On Optimal Anticodes over Permutations with the Infinity Norm

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***Abstract*—Motivated by the set-antiset method for codes over permutations under the infinity norm, we study anticodes under this metric. For half of the parameter range we classify all the optimal anticodes, which is equivalent to finding the maximum permanent of certain $(0,1)$-matrices. For the rest of the cases we show constraints on the structure of optimal anticodes.**

I. Introduction

Codes over permutations have attracted recent interest due storage schemes for flash memories [1], [4], [6], [8]. In particular, correcting limited-magnitude in the rank modulation scheme was discussed in [8], where the metric of choice is the ℓ_∞-metric induced by the distance measure $d(f,g) = \max_{1\leqslant i\leqslant n} |f(i)-g(i)|$ defined between any two permutations $f,g\in S_n$.

In analogy to the definition of a code, a subset $A\subseteq S_n$ is an *anticode* with maximal distance d, if any two of its members are at distance at most d apart. The following theorem, which is sometimes referred to as the *set-antiset theorem*, motivates us to explore anticodes of maximum size. This theorem was also used before over different spaces and distance measures (see [2], [3], [8]).

Theorem 1. *[8, Theorem 13] Let $C, A\subseteq S_n$ be a code and an anticode under the ℓ_∞-metric, with minimal distance d and maximal distance $d-1$, respectively. Then $|C|\cdot|A|\leqslant|S_n|$.*

It should be noted that balls are just a special case of anticodes, since a ball of radius r is an anticode with maximal distance $2r$. The size of balls in S_n under the ℓ_∞-metric has been studied in [5], [7].

Let Γ_n^d denote the set of $(0,1)$-matrices of order n with exactly d non-zero entries in each row which form a contiguous block. Since for any optimal anticode $A\in S_n$ we can assume the identity permutation is in A (since the ℓ_∞-metric is right-invariant), and since for any $n\times n$ matrix $B=(b_{i,j})$, $\mathrm{per}(B)=\sum_{f\in S_n}\prod_{i=1}^n b_{i,f(i)}$, it is easily seen that the size of an optimal anticode of maximal distance $d-1$ in S_n is given by the maximum permanent of a matrix in Γ_n^d. We therefore define, $M_n^d=\left\{A\in\Gamma_n^d : \mathrm{per}(A)\geqslant\mathrm{per}(B) \text{ for all } B\in\Gamma_n^d\right\}$.

II. Results

Our results fall into two cases. First we consider the case of $2d\geqslant n$, i.e., $n=d+r$, where $0\leqslant r\leqslant d$. Let $A\in\Gamma_{d+r}^d$, $A=(a_{i,j})$, and for any row index i we define $x_i=\min\left\{j : a_{i,j}=1\right\}$. It can be seen that A is defined uniquely by the vector $(x_1,x_2,\dots,x_{d+r})$, so by abuse of notation we will write $A=(x_1,x_2,\dots,x_{d+r})$.

Theorem 2. *Let $A\in M_{d+r}^d$, $0\leqslant r\leqslant d$, then the only possible configurations of $A=(x_1,x_2,\dots,x_{d+r})$, up to a permutation of the rows and columns, are*

$$x_i=\begin{cases}1 & 1\leqslant i\leqslant\left\lfloor\frac{d+r}{2}\right\rfloor\\ r+1 & \left\lceil\frac{d+r}{2}\right\rceil< i\leqslant d+r.\end{cases}$$

Note that for $n=d+r$ odd, the value of $x_{\lceil\frac{d+r}{2}\rceil}$ is unconstrained. In addition,

$$\mathrm{per}(A)=\binom{2d-n}{\left\lfloor\frac{2d-n}{2}\right\rfloor}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)!\left(\left\lceil\frac{n}{2}\right\rceil\right)!.$$

For the case of $2d<n$ we show periodicity results.

Theorem 3. *Every $A\in M_n^d$ is of the form*

$$A\cong a1_{d\times d}\oplus C_1\oplus C_2\cdots\oplus C_h$$

where $a\geqslant 0$ and $0\leqslant h\leqslant d-1$. Moreover $C_i\in M^d_{\mathrm{ord}(C_i)}$, and if in addition C_i does not contain $1_{d\times d}$ as a sub-matrix, then $\mathrm{ord}(C_i)\leqslant b_d^*$, where b_d^* is a constant depending on d only.

Theorem 4. *For each positive integer d there exists μ_d such that M_n^d is periodic for $n\geqslant\mu_d$ in the sense that $A\in M_n^d$ if and only if $A\oplus 1_{d\times d}\in M_{n+d}^d$.*

References

[1] A. Barg and A. Mazumdar, "Codes in permutations and error correction for rank modulation," *IEEE Trans. on Inform. Theory*, vol. 56, no. 7, pp. 3158–3165, Jul. 2010.

[2] P. Delsarte, "An algebraic approach to association schemes of coding theory," *Philips J. Res.*, vol. 10, pp. 1–97, 1973.

[3] M. Deza and P. Frankl, "On maximal numbers of permutations with given maximal or minimal distance," *J. Combin. Theory Ser. A*, vol. 22, 1977.

[4] A. Jiang, M. Schwartz, and J. Bruck, "Correcting charge-constrained errors in the rank-modulation scheme," *IEEE Trans. on Inform. Theory*, vol. 56, no. 5, pp. 2112–2120, May 2010.

[5] T. Kløve, "Generating functions for the number of permutations with limited displacement," *Elec. J. of Comb.*, vol. 16, pp. 1–11, 2009.

[6] T. Kløve, T.-T. Lin, S.-C. Tsai, and W.-G. Tzeng, "Permutation arrays under the Chebyshev distance," *IEEE Trans. on Inform. Theory*, vol. 56, no. 6, pp. 2611–2617, Jun. 2010.

[7] M. Schwartz, "Efficiently computing the permanent and hafnian of some banded toeplitz matrices," *Linear Algebra and its Applications*, vol. 430, no. 4, pp. 1364–1374, Feb. 2009.

[8] I. Tamo and M. Schwartz, "Correcting limited-magnitude errors in the rank-modulation scheme," *IEEE Trans. on Inform. Theory*, vol. 56, no. 6, pp. 2551–2560, Jun. 2010.