\,\varphi\right]d\Gamma$$

(Prove it)!

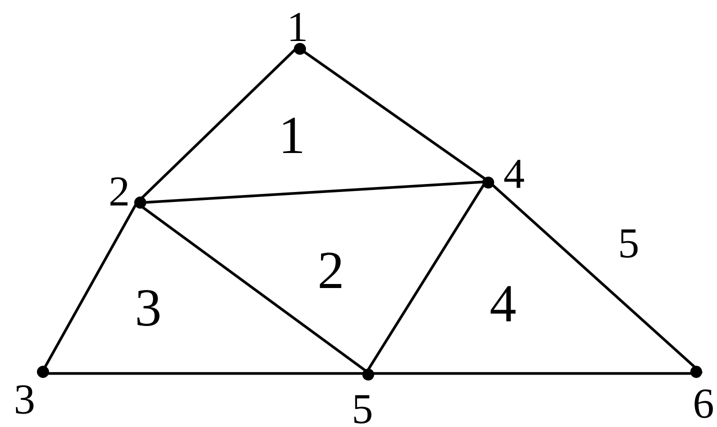
2D FEM Analysis

- The computational domain is divided into triangles (elements)
- Each node has both a local and a global index

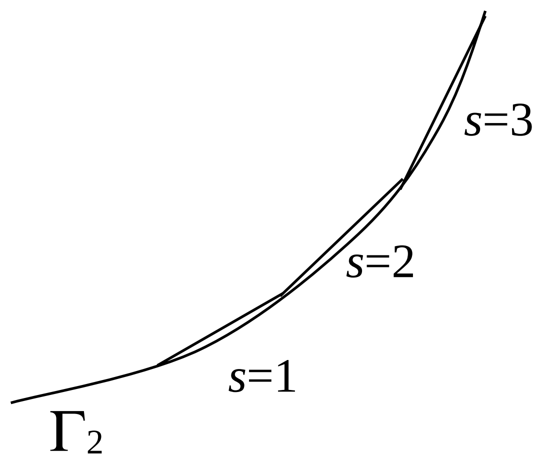


- A connectivity array $n(i,e)$, i=1, 2, 3 and e=1, 2, …, M stores the global indices of the nodes

e \ i	1	2	3
1	2	4	1
2	5	4	2
3	3	5	2
4	5	6	4



2D FEM Analysis (Cont'd)



- We assume that there are M_s line segments on Γ_2
- We store the index array $n_s(i,s)$, i=1, 2 and s=1, 2, …, M_s of global indices of nodes on Γ_2

Input Data to the 2D FEM Analysis

- The coordinates of the nodes $\mathbf{r}_i=(x_i,y_i)$, $i=1, 2, \ldots, N$, where N is the total number of nodes
- The values of α_x, α_y, β and f for each element
- The value of p for each node residing on Γ_1
- The value of γ and q for each segment with nodes on Γ_2
- The two arrays $n(i,e)$, $i=1, 2, 3$ and $e=1, 2, \ldots, M$ and $n_s(i,s)$, $i=1, 2$ and $s=1, 2, \ldots, M_s$

Elemental Interpolation

- Over the *eth* element we utilize the linear approximation
  $$\varphi^e(x,y) = a^e + b^e x + c^e y, \quad (x,y) \in \Omega_e$$
- The three nodes of the *e*th element must satisfy the linear interpolation relation
  $$\varphi_1^e = a^e + b^e x_1^e + c^e y_1^e, \quad \varphi_2^e = a^e + b^e x_2^e + c^e y_2^e,$$
  $$\varphi_3^e = a^e + b^e x_3^e + c^e y_3^e$$
- Solving for a^e, b^e and c^e we obtain $\varphi^e(x,y) = \sum_{j=1}^{3} N_j^e(x,y)\,\varphi_j^e$
  where $N_j^e(x,y) = \dfrac{1}{2A_e}(a_j^e + b_j^e x + c_j^e y), \; j = 1, 2, 3$
  $$a_1^e = x_2^e y_3^e - y_2^e x_3^e, \; b_1^e = y_2^e - y_3^e, \; c_1^e = x_3^e - x_2^e$$
  $$a_2^e = x_3^e y_1^e - y_3^e x_1^e, \; b_2^e = y_3^e - y_1^e, \; c_2^e = x_1^e - x_3^e$$
  $$a_3^e = x_1^e y_2^e - y_1^e x_2^e, \; b_3^e = y_1^e - y_2^e, \; c_3^e = x_2^e - x_1^e$$

Elemental Interpolation (Cont'd)

- A_e is the area of the eth element and is given by

$$A_e = \frac{1}{2}\begin{vmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{vmatrix}$$

- The interpolation functions satisfy

$$N_i^e(x_j^e, y_j^e) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The Homogenous Neuman BC case

- We first consider the case ($\gamma = q = 0$)
- The functional is expressed as a sum of elemental subfunctions

$$F(\varphi) = \sum_{e=1}^{M} F^{e}(\varphi^{e})$$

where

$$F^{e}(\varphi^{e}) = 0.5 \iint_{\Omega e} \alpha_x \left(\frac{d\,\varphi^{e}}{dx}\right)^2 + \alpha_y \left(\frac{d\,\varphi^{e}}{dy}\right)^2 + \beta\,(\varphi^{e})^2\, d\Omega - \iint_{\Omega e} f\ \varphi^{e}\, d\Omega$$

- Substituting with the linear interpolation expression, we write

The Homogenous Neuman BC case

$$F^e(\varphi^e) = 0.5 \iint_{\Omega_e} \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_x \varphi_i^e \frac{d N_i^e}{dx} \frac{d N_j^e}{dx} \varphi_j^e + \alpha_y \varphi_i^e \frac{d N_i^e}{dy} \frac{d N_j^e}{dy} \varphi_j^e \, d\Omega$$

$$+ \iint_{\Omega_e} \sum_{i=1}^{3} \sum_{j=1}^{3} \beta \varphi_i^e N_i^e N_j^e \varphi_j^e d\Omega - \iint_{\Omega_e} f \sum_{i=1}^{3} N_i^e \varphi_i^e \, d\Omega$$

$$\Downarrow$$

$$\frac{\partial F^e}{\partial \varphi_i^e} = \sum_{j=1}^{3} \varphi_j^e \left(\iint_{\Omega_e} \alpha_x \frac{d N_i^e}{dx} \frac{d N_j^e}{dx} + \alpha_y \frac{d N_i^e}{dy} \frac{d N_j^e}{dy} + \beta N_i^e N_j^e \, d\Omega \right)$$

$$- \iint_{\Omega_e} f N_i^e \, d\Omega \qquad i=1, 2, 3$$

$$\Downarrow$$

$$\left\{ \frac{\partial F^e}{\partial \boldsymbol{\varphi}^e} \right\}_{3\times1} = \left[\boldsymbol{K}^e\right]_{3\times3} \left[\boldsymbol{\varphi}^e\right]_{3\times1} - \left[\boldsymbol{b}^e\right]_{3\times1}$$

The Homogenous Neuman BC case (Cont'd)

$$K_{ij}^e = \iint_{\Omega_e} \alpha_x \frac{d\,N_i^e}{dx}\frac{d\,N_j^e}{dx} + \alpha_y \frac{d\,N_i^e}{dy}\frac{d\,N_j^e}{dy} + \beta N_i^e N_j^e \, d\Omega$$

$$b_i^e = \iint_{\Omega_e} f N_i^e \, d\Omega \qquad\qquad i\text{=1, 2, 3 and } j\text{=1, 2, 3}$$

- If α_x, α_y , β and f are taken as constants over each element, and utilizing the property

$$\iint_{\Omega_e} \left(N_1^e\right)^l \left(N_2^e\right)^m \left(N_3^e\right)^n d\Omega = 2\,A_e \frac{l!\,m!\,n!}{l!+m!+n!}$$

we get

$$K_{ij}^e = \frac{1}{4\,A_e}\left(\alpha_x^e b_i^e b_j^e + \alpha_y^e c_i^e c_j^e\right) + \frac{A_e}{12}\beta^e (1+\delta_{ij})$$

$$b_i^e = \frac{A_e}{3} f^e$$

The Process of Assembly

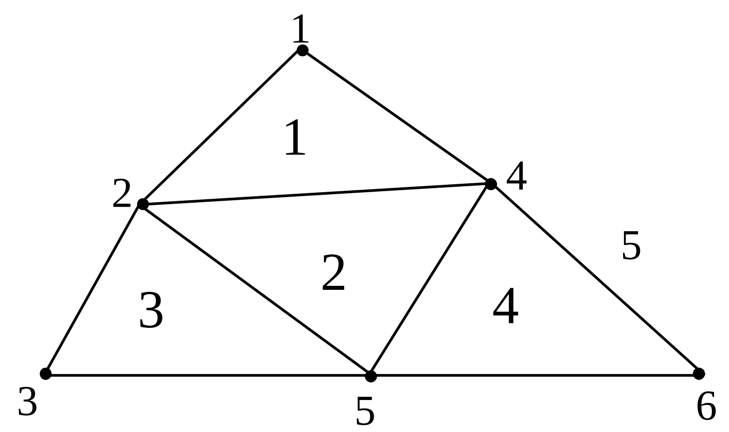
- The process of assembly involves storing the local elemental components into their proper location in the global system of equations

$$\frac{\partial F}{\partial \boldsymbol{\varphi}} = \mathbf{0} \quad \Longrightarrow \quad [\boldsymbol{K}]_{N\times N}[\boldsymbol{\varphi}]_{N\times 1} = [\boldsymbol{b}]_{N\times 1}$$

- The element K_{ij}^{e} is added to $K_{n(i,e),n(j,e)}$
- The element b_{i}^{e} is added to $b_{n(i,e)}$

An Assembly Example

e \ *i*	1	2	3
1	2	4	1
2	5	4	2
3	3	5	2
4	5	6	4



- There are six nodes $\Longrightarrow$ $\boldsymbol{K} \in \Re^{6\times6}$ and $\boldsymbol{b} \in \Re^{6\times1}$
- We initialize $\boldsymbol{K}$ and $\boldsymbol{b}$ with zeros
- Evaluate $\boldsymbol{K}^{(1)}$ and $\boldsymbol{b}^{(1)}$ an add them to the proper locations to get

An Assembly Example (Cont'd)

$$\boldsymbol{K} = \begin{bmatrix} K_{33}^{(1)} & K_{31}^{(1)} & 0 & K_{32}^{(1)} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} & 0 & K_{12}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ K_{23}^{(1)} & K_{21}^{(1)} & 0 & K_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_3^{(1)} \\ b_1^{(1)} \\ 0 \\ b_2^{(1)} \\ 0 \\ 0 \end{bmatrix}$$

- Evaluate $\boldsymbol{K}^{(2)}$ and $\boldsymbol{b}^{(2)}$ an add them to the proper locations

$$\boldsymbol{K} = \begin{bmatrix} K_{33}^{(1)} & K_{31}^{(1)} & 0 & K_{32}^{(1)} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} + K_{33}^{(2)} & 0 & K_{12}^{(1)} + K_{32}^{(2)} & K_{31}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ K_{23}^{(1)} & K_{21}^{(1)} + K_{23}^{(2)} & 0 & K_{22}^{(1)} + K_{22}^{(2)} & K_{21}^{(2)} & 0 \\ 0 & K_{13}^{(2)} & 0 & K_{12}^{(2)} & K_{11}^{(2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_3^{(1)} \\ b_1^{(1)} + b_3^{(2)} \\ 0 \\ b_2^{(1)} + b_2^{(2)} \\ b_1^{(2)} \\ 0 \end{bmatrix}$$

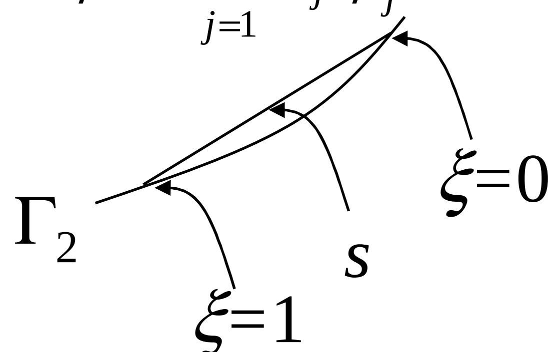
- Repeat the same steps for all elements

Incorporating a Boundary Condition of the 3rd Kind

- In this case γ and q are not zeroes
- The extra subfunctional $F_b(\varphi) = \int_{\Gamma_2} \left[\frac{\gamma}{2}\varphi^2 - q\varphi \right] d\Gamma$ is added to the functional F
- Because Γ_2 is comprised of M_s line segments, we may write

$$F_b(\varphi) = \sum_{s=1}^{M_S} F_b^s(\varphi^s)$$

- We approximate the function φ over the segment s by the linear expression $\varphi^s = \sum_{j=1}^{2} N_j^s \varphi_j^s, \; N_1^s = 1-\xi, \; N_2^s = \xi$



- ξ is the normalized distance from node 1 to node 2

Incorporating a 3rd Kind BC (Cont'd)

$$F_b^s(\varphi) = \int_s \left[\frac{\gamma}{2}\varphi^2 - q\varphi \right] d\Gamma$$

⇓ Use the expansion

$$F_b^s(\varphi) = \int_s \left(\frac{\gamma}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \varphi_i^s N_i^s N_j^s \varphi_j^s - q \sum_{i=1}^{2} N_i^s \varphi_i^s \right) d\Gamma$$

⇓ Differentiate and use d $\Gamma = l^s$ dξ

$$\frac{\partial F_b^s}{\partial \varphi_i^s} = \sum_{j=1}^{2} \varphi_j^s \int_0^1 \gamma N_i^s N_j^s l^s d\xi - \int_0^1 q N_i^s l^s d\xi, \; i = 1,2$$

in matrix form $\dfrac{\partial F_b^s}{\partial \boldsymbol{\varphi}^s} = \left[\boldsymbol{K}^s\right]_{2\times 2} \left[\boldsymbol{\varphi}^s\right]_{2\times 1} - \left[\boldsymbol{b}^s\right]_{2\times 1}$

$$K_{ij}^s = \int_0^1 \gamma N_i^s N_j^s l^s d\xi, \quad b_i^s = \int_0^1 q N_i^s l^s d\xi, \; i = 1,2 \text{ and } j = 1,2$$

- Assembly is then applied to store these coefficients

The Dirichlet Boundary Condition

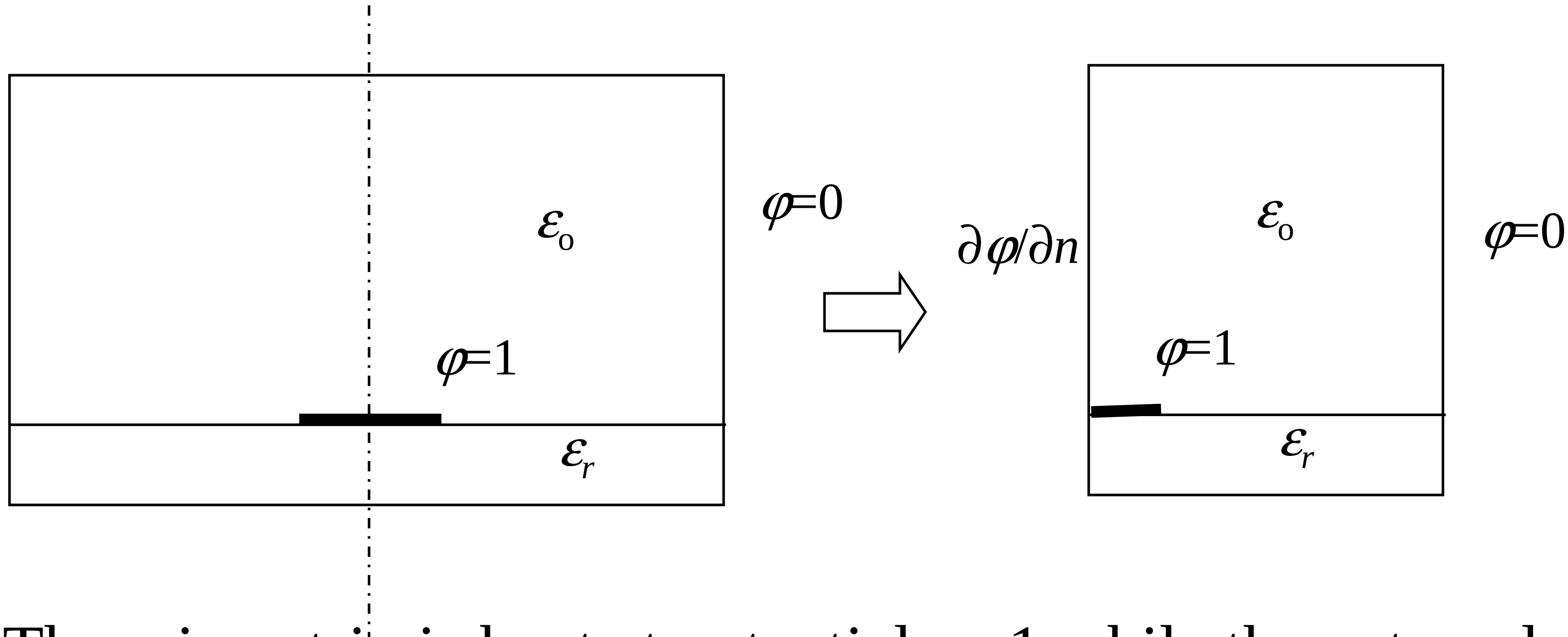
- The Dirichlet boundary conditions are imposed by eliminating the known nodes by substituting for their values

$$\begin{bmatrix} \boldsymbol{K}_{pp} & \boldsymbol{K}_{pu} \\ \boldsymbol{K}_{up} & \boldsymbol{K}_{uu} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_p \\ \boldsymbol{\varphi}_u \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_p \\ \boldsymbol{b}_u \end{bmatrix}$$ Original system

$$\Downarrow$$

$$\boldsymbol{K}_{uu} \boldsymbol{\varphi}_u = \left(\boldsymbol{b}_u - \boldsymbol{K}_{up} \boldsymbol{\varphi}_p \right)$$ Reduced system

An Example: A Shielded Microstrip Line



- The microstrip is kept at potential φ=1 while the external shielding box is kept at potential φ=0
- Symmetry may be employed to reduce the computational domain by one half
- The governing BVP is

An Example: A Shielded Microstrip Line (Cont'd)

$$-\frac{\partial}{\partial x}\left(\varepsilon_r \frac{\partial \varphi}{\partial x}\right)-\frac{\partial}{\partial y}\left(\varepsilon_r \frac{\partial \varphi}{\partial y}\right)=\frac{\rho_c}{\varepsilon_o}$$

with $\varphi = 0$ on the outer conductor, $\varphi = 1$ on the microstrip and $\partial\varphi / \partial n = 0$ on the plane of symmetry

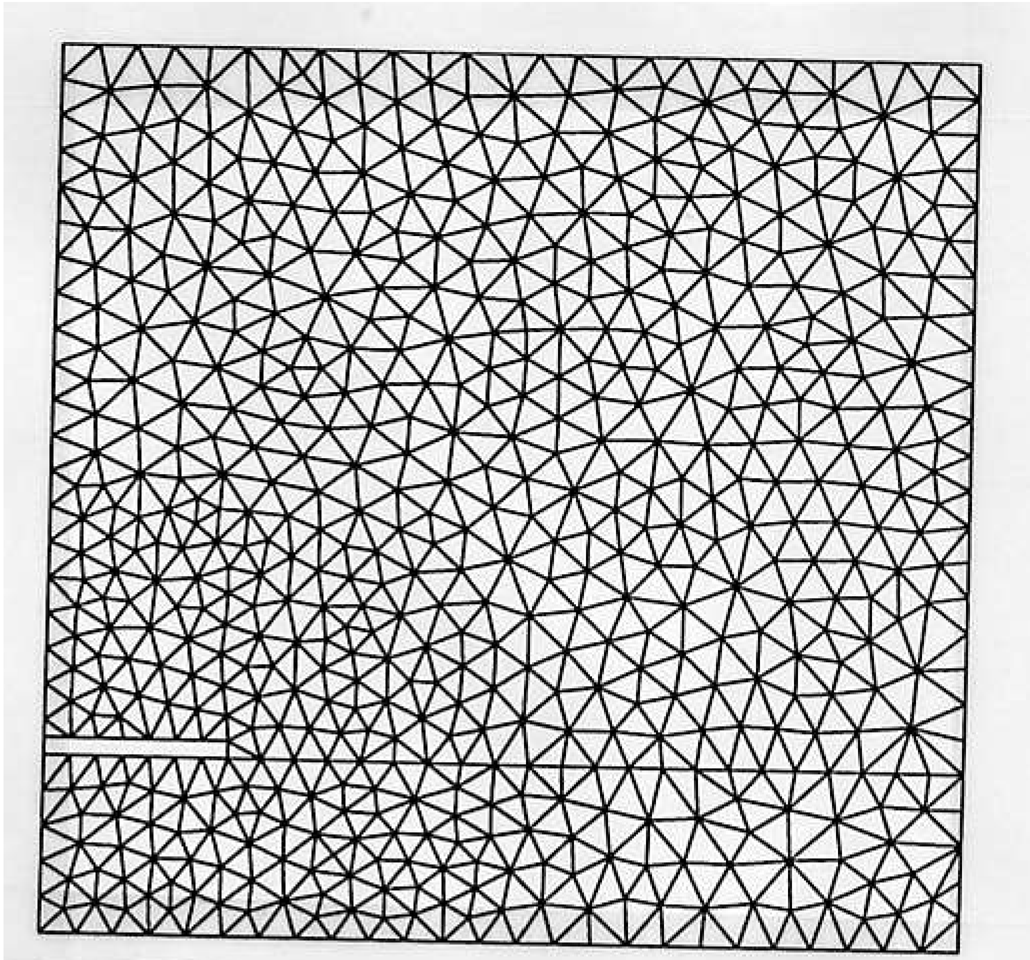
- It follows that we have $\alpha_x = \alpha_y = \varepsilon_r$, $\beta = 0$, $f = 0$
- The electric field is obtained through $\boldsymbol{E} = -\nabla\varphi$. But φ over each element is approximated by

$$\varphi^e(x,y)=\sum_{j=1}^{3} N_j^e(x,y)\,\varphi_j^e,\quad N_j^e(x,y)=\frac{1}{2A_e}(a_j^e+b_j^e x+c_j^e y),\; j=1,2,3$$

$$\Downarrow$$

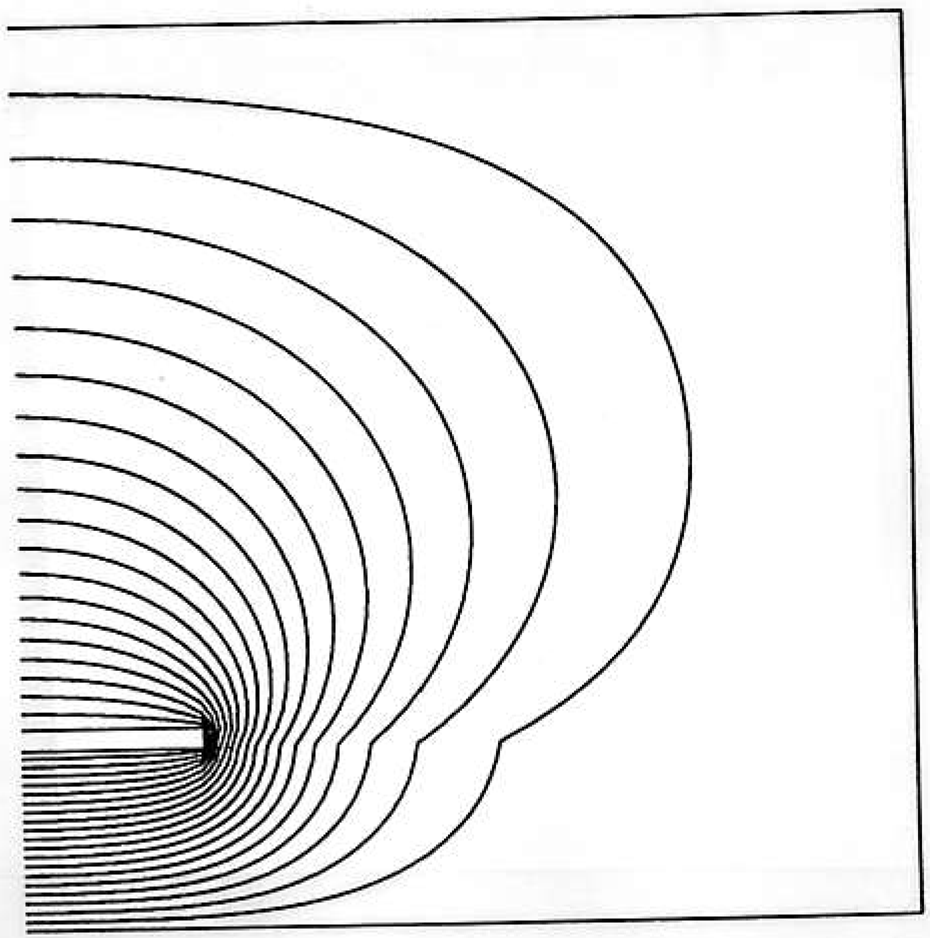
$$\boldsymbol{E}=-\frac{\partial\varphi}{\partial x}\boldsymbol{i}-\frac{\partial\varphi}{\partial y}\boldsymbol{j}=-\frac{1}{2A_e}\sum_{j=1}^{3}(b_j^e\boldsymbol{i}+c_j^e\boldsymbol{j})\,\varphi_j^e \quad \text{Over the } e\text{th element}$$

An Example: A Shielded Microstrip Line (Cont'd)



The Finite Element Method in Electromagnetics, Jianming Jin

An Example: A Shielded Microstrip Line (Cont'd)



The Finite Element Method in Electromagnetics, Jianming Jin

The equi-potential lines