Tristance Anticodes in Certain Graphs

Tuvi Etzion
Technion — Israel Institute of Technology
Department of Computer Science
Technion City, Haifa 32000, Israel
etzion@cs.technion.ac.il

Moshe Schwartz
Technion — Israel Institute of Technology
Department of Computer Science
Technion City, Haifa 32000, Israel
moosh@cs.technion.ac.il

Alexander Vardy
University of California San Diego
Department of Computer Science and Engineering
Department of Electrical and Computer Engineering
9500 Gilman Drive, La Jolla, CA 92093, U.S.A.
vardy@kilimanjaro.ucsd.edu

Abstract

For $z_1, z_2, z_3 \in \mathbb{Z}^n$, the *tristance* $d_3(z_1, z_2, z_3)$ is a generalization of the L_1-distance on $\mathbb{Z}^n$ to a quantity that reflects the relative dispersion of three points rather than two. A *tristance anticode* $\mathcal{A}_d$ of diameter d is a subset of $\mathbb{Z}^n$ with the property that $d_3(z_1, z_2, z_3) \leqslant d$ for all $z_1, z_2, z_3 \in \mathcal{A}_d$. An anticode is *optimal* if it has the largest possible size for its diameter d. We determine the size and completely classify the optimal tristance anticodes $\mathcal{A}_d$ for all $d \geqslant 1$ in $\mathbb{Z}^2$. We briefly discuss a generalization of these results to two related distance models based on certain graphs.

1. Introduction

Given two points $z = (z_1, z_2, \ldots, z_n)$ and $z' = (z'_1, z'_1, \ldots, z'_n)$ in $\mathbb{Z}^n$, the L_1-distance between z and z' is defined as $d_{L_1}(z, z') = |z_1 - z'_1| + |z_2 - z'_2| + \cdots + |z_n - z'_n|$. Alternatively, let $\mathcal{G}_n^{\blacklozenge} = (V, E)$ denote the *grid graph* of $\mathbb{Z}^n$ whose vertex set is $V = \mathbb{Z}^n$ and whose edges are defined as follows: $\{z, z'\} \in E$ if and only if $d_{L_1}(z, z') = 1$. Then the L_1-distance between z and z' in $\mathbb{Z}^n$ is the number of edges in the shortest path joining z and z' in $\mathcal{G}_n^{\blacklozenge}$. The latter point of view leads to a natural generalization of the L_1-distance on $\mathbb{Z}^n$ to a quantity that reflects the relative dispersion of three points rather than two.

Definition 1. *Let $z_1, z_2, z_3 \in \mathbb{Z}^n$. Then the* tristance *$d_3(z_1, z_2, z_3)$ is defined as the number of edges in a minimal spanning tree for z_1, z_2, z_3 in the grid graph $\mathcal{G}_n^{\blacklozenge}$ of $\mathbb{Z}^n$.*

The notion of tristance defined above can be further generalized. In particular, the *quadristance* $d_4(z_1, z_2, z_3, z_4)$, the *quintistance* $d_5(z_1, z_2, z_3, z_4, z_5)$, and more generally the *r-dispersion* $d_r(z_1, z_2, \ldots, z_r)$ may be defined as the number of edges in a minimal spanning tree for $z_1, z_2, \ldots, z_r$ in the grid graph $\mathcal{G}_n^{\blacklozenge}$. For more detail on this, we refer the reader to [10]. In this paper, however, we consider only the tristance $d_3(z_1, z_2, z_3)$ in the grid graph $\mathcal{G}_2^{\blacklozenge}$. Our notions of $\mathcal{G}_2^{\blacklozenge}$, distance, and tristance are illustrated in Figure 1.

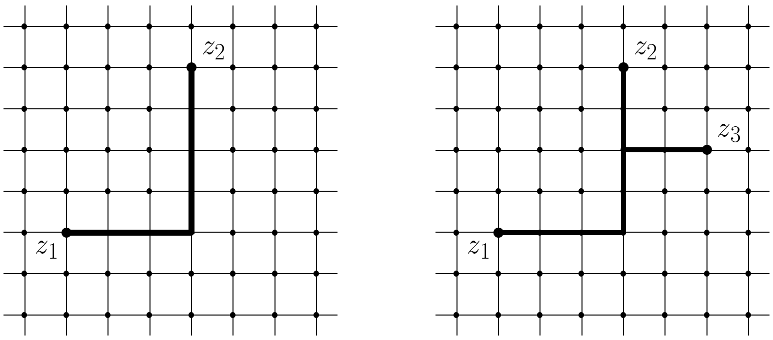

Figure 1. *Distance $d_{L_1}(z_1, z_2)$ and tristance $d_3(z_1, z_2, z_3)$ in the grid graph $\mathcal{G}_2^{\blacklozenge}$*

Given a set $\mathcal{S}$ and a definition of distance between points of $\mathcal{S}$, a *code* $\mathcal{C} \subseteq \mathcal{S}$ of minimum distance d is characterized by the property that the distance between any two distinct points of $\mathcal{C}$ is *at least* d. Similarly, an *anticode* $\mathcal{A} \subseteq \mathcal{S}$ of diameter d is characterized by the property that the distance between any two distinct points of $\mathcal{A}$ is *at most* d. One is usually interested in codes and anticodes of the largest possible size for a given minimum distance or diameter — such codes/anticodes are said to be *optimal*. An encyclopedic survey of optimal codes in the Hamming graph may be found in [12, 14]; for codes in other graphs, see [4, 5, 8, 11, 15]. Optimal anticodes in the Hamming metric and related distance-regular graphs have been studied in [1, 2, 3, 8, 13, 15] and other papers.

The concepts of a code and an anticode can be generalized using the notion of tristance in Definition 1. Thus a tristance code $\mathcal{C} \subseteq \mathbb{Z}^n$ of minimum tristance d such that $d_3(z_1, z_2, z_3) \geqslant d$ for all $z_1, z_2, z_3 \in \mathcal{C}$. Numerous results on optimal tristance codes in $\mathbb{Z}^2$ can be found in [10]. Optimal tristance *anticodes* are the subject matter of this paper.

Definition 2. *A set $\mathcal{A}_d \subset \mathbb{Z}^n$ is a tristance anticode of diameter d if $d_3(z_1, z_2, z_3) \leqslant d$ for all $z_1, z_2, z_3 \in \mathcal{A}_d$; it is optimal if it has the largest possible cardinality for its diameter d.*

One can also define tristance anticodes *centered* about a given point or a pair of points. Given $z_0 \in \mathbb{Z}^n$, a set $\mathcal{A}_d(z_0) \subset \mathbb{Z}^n$ is said to be a tristance anticode of diameter d *centered about* z_0 if $d_3(z_0, z_1, z_2) \leqslant d$ for all $z_1, z_2 \in \mathcal{A}_d(z_0)$. Given distinct $z_1, z_2 \in \mathbb{Z}^n$, a set $\mathcal{A}_d(z_1, z_2) \subset \mathbb{Z}^n$ is said to be a tristance anticode of diameter d *centered about* z_1 *and* z_2 if $d_3(z_1, z_2, z) \leqslant d$ for all $z \in \mathcal{A}_d(z_1, z_2)$. Once again, we are interested in *optimal* centered tristance anticodes that have the largest possible cardinality for their diameter and center(s). We note that the corresponding problem for conventional (distance) anticodes is trivial. The unique optimal anticode of diameter d centered about a given point $z_0 \in \mathbb{Z}^n$ is $\mathscr{S}_d(z_0) = \{z \in \mathbb{Z}^n : d_{L_1}(z, z_0) \leqslant d\}$, which is just a *sphere of radius* d *about* z_0.

In the next section we classify optimal tristance anticodes in $\mathbb{Z}^2$. Section 3 contains a brief discussion of further results concerning generalizations of the L_1 connectivity model.

2. Optimal tristance anticodes in the grid graph

In this section we completely classify the optimal tristance anticodes in the grid graph. We will first need some auxiliary results. Trivially, the L_1-distance between (x_1, y_1) and (x_2, y_2) in $\mathbb{Z}^2$ can be written as $(\max_{1\leqslant i\leqslant 2} x_i - \min_{1\leqslant i\leqslant 2} x_i) + (\max_{1\leqslant i\leqslant 2} y_i - \min_{1\leqslant i\leqslant 2} y_i)$. The following theorem of [10] shows that a similar expression holds for tristance in $\mathbb{Z}^2$.

Theorem 1. *Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, $z_3 = (x_3, y_3)$ be distinct points in $\mathbb{Z}^2$. Then*

$$d_3(z_1, z_2, z_3) \;=\; \left(\max_{1\leqslant i\leqslant 3} x_i - \min_{1\leqslant i\leqslant 3} x_i\right) \;+\; \left(\max_{1\leqslant i\leqslant 3} y_i - \min_{1\leqslant i\leqslant 3} y_i\right) \tag{1}$$

Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2), \ldots, z_n = (x_n, y_n)$ be n distinct points in $\mathbb{Z}^2$. The *bounding rectangle* of $z_1, z_2, \ldots, z_n$ is defined as the smallest rectangle $\mathcal{R}(z_1, \ldots, z_n)$ with edges parallel to the axes that contains all the n points. Explicitly, let $x_{\max} = \max\{x_1, x_2, \ldots, x_n\}$, $x_{\min} = \min\{x_1, x_2, \ldots, x_n\}$, $y_{\max} = \max\{y_1, y_2, \ldots, y_n\}$, and $y_{\min} = \min\{y_1, y_2, \ldots, y_n\}$. Then, the bounding rectangle is

$$\mathcal{R}(z_1, \ldots, z_n) \;\stackrel{\text{def}}{=}\; \bigl\{(x, y) \in \mathbb{Z}^2 \;:\; x_{\min} \leqslant x \leqslant x_{\max} \;\text{ and }\; y_{\min} \leqslant y \leqslant y_{\max}\bigr\}$$

By Theorem 1, the tristance $d_3(z_1, z_2, z_3)$ is precisley one-half of the perimeter of the bounding rectangle $\mathcal{R}(z_1, z_2, z_3)$. Furthermore, the tristance of *any* three points in this rectangle is at most $d_3(z_1, z_2, z_3)$. The latter observation leads to the following result.

Lemma 2. *Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ be arbitrary distinct points in $\mathbb{Z}^2$ and assume, without loss of generality, that $x_1 \leqslant x_2$ and $y_1 \geqslant y_2$. Let $d \geqslant d_{L_1}$, and let $\mathcal{A}_d(z_1, z_2)$ be the optimal tristance anticode of diameter d centered about z_1 and z_2. Let $c = d - d_{L_1}$. Then $\mathcal{A}_d(z_1, z_2)$ consists of all $z = (x, y)$ in $\mathbb{Z}^2$ such that*

$$x_1 - c \;\leqslant\; x \;\leqslant\; x_2 + c\,, \qquad x_1 + y_2 - c \;\leqslant\; x + y \;\leqslant\; x_2 + y_1 + c \tag{2}$$

$$y_2 - c \;\leqslant\; y \;\leqslant\; y_1 + c\,, \qquad x_1 - y_1 - c \;\leqslant\; x - y \;\leqslant\; x_2 - y_2 + c \tag{3}$$

Proof. The given points z_1, z_2 completely determine all the other points in $\mathcal{A}_d(z_1, z_2)$. Specifically $z \in \mathcal{A}_d(z_1, z_2)$ if and only if the L_1-distance from z to the bounding rectangle $\mathcal{R}(z_1, z_2) = \{(x, y) \in \mathbb{Z}^2 :\; x_1 \leqslant x \leqslant x_2,\; y_2 \leqslant y \leqslant y_1\}$ is at most $c = d - d_{L_1}(z_1, z_2)$. ∎

The centered anticode $\mathcal{A}_d(z_1, z_2)$ in (2)-(3) is an example of the set we call an *octagon*. This set is illustrated in Figure 2 and defined below.

Definition 3. *Let $\alpha_1, \alpha_2, \ldots, \alpha_8 \in \mathbb{R}$ be some arbitrary given constants. Then an octagon $\mathscr{O}(\alpha_1, \alpha_2, \ldots, \alpha_8)$ is a subset of $\mathbb{Z}^2$ defined by the inequalities*

$$\alpha_1 \leqslant x \leqslant \alpha_5\,, \qquad \alpha_3 \leqslant x + y \leqslant \alpha_7 \tag{4}$$

$$\alpha_2 \leqslant y \leqslant \alpha_6\,, \qquad \alpha_4 \leqslant x - y \leqslant \alpha_8 \tag{5}$$

Observe that an octagon $\mathscr{O}(\alpha_1, \alpha_2, \ldots, \alpha_8)$ may have fewer than eight sides (for example, if $\alpha_3 \leqslant \alpha_1 + \alpha_2$), or may be empty altogether (say, if $\alpha_5 < \alpha_1$). Octagons will play an important role in this section. In addition to the fact that $\mathcal{A}_d(z_1, z_2)$ is an octagon by Lemma 2, the following lemma establishes a useful property of octagons.

Lemma 3. *The intersection of any two octagons is also an octagon.*

Proof. By definition $\mathscr{O}(\alpha_1, \alpha_2, \ldots, \alpha_8) \cap \mathscr{O}(\beta_1, \beta_2, \ldots, \beta_8) = \mathscr{O}(\gamma_1, \gamma_2, \ldots, \gamma_8)$, where $\gamma_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2, 3, 4$ and $\gamma_i = \min\{\alpha_i, \beta_i\}$ for $i = 5, 6, 7, 8$. ∎

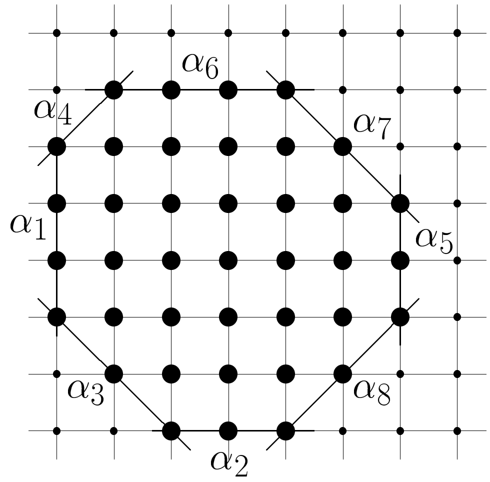

Figure 2. *A generic octagon* $\mathscr{O}(\alpha_1, \alpha_2, \ldots, \alpha_8)$

We are now in a position to begin the classification of (unrestricted) optimal tristance anticodes in $\mathbb{Z}^2$. The next lemma establishes a certain closure property of such anticodes.

Lemma 4. *Let $\mathcal{A}_d$ be an optimal tristance anticode of diameter d in $\mathbb{Z}^2$. Then $\mathcal{A}_d$ is closed under intersection with anticodes centered about pairs of its own points, namely*

$$\mathcal{A}_d \;=\; \bigcap_{z_1, z_2 \in \mathcal{A}_d} \mathcal{A}_d(z_1, z_2)$$

Proof. If $z \in \mathcal{A}_d$, then by definition of an anticode and a centered anticode, we have $d_3(z, z_1, z_2) \leqslant d$ for all $z_1, z_2 \in \mathcal{A}_d$. Thus $z \in \mathcal{A}_d(z_1, z_2)$ for all $z_1, z_2 \in \mathcal{A}_d$, and therefore $z \in \bigcap_{z_1, z_2 \in A} \mathcal{A}_d(z_1, z_2)$. Hence, for *any* tristance anticode $\mathcal{A}$ of diameter d, we have

$$\mathcal{A} \;\subseteq\; \bigcap_{z_1, z_2 \in \mathcal{A}} \mathcal{A}_d(z_1, z_2)$$

Now, if $z \notin \mathcal{A}_d$, then there exist $z_1, z_2 \in \mathcal{A}_d$ such that $d_3(z, z_1, z_2) > d$, because $\mathcal{A}_d$ is optimal (otherwise, we could adjoin z to $\mathcal{A}_d$ to obtain a larger anticode). Therefore, for these z_1, z_2, we have $z \notin \mathcal{A}_d(z_1, z_2)$. Hence $z \notin \bigcap_{z_1, z_2 \in \mathcal{A}_d} \mathcal{A}_d(z_1, z_2)$. ∎

A combination of the foregoing three lemmas makes it possible to determine the *shape* of optimal tristance anticodes in the grid graph of $\mathbb{Z}^2$.

Lemma 5. *Let $\mathcal{A}_d$ be an optimal tristance anticode in the grid graph of $\mathbb{Z}^2$. Then $\mathcal{A}_d$ is an octagon $\mathscr{O}(\alpha_1, \alpha_2, \ldots, \alpha_8)$, for some $\alpha_1, \ldots, \alpha_8 \in \mathbb{Z}$.*

Proof. By Lemma 4, $\mathcal{A}_d = \bigcap_{z_1, z_2 \in \mathcal{A}_d} \mathcal{A}_d(z_1, z_2)$. By Lemma 2, each of the sets $\mathcal{A}_d(z_1, z_2)$ is an octagon. By Lemma 3, an intersection of octagons is again an octagon. ∎

Using translations in $\mathbb{Z}^2$, we may always assume without loss of generality that $\alpha_1 = 0$ in (4) and $\alpha_2 = 0$ in (5). Thus we have $\mathcal{A}_d = \mathscr{O}(0, 0, \alpha_3, \alpha_4, \ldots, \alpha_8)$, and it remains to determine the six integer parameters $\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ as a function of the diameter d.

To do so, we first need to rewrite the definition of an octagon $\mathscr{O}(0, 0, \alpha_3, \alpha_4, \ldots, \alpha_8)$ in a slighly different form. As shown in Figure 3, this octagon can defined by

$$0 \leqslant x \leqslant a, \quad c_0 \leqslant x + y \leqslant a + b - c_2 \tag{6}$$

$$0 \leqslant y \leqslant b, \quad c_3 - b \leqslant x - y \leqslant a - c_1 \tag{7}$$

where $a = \alpha_5$, $b = \alpha_6$, $c_0 = \alpha_3$, $c_1 = a - \alpha_8$, $c_2 = a + b - \alpha_7$, $c_3 = \alpha_4 + b$. We leave a tedious, but easy, proof of the transformation from (4)-(5) to (6)-(7) as an exercise for the reader.

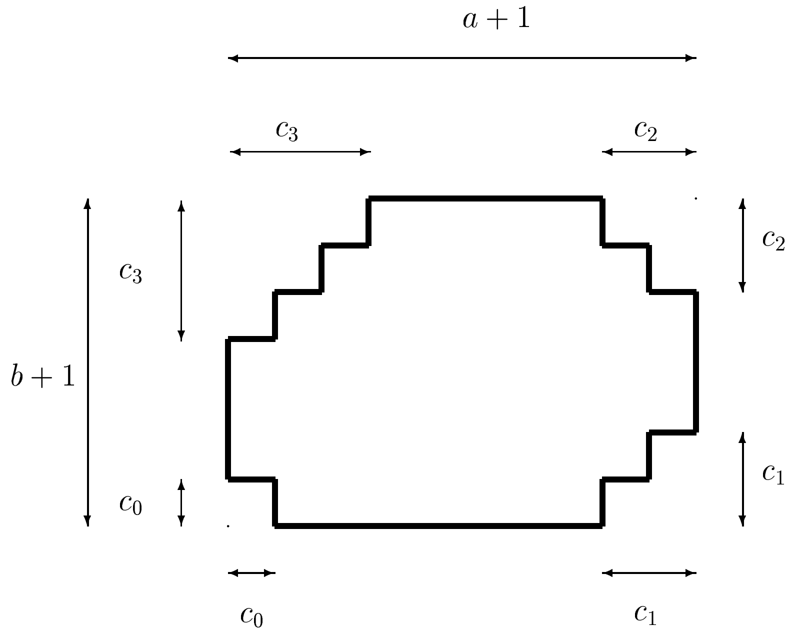

Figure 3. *An octagon with rewritten parameters*

It is now easy to see from Figure 3 that the size of the octagon $\mathscr{O}(0, 0, \alpha_3, \alpha_4, \ldots, \alpha_8) = \mathscr{O}(a, b, c_0, c_1, c_2, c_3)$ is given by

$$\left|\mathscr{O}(a, b, c_0, c_1, c_2, c_3)\right| = (a + 1)(b + 1) - \sum_{i=0}^{3} \frac{c_i(c_i + 1)}{2} \tag{8}$$

The next step is to determine the maximum tristance d of an optimal tristance anticode $\mathcal{A}_d = \mathscr{O}(a, b, c_0, c_1, c_2, c_3)$ as a function of the parameters a, b, c_0, c_1, c_2, c_3.

Lemma 6. *If $\mathcal{A}_d = \mathscr{O}(a, b, c_0, c_1, c_2, c_3)$ is an optimal tristance anticode of diameter d, then there exist points z_1, z_2, z_3 in $\mathcal{A}_d$ such that $d_3(z_1, z_2, z_3) = d$, $z_1 = (x_0, 0)$, and $z_2 = (x_1, b)$ for some $x_0, x_1 \in \mathbb{Z}$.*

Proof. Let $z_0', z_1', z_2' \in \mathcal{A}_d$ be three points achieving the maximal tristance. W.l.o.g., assume that $z_0' = (x_0', y_0')$ has the smallest Y-coordinate among the three points. Clearly $(x_0', y_0' - 1) \notin \mathcal{A}_d$. If $y_0' = 0$ then z_0' is the desired point. Otherwise we have two cases.

Case 1: If $0 \leqslant x'_0 < c_0$ then $y'_0 = c_0 - x'_0$. Replacing z'_0 with $z_0 = (c_0, 0)$ we ensure that $d_3(z'_0, z'_1, z'_2) \leqslant d_3(z_0, z'_1, z'_2)$, since the changed Y-coordinate increases the tristance by exactly y'_0 whereas the changed X-coordinate decreases the new tristance by at most y'_0.

Case 2: For $a - c_1 < x \leqslant a$ the proof is similar. We can again move the point z_0, without decreasing the tristance, until it reaches the desired position.

Finally, a symmetrical argument shows the existence of $z_1 = (x_1, y_1) \in \mathcal{A}_d$ with $y_1 = b$ such that $d_3(z_0, z'_1, z'_2) \leqslant d_3(z_0, z_1, z'_2)$. ∎

We are now left with the task of finding a third point which maximizes the tristance. W.l.o.g., let us assume that $c_0 \leqslant c_1, c_2, c_3$. It is then easy to see that taking

$$z_0 = (c_0, 0), \qquad z_1 = (x_1, b), \qquad z_2 = (a, y_2) \tag{9}$$

for some $c_3 \leqslant x_1 \leqslant a - c_2$ and $c_1 \leqslant y_2 \leqslant b - c_2$, as illustrated in Figure 4, maximizes the tristance within the octagon $\mathcal{A}_d = \mathscr{O}(a, b, c_0, c_1, c_2, c_3)$.

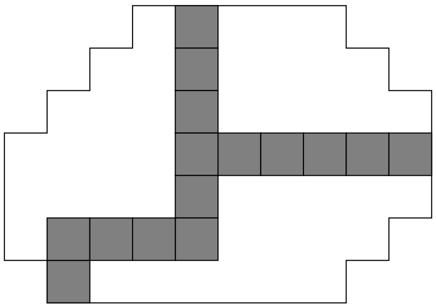

Figure 4. *An example of a maximal tristance configuration within $\mathcal{A}_d$*

Corollary 7. *If $\mathcal{A}_d = \mathscr{O}(a, b, c_0, c_1, c_2, c_3)$ is an optimal tristance anticode of diameter d, then $d = a + b - \min\{c_0, c_1, c_2, c_3\}$.*

Proof. This is the tristance of the three points z_0, z_1, z_2 in (9). ∎

Corollary 8. *If $\mathcal{A}_d = \mathscr{O}(a, b, c_0, c_1, c_2, c_3)$ is an optimal tristance anticode, then*

$$c_0 = c_1 = c_2 = c_3$$

Proof. Obviously, this maximizes the size of $\mathcal{A}_d = \mathscr{O}(a, b, c_0, c_1, c_2, c_3)$ in (8) for the given tristance $d = a + b - \min\{c_0, c_1, c_2, c_3\}$. ∎

Theorem 9. *Let $\mathcal{A}_d$ be an optimal tristance anticode of diameter d in $\mathbb{Z}^2$. Then*

$$|\mathcal{A}_d| \;=\; \left\lceil \frac{2(d^2 + 3d + 2)}{7} \right\rceil \;=\; \left\lceil \frac{2(d+1)(d+2)}{7} \right\rceil$$

Moreover, up to translation, rotation, and reflection, the set $\mathcal{A}_d$ consists of all $(x, y) \in \mathbb{Z}^2$ such that

$$\begin{array}{ccccccccc} 0 & \leqslant & x & \leqslant & a & \qquad c & \leqslant & x+y & \leqslant & a+b-c \\ 0 & \leqslant & y & \leqslant & b & \qquad c-b & \leqslant & x-y & \leqslant & a-c \end{array}$$

where the parameters a, b, and c are given as a function of d in Table 1 on the next page.

$d \pmod 7$	a	b	c	$\lvert\mathcal{A}_d\rvert$
0	$\frac{4d}{7}$	$\frac{4d}{7}$	$\frac{d}{7}$	$\frac{2d^2+6d+7}{7}$
1	$\frac{4d+3}{7}$	$\frac{4d-4}{7}$	$\frac{d-1}{7}$	$\frac{2d^2+6d+6}{7}$
2	$\frac{4d-1}{7}$	$\frac{4d-1}{7}$	$\frac{d-2}{7}$	$\frac{2d^2+6d+8}{7}$
3	$\frac{4d+2}{7}$	$\frac{4d-5}{7}$	$\frac{d-3}{7}$	$\frac{2d^2+6d+6}{7}$
4	$\frac{4d-2}{7}$	$\frac{4d-2}{7}$	$\frac{d-4}{7}$	$\frac{2d^2+6d+7}{7}$
5	$\frac{4d+1}{7}$ $\frac{4d+1}{7}$	$\frac{4d+1}{7}$ $\frac{4d-6}{7}$	$\frac{d+2}{7}$ $\frac{d-5}{7}$	$\frac{2d^2+6d+4}{7}$
6	$\frac{4d-3}{7}$ $\frac{4d+4}{7}$	$\frac{4d-3}{7}$ $\frac{4d-3}{7}$	$\frac{d-6}{7}$ $\frac{d+1}{7}$	$\frac{2d^2+6d+4}{7}$

Table 1. *Parameters of optimal tristance anticodes in the grid graph $\mathcal{G}_2^{\blacklozenge}$*

Proof. It follows from Corollary 7 and Corollary 8 that an optimal tristance anticode $\mathcal{A}_d$ of diameter d is an octagon of the form $\mathscr{O}(a,b,c,c,c,c)$. Its size is

$$|\mathcal{A}_d| \;=\; (a+1)(b+1) - 2c(c+1)$$

and its maximum tristance is $d = a+b-c$. To complete the proof, it remains to maximize the size $(a+1)(b+1) - 2c(c+1)$ subject to a single constraint, namely $a+b-c = d$, for $d = 2,3,\ldots$. The solution to this simple optimization problem is given in Table 1. ∎

Theorem 9 is our main result in this paper. It completely characterizes the optimal tristance anticodes of a given diameter in the grid graph of $\mathbb{Z}^2$.

3. Discussion and further results

In an upcoming paper [9], we generalize the result of Theorem 9 to three dimensions: we classify the optimal tristance anticodes of a given diameter in the grid graph of $\mathbb{Z}^3$.

Another way to generalize our results is to replace the grid graph $\mathcal{G}_n^{\blacklozenge}$ in Definition 1 by a different graph. In [9], we consider two alternative graphs $\mathcal{G}_2^{\ast}$ and $\mathcal{G}_2^{\star}$ that are useful in applications to two-dimensional interleaving with repetitions [6, 10]. The graph $\mathcal{G}_2^{\ast}$ has $\mathbb{Z}^2$ as its vertex set, with $z=(x,y)$ and $z'=(x',y')$ in $\mathbb{Z}^2$ being adjacent if and only if

$$d^{\infty}(z,z') \;=\; \max\{|x-x'|,|y-y'|\} \;=\; 1$$

Thus tristance in $\mathcal{G}_2^{\ast}$ may be thought of as a generalization of the L_∞-distance on $\mathbb{Z}^2$. The vertex set of the graph $\mathcal{G}_2^{\star}$ is the hexagonal lattice $A_2 = \{(1/2\,v, u+\sqrt{3}/2\,v) : u,v \in \mathbb{Z}\}$ with two points $z=(x,y)$ and $z'=(x',y')$ in A_2 being adjacent iff

$$d_{\mathrm{E}}(z,z') \;=\; \sqrt{(x-x')^2+(y-y')^2} \;=\; 1$$

The three graphs $\mathcal{G}_2^{\blacklozenge}$, $\mathcal{G}_2^{\ast}$, and $\mathcal{G}_2^{\star}$ are illustrated in Figure 5. We refer to the graphs $\mathcal{G}_2^{\blacklozenge}$, $\mathcal{G}_2^{\ast}$, and $\mathcal{G}_2^{\star}$ as the grid graph, the *star graph*, and the *hexagonal graph*, respectively.

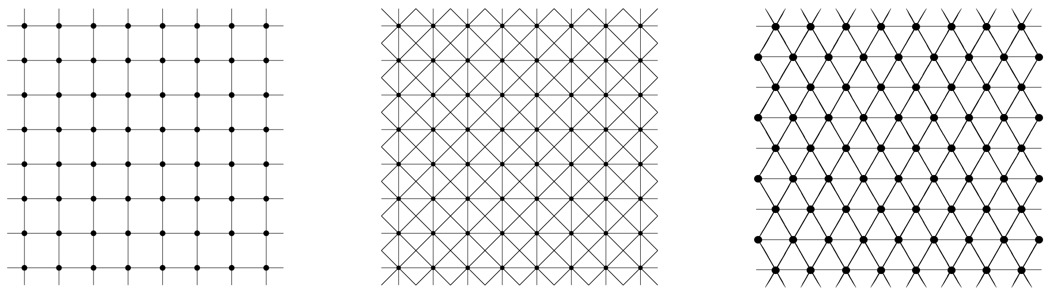

Figure 5. *The grid graph $\mathcal{G}_2^{\blacklozenge}$, the star graph $\mathcal{G}_2^{\ast}$, and the hexagonal graph $\mathcal{G}_2^{\star}$*

Observe that Definition 2 can be extended in the obvious way to other graphical models (such as the $\ast$ and the $\star$ models) as well as to higher dispersions (such as quadristance). A complete classification of optimal tristance anticodes in all the three models above is available in [9]. Applications of these results to two-dimensional interleaving schemes with repetitions [10], as well as to the game of Go, are also discussed in detail in [9].

References

[1] R. Ahlswede, H.K. Aydinian, and L.H. Khachatrian, On perfect codes and related concepts, *Designs, Codes, and Crypt.*, **22**, (2001), 221–237.

[2] R. Ahlswede and L.H. Khachatrian, The complete nontrivial-intersection theorem for systems of finite sets, *J. Combin. Theory Ser. A*, **76**, (1996), 121–138.

[3] R. Ahlswede and L.H. Khachatrian, The diametric theorem in Hamming spaces — optimal anticodes, *Adv. Appl. Math.*, **20**, (1998), 429–449.

[4] E. Bannai, Codes in bi-partite distance-regular graphs, *J. London Math. Soc.*, **16**, (1977), 197–202.

[5] N.L. Biggs, Perfect codes in graphs, *J. Combin. Theory Ser. B*, **15**, (1973), 289–296.

[6] M. Blaum, J. Bruck, and P.G. Farrell, Two-dimensional interleaving schemes with repetitions, Technical Report RJ 10047 (90543), IBM Corp., 1996.

[7] M. Blaum, J. Bruck, and A. Vardy, Interleaving schemes for multidimensional cluster errors, *IEEE Trans. Inform. Theory*, **44**, (1998), 730–743.

[8] Ph. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips J. Research*, **10**, (1973), 1–97.

[9] T. Etzion, M. Schwartz, and A. Vardy, Optimal tristance anticodes, manuscript in preparation, October 2002.

[10] T. Etzion and A. Vardy, Two-dimensional interleaving schemes with repetitions: constructions and bounds, *IEEE Trans. Inform. Theory*, **48**, (2002), 428–457.

[11] J. Kratochvíl, Perfect codes over graphs, *J. Combin. Theory*, **40**, (1986), 224–228.

[12] F.J. MacWilliams and N.J.A. Sloane, *The Theory of Error Correcting Codes*, Amsterdam: North-Holland/Elsevier, 1977.

[13] W.J. Martin and X.J. Zhu, Anticodes for the Grassman and bilinear forms graphs, *Designs, Codes, and Crypt.*, **6**, (1995), 73–79.

[14] V.S. Pless and W.C. Huffman (Editors), *Handbook of Coding Theory*, Amsterdam: North-Holland/Elsevier, 1998.

[15] M. Schwartz and T. Etzion, Codes and anticodes in the Grassman graph, *J. Combin. Theory Ser. A*, **97**, (2002), 27–42.