EE750
Advanced Engineering Electromagnetics
Lecture 7

Auxiliary Vector Potentials

- Auxiliary vector potentials are introduced to simplify the solution of Maxwell's equations
- A two step procedure is applied: First we find ***A*** and ***F***, the magnetic and electric potentials
- Second, from ***A*** and ***F*** we obtain the electric and magnetic fields
- Starting with Maxwell's equations we have:

$$\nabla \times \boldsymbol{E} = -\boldsymbol{M} - j\omega\mu\boldsymbol{H} \qquad \nabla . \boldsymbol{E} = q_{ev} / \varepsilon$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega\varepsilon\boldsymbol{E} \qquad \nabla . \boldsymbol{H} = q_{mv} / \mu$$

Auxiliary Vector Potentials (Cont'd)

- utilizing the vector identity $\nabla\times\nabla\times\boldsymbol{V}=\nabla(\nabla.\boldsymbol{V})-\nabla^2\boldsymbol{V}$

we have $\nabla\times\nabla\times\boldsymbol{E}=-\nabla\times\boldsymbol{M}-j\omega\mu\nabla\times\boldsymbol{H}$

$$\Downarrow$$

$$\nabla(\nabla.\boldsymbol{E})-\nabla^2\boldsymbol{E}=-\nabla\times\boldsymbol{M}-j\omega\mu(\boldsymbol{J}+j\omega\varepsilon\boldsymbol{E})$$

$$\Downarrow$$

$$\nabla^2\boldsymbol{E}+\omega^2\mu\varepsilon\boldsymbol{E}=\nabla\times\boldsymbol{M}+j\omega\mu\boldsymbol{J}+\nabla(q_{ev}/\varepsilon)$$

(Vector wave equation for the electric field)

- Similarly, we obtain for the magnetic field

$$\nabla^2\boldsymbol{H}+\omega^2\mu\varepsilon\boldsymbol{H}=-\nabla\times\boldsymbol{J}+j\omega\mu\boldsymbol{\mu}+\nabla(q_{mv}/\mu)$$

Magnetic Vector Potential (*A*)

- ***A*** represents the fields due to electric sources only
- For regions where ***M***=**0** and q_{mv}=0, we have $\nabla.\boldsymbol{B}=0$
- Using the vector identity ($\nabla.(\nabla\times\boldsymbol{V})=0$) , ***B*** can be represented as $\boldsymbol{B}=\nabla\times\boldsymbol{A}$ ⟹ $\boldsymbol{H}=(1/\mu)\nabla\times\boldsymbol{A}$
- But for the case (***M***=**0** and q_{mv}=0), we have

  $$\nabla\times\boldsymbol{E}=-j\omega\mu\boldsymbol{H}=-j\omega(\nabla\times\boldsymbol{A}) \Longrightarrow \nabla\times(\boldsymbol{E}+j\omega\boldsymbol{A})=\boldsymbol{0}$$

- Using the vector identity ($\nabla\times(\nabla\varphi)=0$), we can write

  $$\boldsymbol{E}+j\omega\boldsymbol{A}=-\nabla\varphi_e \Longrightarrow \boldsymbol{E}=-j\omega\boldsymbol{A}-\nabla\varphi_e$$

- It follows that by knowing ***A*** and φ_e , we can evaluate ***H*** and ***E*** ⟹ How do we get ***A*** and φ_e ?

Magnetic Vector Potential (Cont'd)

- We derive a second-order differential equation for $\boldsymbol{A}$

$$\nabla \times \boldsymbol{H} = (1/\mu)(\nabla \times \nabla \times \boldsymbol{A})$$

- Using $\nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega\varepsilon\boldsymbol{E}$ and the vector identity

$$\nabla \times \nabla \times \boldsymbol{V} = \nabla(\nabla.\boldsymbol{V}) - \nabla^2 \boldsymbol{V}$$, we get

$$\boldsymbol{J} + j\omega\varepsilon\boldsymbol{E} = (1/\mu)(\nabla(\nabla.\boldsymbol{A}) - \nabla^2 \boldsymbol{A})$$

$$\Downarrow$$

$$\boldsymbol{J} + j\omega\varepsilon(-\nabla\varphi_e - j\omega\boldsymbol{A}) = (1/\mu)(\nabla(\nabla.\boldsymbol{A}) - \nabla^2 \boldsymbol{A})$$

$$\Downarrow$$

$$\nabla^2 \boldsymbol{A} + \beta^2 \boldsymbol{A} = -\mu\boldsymbol{J} + \nabla(\nabla.\boldsymbol{A} + j\omega\mu\varepsilon\varphi_e), \beta = \omega\sqrt{\mu\varepsilon}$$

Magnetic Vector Potential (Cont'd)

- The magnetic vector potential is defined by both its curl and divergence. To simplify the analysis we utilize the Lorentz gauge $\nabla.\boldsymbol{A} = -j\omega\mu\varepsilon\varphi_e \Longrightarrow \varphi_e = -(\nabla.\boldsymbol{A})/(j\omega\mu\varepsilon)$
- It follows that the magnetic potential satisfies the magnetic potential vector wave equation

$$\nabla^2\boldsymbol{A} + \beta^2\boldsymbol{A} = -\mu\boldsymbol{J}$$

Magnetic Vector Potential (Cont'd)

- Summary of steps for the case (***M***=**0** and q_{mv}=0)

1. Solve $\nabla^2 \boldsymbol{A} + \beta^2 \boldsymbol{A} = -\mu \boldsymbol{J}$, for ***A***
2. Obtain $\boldsymbol{H} = (1/\mu)\nabla \times \boldsymbol{A}$
3. Obtain $\boldsymbol{E} = -j\omega \mathbf{A} - \nabla \varphi_e = -j\omega \mathbf{A} - (j/\omega\mu\varepsilon)(\nabla(\nabla.\boldsymbol{A}))$

- In many cases, this two step procedure is much easier than the one step procedure

- The magnetic potential vector wave equation can be solved for any coordinate system

The Electric Vector Potential *F*

- ***F*** represents the fields due to magnetic sources only
- For regions where ***J***=**0** and q_{ev}=0, we have

$$\nabla.\boldsymbol{D} = 0$$

- Using the vector identity ($\nabla.(\nabla\times\boldsymbol{V}) = 0$) , ***D*** can be represented as $\boldsymbol{D} = -\nabla\times\boldsymbol{F} \Longrightarrow \boldsymbol{E} = (-1/\varepsilon)\nabla\times\boldsymbol{F}$
- But for the case (***J***=**0** and q_{ev}=0), we have

$$\nabla\times\boldsymbol{H} = j\omega\varepsilon\boldsymbol{E} = -j\omega(\nabla\times\boldsymbol{F}) \Longrightarrow \nabla\times(\boldsymbol{H} + j\omega\boldsymbol{F}) = \boldsymbol{0}$$

- Using the vector identity ($\nabla\times(\nabla\varphi) = 0$), we can write

$$\boldsymbol{H} + j\omega\boldsymbol{F} = -\nabla\varphi_m \Longrightarrow \boldsymbol{H} = -j\omega\boldsymbol{F} - \nabla\varphi_m$$

- It follows that by knowing ***F*** and φ_m, we can evaluate ***E*** and ***H*** $\Longrightarrow$ How do we get ***A*** and φ_e?

Electric Vector Potential (Cont'd)

- We derive a second-order differential equation for ***F***

$$\nabla\times\boldsymbol{E} = (-1/\varepsilon)(\nabla\times\nabla\times\boldsymbol{F})$$

- Using $\nabla\times\boldsymbol{E} = -\boldsymbol{M} - j\omega\mu\boldsymbol{H}$ and the vector identity

$\nabla\times\nabla\times\boldsymbol{V} = \nabla(\nabla.\boldsymbol{V}) - \nabla^2\boldsymbol{V}$, we get

$$-\boldsymbol{M} - j\omega\mu\boldsymbol{H} = (-1/\varepsilon)(\nabla(\nabla.\boldsymbol{F}) - \nabla^2\boldsymbol{F})$$

$$\Downarrow$$

$$-\boldsymbol{M} - j\omega\mu(-\nabla\varphi_m - j\omega\boldsymbol{F}) = (-1/\varepsilon)(\nabla(\nabla.\boldsymbol{F}) - \nabla^2\boldsymbol{F})$$

$$\Downarrow$$

$$\nabla^2\boldsymbol{F} + \beta^2\boldsymbol{F} = -\varepsilon\boldsymbol{M} + \nabla(\nabla.\boldsymbol{F} + j\omega\mu\varepsilon\varphi_m)\ ,\quad \beta = \omega\sqrt{\mu\varepsilon}$$

Electric Vector Potential (Cont'd)

- The electric vector potential is defined by both its curl and divergence. To simplify the analysis we utilize the Lorentz gauge $\nabla.\boldsymbol{F} = -j\omega\mu\varepsilon\varphi_m \Longrightarrow \varphi_m = -(\nabla.\boldsymbol{F})/(j\omega\mu\varepsilon)$
- It follows that the electric potential satisfies the vector wave equation

$$\nabla^2 \boldsymbol{F} + \beta^2 \boldsymbol{F} = -\varepsilon \boldsymbol{M}$$

Electric Vector Potential (Cont'd)

- Summary of Steps for the case ($\boldsymbol{J}$=**0** and q_{ev}=0)

1. Solve $\nabla^2 \boldsymbol{F} + \beta^2 \boldsymbol{F} = -\varepsilon \boldsymbol{M}$, for $\boldsymbol{A}$
2. Obtain $\boldsymbol{E} = (-1/\varepsilon)\nabla \times \boldsymbol{F}$
3. Obtain $\boldsymbol{H} = -j\omega \boldsymbol{F} - \nabla \varphi_m = -j\omega \boldsymbol{F} - (j/\omega\mu\varepsilon)(\nabla(\nabla.\boldsymbol{F}))$

- Again, in many cases this two step procedure is much easier than the one step procedure
- The electric potential vector wave equation can be solved for any coordinate system

Magnetic and Electric Vector Potentials

- When both electric and magnetic sources exist, superposition should be applied as follows:

1. Solve $\nabla^2 \boldsymbol{A} + \beta^2 \boldsymbol{A} = -\mu \boldsymbol{J}$, for $\boldsymbol{A}$
2. Solve $\nabla^2 \boldsymbol{F} + \beta^2 \boldsymbol{F} = -\varepsilon \boldsymbol{M}$, for $\boldsymbol{F}$
3. Obtain fields due to electric sources

   $\boldsymbol{H}_A = (1/\mu)\nabla \times \boldsymbol{A}$ and $\boldsymbol{E}_A = -j\omega \boldsymbol{A} - (j/\omega\mu\varepsilon)(\nabla(\nabla.\boldsymbol{A}))$
4. Obtain fields due to magnetic sources

   $\boldsymbol{E}_F = (-1/\varepsilon)\nabla \times \boldsymbol{F}$ and $\boldsymbol{H}_F = -j\omega \boldsymbol{F} - (j/\omega\mu\varepsilon)(\nabla(\nabla.\boldsymbol{F}))$
5. Total fields are obtained through superposition

   $\boldsymbol{E}=\boldsymbol{E}_A+\boldsymbol{E}_F$ and $\boldsymbol{H}=\boldsymbol{H}_A+\boldsymbol{H}_F$

Construction of Solutions (Source-Free Case)

- We limit our discussion to the rectangular coordinate system

$$\nabla^2 \boldsymbol{A} + \beta^2 \boldsymbol{A} = \boldsymbol{0} \Longrightarrow \nabla^2 A_x + \beta^2 A_x = 0 \ , \ \nabla^2 A_y + \beta^2 A_y = 0 \ ,$$

$$\nabla^2 A_z + \beta^2 A_z = 0 \ , \ \boldsymbol{A} = A_x \boldsymbol{a}_x + A_y \boldsymbol{a}_y + A_z \boldsymbol{a}_z$$

- Similarly, we have for the electric vector potential

$$\nabla^2 \boldsymbol{F} + \beta^2 \boldsymbol{F} = \boldsymbol{0} \Longrightarrow \nabla^2 F_x + \beta^2 F_x = 0 \ , \nabla^2 F_y + \beta^2 F_y = 0 \ ,$$

$$\nabla^2 F_z + \beta^2 F_z = 0 \ , \ \boldsymbol{F} = F_x \boldsymbol{a}_x + F_y \boldsymbol{a}_y + F_z \boldsymbol{a}_z$$

- Starting with the expressions

$$\boldsymbol{E} = -j\omega\boldsymbol{A} - (j/\omega\mu\varepsilon)(\nabla(\nabla.\boldsymbol{A})) - (1/\varepsilon)\nabla\times\boldsymbol{F} \ ,$$

$$\boldsymbol{H} = -j\omega\boldsymbol{F} - (j/\omega\mu\varepsilon)(\nabla(\nabla.\boldsymbol{F})) + (1/\mu)\nabla\times\boldsymbol{A}$$

Construction of Solutions (Cont'd)

We obtain the expressions

$$\boldsymbol{E} = \left[-j\omega A_x - \frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} \right) - \frac{1}{\varepsilon}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \right] \boldsymbol{a}_x$$

$$+ \left[-j\omega A_y - \frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x \partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y \partial z} \right) - \frac{1}{\varepsilon}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \right] \boldsymbol{a}_y$$

$$+ \left[-j\omega A_z - \frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial z \partial y} + \frac{\partial^2 A_z}{\partial z^2} \right) - \frac{1}{\varepsilon}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \right] \boldsymbol{a}_z$$

$$\boldsymbol{H} = \left[-j\omega F_x - \frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_y}{\partial x \partial y} + \frac{\partial^2 F_z}{\partial x \partial z} \right) + \frac{1}{\mu}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \right] \boldsymbol{a}_x$$

$$+ \left[-j\omega F_y - \frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x \partial y} + \frac{\partial^2 F_y}{\partial y^2} + \frac{\partial^2 F_z}{\partial y \partial z} \right) + \frac{1}{\mu}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right] \boldsymbol{a}_y$$

$$+ \left[-j\omega F_z - \frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 F_x}{\partial x \partial z} + \frac{\partial^2 F_y}{\partial z \partial y} + \frac{\partial^2 F_z}{\partial z^2} \right) + \frac{1}{\mu}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] \boldsymbol{a}_z$$

The TEM$_z$ Case

- In this case we must have $E_z=H_z=0$. Considering the equations for E_z and H_z, there are three possible ways to get a TEM$_z$ wave:

1. Put $A_x=A_y=F_x=F_y=0$, $A_z\neq 0$, $F_z\neq 0$, $\partial/\partial x \neq 0$, $\partial/\partial y \neq 0$, to get

$$E_z=-j\omega A_z-\frac{j}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial z^2}=0 \Rightarrow A_z(x,y)=A_z^{+}\exp(-j\beta z)+A_z^{-}\exp(j\beta z)$$

$$H_z=-j\omega F_z-\frac{j}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial z^2}=0 \Rightarrow F_z(x,y)=F_z^{+}\exp(-j\beta z)+F_z^{-}\exp(j\beta z)$$

$$E_x=E_x^{+}+E_x^{-}\ ,\ H_y=H_y^{+}+H_y^{-}\ ,\ E_x^{+}/H_y^{+}=-E_x^{-}/H_y^{-}=\sqrt{\mu/\varepsilon}$$

$$E_y=E_y^{+}+E_y^{-}\ ,\ H_x=H_x^{+}+H_x^{-}\quad -E_y^{+}/H_x^{+}=E_y^{-}/H_x^{-}=\sqrt{\mu/\varepsilon}$$

Prove it!

The TEM_z Case (Cont'd)

2. Put A_x=A_y= A_z =F_x=F_y=0, F_z≠0, ∂/∂x ≠0, ∂/∂y ≠0, to get

$$E_z = 0$$

$$H_z = -j\omega F_z - \frac{j}{\omega\mu\varepsilon}\frac{\partial^2 F_z}{\partial z^2} = 0 \Rightarrow F_z(x,y) = F_z^+ \exp(-j\beta z) + F_z^- \exp(j\beta z)$$

$$E_x = E_x^+ + E_x^-, \quad H_y = H_y^+ + H_y^-, \quad E_x^+ / H_y^+ = -E_x^- / H_y^- = \sqrt{\mu / \varepsilon}$$

$$E_y = E_y^+ + E_y^-, \quad H_x = H_x^+ + H_x^-, \quad -E_y^+ / H_x^+ = E_y^- / H_x^- = \sqrt{\mu / \varepsilon}$$

3. Put A_x=A_y=F_x=F_y= F_z =0, A_z≠0, ∂/∂x ≠0, ∂/∂y ≠0, to get

$$E_z = -j\omega A_z - \frac{j}{\omega\mu\varepsilon}\frac{\partial^2 A_z}{\partial z^2} = 0 \Rightarrow A_z(x,y) = A_z^+ \exp(-j\beta z) + A_z^- \exp(j\beta z)$$

$$H_z = 0$$

$$E_x = E_x^+ + E_x^-, \quad H_y = H_y^+ + H_y^-, \quad E_x^+ / H_y^+ = -E_x^- / H_y^- = \sqrt{\mu / \varepsilon}$$

$$E_y = E_y^+ + E_y^-, \quad H_x = H_x^+ + H_x^-, \quad -E_y^+ / H_x^+ = E_y^- / H_x^- = \sqrt{\mu / \varepsilon}$$

The $\mathrm{TM_z}$ Case

- To obtain $E_z \neq 0$, $H_z = 0$, we must have

$\boldsymbol{A} = A_z(x,y,z)\,\boldsymbol{a}_z$, $\boldsymbol{F} = \mathbf{0}$

- Only one scalar wave equation to be solved

$$\nabla^2 A_z + \beta^2 A_z = 0 \quad \Longrightarrow \quad \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z = 0$$

$$\boldsymbol{E} = -\frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_z}{\partial x \partial z}\right)\boldsymbol{a}_x + \frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_z}{\partial y \partial z}\right)\boldsymbol{a}_y + \left[-j\omega A_z - \frac{j}{\omega\mu\varepsilon}\left(\frac{\partial^2 A_z}{\partial z^2}\right)\right]\boldsymbol{a}_z$$

$$\boldsymbol{H} = \frac{1}{\mu}\frac{\partial A_z}{\partial y}\boldsymbol{a}_x + -\frac{1}{\mu}\left(\frac{\partial A_z}{\partial x}\right)\boldsymbol{a}_y$$

- Once the A_z component has been determined according to the boundary conditions, all field components may be found

A Waveguide Example

- Starting with $$\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2}+\beta^2 A_z=0$$

We apply separation of variables $A_z(x,y,z)=f(x)g(y)h(z)$

$$\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2}+\frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2}+\frac{1}{h(z)}\frac{\partial^2 h(z)}{\partial z^2}+\beta^2=0$$

each one of these terms must be equal to a constant

$$\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2}=-\beta_x^2 \quad , \quad \frac{1}{g(y)}\frac{\partial^2 g(y)}{\partial y^2}=-\beta_y^2 \; ,$$

$$\frac{1}{h(z)}\frac{\partial^2 h(z)}{\partial z^2}=-\beta_z^2 \quad , \quad \beta^2=\beta_x^2+\beta_y^2+\beta_z^2$$

For the transversal *x* and *y* directions we select a standing wave. A travelling wave is selected for the *z* direction

A Waveguide Example (Cont'd)

- It follows that we have

$$f(x) = C_1 Cos(\beta_x x) + D_1 Sin(\beta_x x)$$

$$g(y) = C_2 Cos(\beta_y y) + D_2 Sin(\beta_y y)$$

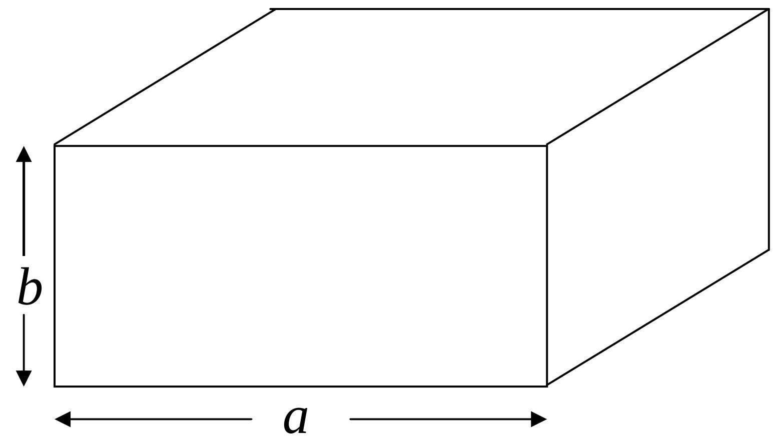
$$h(z) = A_3 \exp(-j\beta_z z)$$ (we consider only incident wave)

$$A^+(x, y, z) = (C_1 Cos(\beta_x x) + D_1 Sin(\beta_x x))(C_2 Cos(\beta_y y)$$
$$+ D_2 Sin(\beta_y y))\, A_3 \exp(-j\beta_z z)$$

- All constants are determined through the boundary conditions

A Waveguide Example (Cont'd)

$$E_z = \frac{-j}{\omega\mu\varepsilon}\left(\frac{\partial^2}{\partial z^2} + \beta^2\right) A_z$$



$$\Downarrow$$

$$E_z^+ = \frac{-j}{\omega\mu\varepsilon}(C_1 Cos(\beta_x x) + D_1 Sin(\beta_x x))(C_2 Cos(\beta_y y)$$

$$+ D_2 Sin(\beta_y y)) A_3 \exp(-j\beta_z z)$$

$$E_z^+(x=0) = 0 \Longrightarrow C_1 = 0$$

$$E_z^+(x=a) = 0 \Longrightarrow Sin(\beta_x a) = 0 \Longrightarrow \beta_x a = m\pi$$

$$\beta_x = m\pi/a$$

A Waveguide Example (Cont'd)

- Similarly, using $E_z^+(y=0)=0 \Rightarrow C_2=0$

  $E_z^+(y=b)=0 \Rightarrow Sin(\beta_y b)=0 \Rightarrow \beta_y b=n\pi$

  $\beta_y=n\pi/b$

- It follows that we have

  $$A_z^+ = A_{mn}\, sin(m\pi\pi / a) sin(n\pi\pi / b)\exp(-j\beta z)$$

  m=1, 2, … and n=1, 2, …

- All field components can be obtained in terms of the magnetic vector potential