Optimal 2-Dimensional 3-Dispersion Lattices

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Abstract. We examine 2-dimensional 3-dispersion lattice interleavers in three connectivity models: the rectangular grid with either 4 or 8 neighbors, and the hexagonal grid. We provide tight lower bounds on the interleaving degree in all cases and show lattices which achieve the bounds.

1 Introduction

In some relatively new applications, two-dimensional error-correcting codes are used. The codewords are written on the plane, and their coordinates are indexed by $\mathbb{Z}^2$. Several models of two-dimensional bursts of errors are handled in the literature. The most common burst type studied involves the rectangular grid and rectangular bursts [1,2,3,4,5]. The general two-dimensional case was studied in [6] and later in [7]. In the general case, an unrestricted burst (also called a cluster) is a connected set of points in $\mathbb{Z}^2$. The only parameter associated with such a burst is its size.

Since a burst is a connected set of points of $\mathbb{Z}^2$, we must consider several connectivity models. The simplest one is the $+$ model in which the neighbors of a given point $(x, y) \in \mathbb{Z}^2$ are, $\{(x+1, y), (x-1, y), (x, y+1), (x, y-1)\}$. A natural variation on the $+$ model is the $\ast$ model in which a point $(x, y) \in \mathbb{Z}^2$ has the following neighbor set, $\{(x+a, y+b) \mid a, b \in \{-1, 0, 1\}, |a|+|b| \neq 0\}$. Finally, another model of interest to us is the hexagonal model. Instead of the rectangular grid, we define the following grid: we start by tiling the plane $\mathbb{R}^2$ with regular hexagons. The vertices of the grid are the center points of the hexagons. We connect two vertices if and only if their respective hexagons are adjacent. This way, each vertex has exactly 6 neighboring vertices.

Given some connectivity model and r-points, $p_1, \ldots, p_r \in \mathbb{Z}^2$, we define $d_r(p_1, \ldots, p_r)$, also called the r-dispersion, to be the size (minus one) of the smallest burst containing all r points. The function d_2 is the known distance, while d_3 is called the *tristance*.

Bursts of errors are usually handled by interleaving several codewords together. An interleaving scheme, $\Gamma : \mathbb{Z}^2 \to \{1, 2, \ldots, m\}$ is denoted $A(t, r)$ if every burst of size t contains no more than r instances of the same integer from $\{1, 2, \ldots, m\}$. The number m of codewords needed for the interleaving, is the *interleaving degree* of Γ denoted by $\deg(\Gamma)$. If we take m codewords of an

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r-error-correcting code and write the i-th codeword in coordinates which are mapped by Γ to i, then a burst of size t generates no more than r errors in each of the codewords.

A simple way of creating an interleaving scheme is by taking a lattice Λ, i.e., a subspace of $\mathbb{Z}^2$, and mapping it, and each of its cosets to a unique integer. It was shown in [7] that if the $(r+1)$-dispersion of any $r+1$ points of Λ is at least t, then the interleaving scheme induced by Λ is an $A(t,r)$. Its degree is the index of Λ in $\mathbb{Z}^2$, also called the *volume* of Λ. The lattice Λ is always the span of a 2×2 matrix $\mathbf{G}$ over $\mathbb{Z}^2$ and the index of Λ in $\mathbb{Z}^2$ is also given by $|\mathbf{G}|$.

In this paper we describe optimal lattice interleavers for 2 repetitions. That is, for a given tristance d_3 we build lattices with minimal volume for which the tristance between any three of its points is at least d_3. The following three sections describe optimal lattices in each of the three connectivity models.

2 The + Model

2.1 Preliminaries

In the + model, a point $(x,y) \in \mathbb{Z}^2$ is connected to $(x+1,y), (x-1,y), (x,y+1)$, and $(x,y-1)$. We note that the distance in this model coincides with the definition of the L_1 distance between two points. Thus, for $p_i = (x_i, y_i)$, $1 \le i \le r$ we have

$$d_2(p_1,p_2) = |x_1 - x_2| + |y_1 - y_2| = \max_{1\le i\le 2} x_i - \min_{1\le i\le 2} x_i + \max_{1\le i\le 2} y_i - \min_{1\le i\le 2} y_i \ .$$

Lemma 1 (Theorem 2.4, [7]). *If $p_i = (x_i, y_i)$, $1 \le i \le 3$, are three points in $\mathbb{Z}^2$, then their tristance equals,*

$$d_3(p_1,p_2,p_3) = \max_{1\le i\le 3} x_i - \min_{1\le i\le 3} x_i + \max_{1\le i\le 3} y_i - \min_{1\le i\le 3} y_i \ .$$

In [7], Etzion and Vardy give constructions for lattice interleavers with 2 repetitions in the + model. The generator matrices for the interleavers are,

$$\mathbf{G}_{4k} = \begin{pmatrix} k & k \\ 0 & 3k \end{pmatrix} \qquad \mathbf{G}_{4k+1} = \begin{pmatrix} k & k+1 \\ 0 & 3k+2 \end{pmatrix}$$

$$\mathbf{G}_{4k+2} = \begin{pmatrix} k+1 & k \\ 1 & 3k+1 \end{pmatrix} \qquad \mathbf{G}_{4k+3} = \begin{pmatrix} k+1 & k+1 \\ 0 & 3k+2 \end{pmatrix}$$

for $k \ge 1$, and the resulting lattices are denoted Λ_{4k+i}, for $0 \le i \le 3$. It was shown ([7], Theorem 3.1) that for all $k \ge 1$ and $0 \le i \le 3$,

$$d_3(\Lambda_{4k+i}) = 4k + i \ .$$

Furthermore, the following theorem shows that Λ_{4k} and Λ_{4k+2} are optimal.

Theorem 1 ([7], Theorem 3.6). *Let Λ be any sublattice of $\mathbb{Z}^2$ with tristance $d_3(\Lambda) = t$. Set $k = \lfloor t/4 \rfloor$. Then the volume of Λ is bounded from below as follows:*

$$\begin{aligned} V(\Lambda) &\geq 3k^2 && \text{if } t \equiv 0 \pmod 4 \\ V(\Lambda) &\geq 3k^2 + \frac{3}{2}k + \frac{1}{2} && \text{if } t \equiv 1 \pmod 4 \\ V(\Lambda) &\geq 3k^2 + 3k + 1 && \text{if } t \equiv 2 \pmod 4 \\ V(\Lambda) &\geq 3k^2 + \frac{9}{2}k + \frac{5}{2} && \text{if } t \equiv 3 \pmod 4 \end{aligned}$$

In the following subsection we improve on the second and fourth cases, and show that Λ_{4k+1} and Λ_{4k+3} are also optimal.

2.2 Lower Bounds

Theorem 2. *Let Λ be a sublattice of $\mathbb{Z}^2$ with $d_3(\Lambda) = 4k + 1 + 2i$, where $i \in \{0, 1\}$, then $V(\Lambda) \geq (3k + 2)(k + i)$.*

Proof. We first note that $d_2(\Lambda) \geq 2k+1+i$. Otherwise, let $p_0 = (0, 0)$, and $p' = (x', y')$ be two points in Λ such that $d_2(p_0, p') \leq 2k+i$, and then $d_3(p_0, p', 2p') = 4k + 2i$, so $d_3(\Lambda) \leq 4k + 2i$ which is a contradiction.

Let $p_0 = (0, 0)$, $p_1 = (x_1, y_1)$, and $p_2 = (x_2, y_2)$, for which $x_2 \geq x_1 \geq 0$, and $d_3(p_0, p_1, p_2) = d_3(\Lambda) = 4k + 1 + 2i$. We start by showing that we should only prove the case where $y_1 > y_2 \geq 0$.

If $y_1 < 0$ we take a mirror image of the lattice along the X axis and continue with the same proof. Hence we may assume that $y_1 \geq 0$. Now, if $y_2 < 0$, we move p_2 to the origin and take a mirror image of the lattice along the Y axis to achieve the required configuration, and then continue with the same proof. Therefore we may also assume that $y_2 \geq 0$. The last case is that of $y_2 \geq y_1$. In that case,

$$d_3(p_0, p_1, p_2) = d_2(p_0, p_1) + d_2(p_1, p_2) \geq 2d_2(\Lambda) \geq 4k + 2 + 2i \ ,$$

which contradicts our assumption. Thus, $y_1 > y_2 \geq 0$ is the only case left for us to handle.

We start by sharpening the inequalities. If $x_1 = x_2$ then again,

$$d_3(p_0, p_1, p_2) = d_2(p_0, p_2) + d_2(p_2, p_1) \geq 2d_2(\Lambda) \geq 4k + 2 + 2i \ ,$$

which is a contradiction. Hence $x_2 > x_1$. We now show that p_0, p_1, p_2, and $p_2 - p_1$, define a fundamental region. We actually prove a slightly stronger claim: there are no points of Λ in the rectangle $R = \{(x, y) \mid 0 < x < x_2, \quad y_2 - y_1 < y < y_1\}$. Let us assume the contrary, i.e., that there exists $p = (x, y) \in \Lambda \cap R$. Now, if $y \geq 0$ then $d_3(p_0, p_2, p) = x_2 + \max\{y_2, y\} < x_2 + y_1 = 4k + 1 + 2i$, since $y, y_2 < y_1$. This is a contradiction, since $d_3(\Lambda) = 4k + 1 + 2i$. In the same manner, if $y < 0$, then $d_3(p_0, p_2, p) = x_2 + y_2 - y < x_2 + y_1 = 4k + 1 + 2i$, since $y > y_2 - y_1$, again a contradiction. Thus, p_0, p_1, p_2, and $p_2 - p_1$, define a fundamental region.

In the current configuration, $d_3(\Lambda) = 4k + 1 + 2i = x_2 + y_1$. Since one of the two summands must be strictly greater than the other, we may assume that $x_2 > y_1$, or else we exchange the X and Y axes and repeat the proof. We may therefore denote $x_2 = 2k + 1 + i + \delta$, and $y_1 = 2k + i - \delta$ for some integer $\delta \geq 0$. With the fundamental region defined above we have,

$$\begin{aligned} V(\Lambda) = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_1 & y_1 \end{vmatrix} &= x_2 y_1 - x_1 y_2 \\ &= (3k+2)(k+i) + i(i-1) + k(k+i) - (\delta^2 + \delta + x_1 y_2) \\ &= (3k+2)(k+i) + k(k+i) - (\delta^2 + \delta + x_1 y_2) \qquad \text{since } i \in \{0,1\} \ . \end{aligned}$$

All we have to do now, is show that $\delta^2 + \delta + x_1 y_2 \leq k(k+i)$.

Using the fact that $d_2(\Lambda) \geq 2k + 1 + i$ we get the following inequalities:

$$2k + 1 + i \leq d_2(p_0, p_1) = x_1 + 2k + i - \delta \iff 0 \leq \delta \leq x_1 - 1 \tag{1}$$

$$2k + 1 + i \leq d_2(p_1, p_2) = 4k + 1 + 2i - (x_1 + y_2) \iff x_1 + y_2 \leq 2k + i \tag{2}$$

Two more inequalities are achieved by examining p_1, p_2, and $2p_1$. If $2x_1 \leq x_2$ then,

$$4k + 1 + 2i \leq d_3(p_1, p_2, 2p_1) \iff y_2 \leq 2k + i - x_1 - \delta \ . \tag{3}$$

Otherwise, if $2x_1 > x_2$, then,

$$4k + 1 + 2i \leq d_3(p_1, p_2, 2p_1) \iff x_1 - y_2 \geq 2\delta + 1 \ . \tag{4}$$

If $2x_1 \leq x_2$ then,

$$\begin{aligned} \delta^2 + \delta + x_1 y_2 &\leq \delta^2 + \delta + x_1(2k + i - x_1 - \delta) && \text{by (3)} \\ &\leq x_1(2k + i - x_1) && \text{maximized at } \delta = 0, x_1 - 1 \text{ by (1)} \\ &\leq k(k+i) && \text{maximized at } x_1 = k, k+i \ . \end{aligned}$$

Otherwise, $2x_1 > x_2$ and then,

$$\begin{aligned} \delta^2 + \delta + x_1 y_2 &\leq \delta^2 + \delta + (k + \delta + 1)(k + i - \delta - 1) && \text{by (2) and (4)} \\ &= k(k+i) + (\delta + 1)(i - 1) \leq k(k+i) && \text{since } \delta \geq 0, \text{ and } i \in \{0,1\} \ . \end{aligned}$$

□

Corollary 1. *The lattices Λ_{4k+1} and Λ_{4k+3} are optimal.*

3 The Hexagonal Model

3.1 Preliminaries

Another model of interest to us is the hexagonal model. We follow the same notations as in [8]. Instead of the rectangular grid we used up to now, we define the following graph. We start by tiling the plane $\mathbb{R}^2$ with regular hexagons. The vertices of the graph are the center points of the hexagons. We connect two vertices if and only if their respective hexagons are adjacent. This way, each vertex has exactly 6 neighboring vertices.

Since handling this grid directly is hard, we prefer an isomorphic representation of the model. This representation includes $\mathbb{Z}^2$ as the set of vertices. Each point $(x, y) \in \mathbb{Z}^2$ has the following neighboring vertices,

$$\{(x+a, y+b) \mid a, b \in \{-1, 0, 1\}, a+b \neq 0\} \quad .$$

It may be shown that the two models are isomorphic by using the mapping $\xi : \mathbb{R}^2 \to \mathbb{Z}^2$, which is defined by $\xi(x, y) = (\frac{x}{\sqrt{3}} + \frac{y}{3}, \frac{2y}{3})$. The effect of the mapping on the neighbor set is shown in Fig. 1. From now on, by abuse of notation, we will also call the last model – the hexagonal model.

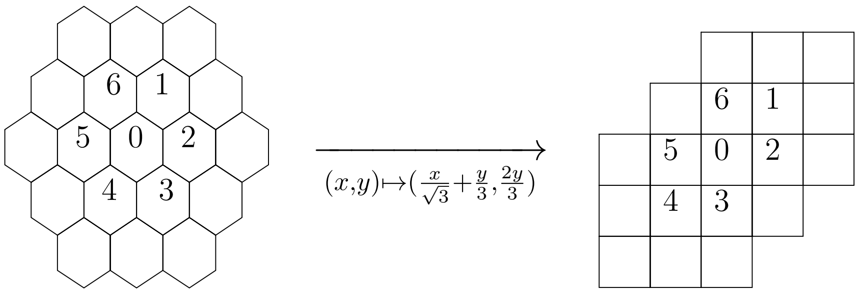

Fig. 1. The hexagonal model translation

Obviously, the distance d_2^{hex} between two points $p_i = (x_i, y_i)$, $i = 1, 2$, is

$$d_2^{\mathrm{hex}}(p_1, p_2) = \begin{cases} \max\{|x_1 - x_2|, |y_1 - y_2|\} & (x_1 - x_2)(y_1 - y_2) \geq 0 \\ |x_1 - x_2| + |y_1 - y_2| & \text{otherwise} \end{cases} .$$

Handling the tristance in the hexagonal model is a little more complicated.

Theorem 3 ([8], Theorem 6). *Let $p_i = (x_i, y_i)$, $1 \leq i \leq 3$ be points in $\mathbb{Z}^2$ for which, W.l.o.g., $x_1 \leq x_2 \leq x_3$ then,*

$$d_3^{\mathrm{hex}}(p_1, p_2, p_3) = \begin{cases} d_2^{\mathrm{hex}}(p_1, p_2) + d_2^{\mathrm{hex}}(p_2, p_3) & y_1 \leq y_2 \leq y_3 \\ d_2^{\mathrm{hex}}(p_1, \min(p_2, p_3)) + d_2^{\mathrm{hex}}(p_2, p_3) & y_1 \leq y_3 \leq y_2 \\ d_3(p_1, p_2, p_3) & y_3 \leq y_1 \leq y_2 \\ d_3(p_1, p_2, p_3) & y_3 \leq y_2 \leq y_1 \\ d_3(p_1, p_2, p_3) & y_2 \leq y_3 \leq y_1 \\ d_2^{\mathrm{hex}}(p_1, p_2) + d_2^{\mathrm{hex}}(\max(p_1, p_2), p_3) & y_2 \leq y_1 \leq y_3 \end{cases} ,$$

where max *(*min*) of two points is a component-wise* max *(*min*).*

An important thing to observe is that the hexagonal tristance allows scaling.

Theorem 4. *Let $p_1, p_2, p_3 \in \mathbb{Z}^2$ be three points and $k \geq 0$ an integer, then*

$$d_3^{\mathrm{hex}}(kp_1, kp_2, kp_3) = k \cdot d_3^{\mathrm{hex}}(p_1, p_2, p_3) \ .$$

Proof. The theorem simply results from Theorem 3 and the fact that the tristance in the + model also allows scaling, as stated in [7]. □

3.2 Constructions

For each integer $k \geq 1$ we define the lattices $\Lambda_{2k}^{\mathrm{hex}}$ and $\Lambda_{2k+1}^{\mathrm{hex}}$ by their respective generator matrices:

$$\mathbf{G}_{2k}^{\mathrm{hex}} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \qquad \mathbf{G}_{2k+1}^{\mathrm{hex}} = \begin{pmatrix} k & -1 \\ 1 & k+1 \end{pmatrix} \ .$$

Theorem 5.

$$\begin{aligned} d_3^{\mathrm{hex}}(\Lambda_{2k}^{\mathrm{hex}}) &= 2k & \qquad V(\Lambda_{2k}^{\mathrm{hex}}) &= k^2 \\ d_3^{\mathrm{hex}}(\Lambda_{2k+1}^{\mathrm{hex}}) &= 2k+1 & \qquad V(\Lambda_{2k+1}^{\mathrm{hex}}) &= k^2 + k + 1 \ . \end{aligned}$$

Proof. The volumes of the lattices are easily calculated by the determinants of generator matrices. Therefore, we turn to prove the minimal tristance of the lattices is as specified.

The simple case is the lattice $\Lambda_{2k}^{\mathrm{hex}}$. This lattice is a scaling up of the trivial lattice $\mathbb{Z}^2$, by a factor of k. Since $d_3^{\mathrm{hex}}(\mathbb{Z}^2) = 2$, we immediately get $d_3^{\mathrm{hex}}(\Lambda_{2k}^{\mathrm{hex}}) = 2k$.

The last case requires more care. Given three points which achieve the minimal tristance in $\Lambda_{2k+1}^{\mathrm{hex}}$, we may always move the leftmost point to the origin. Hence, we may assume that the three points are, $p_0 = (0,0)$, $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, and $x_1, x_2 \geq 0$.

We now note that both $p'_1 = (k, -1)$ and $p'_2 = (k+1, k)$ belong to $\Lambda_{2k+1}^{\mathrm{hex}}$, and that $d_3^{\mathrm{hex}}(p_0, p'_1, p'_2) = 2k+1$. Hence, as potential candidates for p_1 and p_2, we need to examine only points $p = (x, y)$ of $\Lambda_{2k+1}^{\mathrm{hex}}$ for which, $x \geq 0$ and $d_2^{\mathrm{hex}}(p_0, p) \leq 2k+1$. The only such points of $\Lambda_{2k+1}^{\mathrm{hex}}$ are easily seen to be,

$$(k, -1), \quad (1, k+1), \quad (k+1, k), \quad (k-1, -k-2) \ .$$

Going over the 6 possible choices of pairs of points from the list, one may verify that $d_3^{\mathrm{hex}}(p_0, p_1, p_2) \geq 2k+1$ in all cases. □

3.3 Lower Bounds

We now show that both $\Lambda_{2k}^{\mathrm{hex}}$ and $\Lambda_{2k+1}^{\mathrm{hex}}$ are optimal in the sense that they have the lowest possible interleaving degree. We do so by explicitly proving $\Lambda_{2k}^{\mathrm{hex}}$ to be optimal, and deducing that $\Lambda_{2k+1}^{\mathrm{hex}}$ must be also optimal.

Due to lack of space, the proof that $\Lambda_{2k}^{\mathrm{hex}}$ is optimal is omitted. The complete proof may be found in [9].

Theorem 6. *Let Λ be a sublattice of $\mathbb{Z}^2$ with $d_3^{\text{hex}}(\Lambda) = 2k$, and $d_2^{\text{hex}}(\Lambda) = k$. In that case, $V(\Lambda) \geq k^2$.*

Theorem 7. *Let Λ be a sublattice of $\mathbb{Z}^2$ with $d_3^{\text{hex}}(\Lambda) = 2k$, and $d_2^{\text{hex}}(\Lambda) > k$. In that case, $V(\Lambda) > k^2$.*

Corollary 2. *Let Λ be a sublattice of $\mathbb{Z}^2$ with $d_3^{\text{hex}}(\Lambda) = 2k$. So then, $V(\Lambda) \geq k^2$.*

We now show that $\Lambda_{2k+1}^{\text{hex}}$ is optimal also.

Theorem 8. *Let Λ be a sublattice of $\mathbb{Z}^2$ with $d_3^{\text{hex}}(\Lambda) = 2k+1$. So then, $V(\Lambda) \geq k^2 + k + 1$.*

Proof. Let us assume to the contrary, that $V(\Lambda) \leq k^2 + k$. Let Λ' be a scaling up of Λ by a factor of 2. Hence, $d_3^{\text{hex}}(\Lambda') = 2d_3^{\text{hex}}(\Lambda) = 2(2k+1)$, and $V(\Lambda') = 4V(\Lambda) \leq 4k^2 + 4k$. However, according to Corollary 2, $V(\Lambda') \geq (2k+1)^2 = 4k^2 + 4k + 1$, a contradiction. $\square$

4 The $*$ Model

4.1 Preliminaries

The $*$ model uses the rectangular grid as the previous $+$ model does, but each point $(x, y) \in \mathbb{Z}^2$ has eight neighboring points forming the set

$$\left\{(x+a, y+b) \in \mathbb{Z}^2 \mid a, b \in \{-1, 0, 1\}, |a| + |b| \neq 0\right\} \ .$$

We denote the r-dispersion in the $*$ model as d^* and in general, by affixing the $*$ to a notation we refer to its $*$ model counterpart. Etzion and Vardy [7] construct the lattices Λ_{4k}^*, Λ_{4k+1}^*, Λ_{4k+2}^*, los_{4k+3}, by providing their respective generator matrices,

$$\mathbf{G}_{4k}^* = \begin{pmatrix} k & 3k \\ 0 & 6k-1 \end{pmatrix} \qquad \mathbf{G}_{4k+1}^* = \begin{pmatrix} k+1 & 3k+1 \\ 1 & 6k \end{pmatrix}$$
$$\mathbf{G}_{4k+2}^* = \begin{pmatrix} k+1 & 3k+1 \\ 1 & 6k+2 \end{pmatrix} \qquad \mathbf{G}_{4k+3}^* = \begin{pmatrix} k+2 & 3k+2 \\ 2 & 6k+3 \end{pmatrix} \ .$$

It was shown ([7], Theorem 7.2) that for all $k \geq 1$ and $0 \leq i \leq 3$,

$$d_3^*(\Lambda_{4k+i}^*) = 4k + i \ .$$

However, no proof is given to show that the lattices are optimal.

Our main tool for handling the $*$ model is the function φ defined in [7]. Let us denote the sublattice of $\mathbb{Z}^2$ defined as,

$$D_2 = \{(x, y) \mid x + y \equiv 0 \pmod 2\} \ .$$

The mapping $\varphi : \mathbb{Z}^2 \to D_2$ is defined as

$$\varphi(x, y) = (x - y, x + y) \ .$$

In essence, φ rotates the plane counterclockwise by an angle of $\pi/4$ and scales it up by a factor of $\sqrt{2}$.

If Λ is a sublattice of $\mathbb{Z}^2$, then $\Lambda' = \varphi(\Lambda)$ is obviously a sublattice of D_2. By [7] Theorem 7.1,

$$d_3^*(\Lambda) = \left\lceil \frac{d_3(\Lambda')}{2} \right\rceil \ .$$

By the nature of φ, it is also easy to show that

$$V(\Lambda) = \frac{V(\Lambda')}{2} \ .$$

4.2 Lower Bounds

Theorem 9. *Let Λ be a sublattice of $\mathbb{Z}^2$ with $d_3^*(\Lambda) = 4k$, then $V(\Lambda) \geq 6k^2 - k$.*

Proof. Let us assume the contrary, i.e., that $d_3^*(\Lambda) = 4k$, and $V(\Lambda) < 6k^2 - k$. Let $\Lambda' = \varphi(\Lambda)$, so then $d_3(\Lambda')$ is either $8k - 1$ or $8k$, and $V(\Lambda') < 12k^2 - 2k$.

If $d_3(\Lambda') = 8k - 1$, then by Theorem 2, $V(\Lambda') \geq 12k^2 - 2k$. If $d_3(\Lambda') = 8k$, then by Theorem 1, $V(\Lambda') \geq 12k^2$. Either way, we have a contradiction. □

Theorem 10. *Let Λ be a sublattice of $\mathbb{Z}^2$ with $d_3^*(\Lambda) = 4k + 2$, then $V(\Lambda) \geq 6k^2 + 5k + 1$.*

Proof. Let us assume the contrary, i.e., that $d_3^*(\Lambda) = 4k + 2$, and $V(\Lambda) < 6k^2 + 5k + 1$. Let $\Lambda' = \varphi(\Lambda)$, so then $d_3(\Lambda')$ is either $8k + 3$ or $8k + 4$, and $V(\Lambda') < 12k^2 + 10k + 2$.

If $d_3(\Lambda') = 8k + 3$, then by Theorem 2, $V(\Lambda') \geq 12k^2 + 10k + 2$. If $d_3(\Lambda') = 8k + 4$, then by Theorem 1, $V(\Lambda') \geq 12k^2 + 12k + 3$. Either way, we have a contradiction. □

Corollary 3. *The lattices Λ^*_{4k} and Λ^*_{4k+2} are optimal.*

The two cases left require some more work. If we try to apply the method used in the last two theorems, we find that the bound we achieve is not tight. This stems from the fact that by examining $\varphi(\Lambda)$, we restrict ourselves to sublattices of D_2. We now state the equivalent theorem to Theorem 2 which refers to sublattices of D_2.

Theorem 11. *Let Λ be a sublattice of D_2 with $d_3(\Lambda) = 4k + 1$, then $V(\Lambda) \geq 3k^2 + 3k - 2$.*

Proof. The proof proceeds in a similar fashion to the proof of Theorem 2, so we will only point out the differences. The first one is the fact that in D_2, the distance between any two points is even. Hence, $d_2(\Lambda) \geq 2k+2$.

This, in turn, changes inequalities (1) and (2) to the following:

$$2k+2 \leq d_2(p_0,p_1) = x_1 + 2k - \delta \iff 0 \leq \delta \leq x_1 - 2 \tag{5}$$
$$2k+2 \leq d_2(p_1,p_2) = 4k+1-(x_1+y_2) \iff x_1 + y_2 \leq 2k-1 \tag{6}$$

We now remind that $x_2 = 2k+1+\delta$ and $y_1 = 2k-\delta$, so x_2 and y_1 have different parity. This means that x_1 and y_2 also have different parity. We distinguish between two cases:

Case 1: $2x_1 \leq x_2$. There are two subcases according to the parity of δ.

Case 1a: δ is even. Since the parity of x_1 and y_2 is different, (3) is sharper and we get

$$y_2 \leq 2k - x_1 - \delta - 1 \ . \tag{7}$$

Now,

$$\begin{aligned} \delta^2 + \delta + x_1 y_2 &\leq \delta^2 + \delta + x_1(2k - x_1 - \delta - 1) && \text{by (7)} \\ &\leq x_1(2k - x_1 - 1) && \text{maximized at } \delta = 0 \text{ by (5)} \\ &\leq k^2 - k && \text{maximized at } x_1 = k-1, k \ . \end{aligned}$$

Case 1b: δ is odd. Hence $\delta \geq 1$, so then,

$$\begin{aligned} \delta^2 + \delta + x_1 y_2 &\leq \delta^2 + \delta + x_1(2k - x_1 - \delta) && \text{by (3)} \\ &\leq 2 + x_1(2k - x_1 - 1) && \text{maximized at } \delta = 1, x_1 - 2 \\ &\leq k^2 - k + 2 && \text{maximized at } x_1 = k-1, k \ . \end{aligned}$$

Case 2: $2x_1 > x_2$. Then,

$$\begin{aligned} \delta^2 + \delta + x_1 y_2 &\leq \delta^2 + \delta + (k+\delta)(k-\delta-1) && \text{by (6) and (4)} \\ &= k^2 - k \end{aligned}$$

We see that in any case, $\delta^2 + \delta + x_1 y_2 \leq k^2 - k + 2$. Like in the proof of Theorem 2,

$$\begin{aligned} V(\Lambda) &= \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_1 & y_1 \end{vmatrix} \\ &= x_2 y_1 - x_1 y_2 = 4k^2 + 2k - (\delta^2 + \delta + x_1 y_2) \geq 3k^2 + 3k - 2 \ . \end{aligned}$$

□

Note that for $k = 1$, the bound of Theorem 11 is worse than the bound of Theorem 2. This does not interfere with the following theorems which do not reach that case.

Theorem 12. *Let Λ be a sublattice of $\mathbb{Z}^2$ with $d_3^*(\Lambda) = 4k+1$ and $k \geq 1$, then $V(\Lambda) \geq 6k^2 + 3k - 1$.*

Proof. Let us assume the contrary, i.e., that $d_3^*(\Lambda) = 4k+1$, and $V(\Lambda) < 6k^2 + 3k - 1$. Let $\Lambda' = \varphi(\Lambda)$, so then $d_3(\Lambda')$ is either $8k+1$ or $8k+2$, and $V(\Lambda') < 12k^2 + 6k - 2$.

Note that Λ' is a sublattice of D_2. Therefore, if $d_3(\Lambda') = 8k+1$, then by Theorem 11, $V(\Lambda') \geq 12k^2 + 6k - 2$. If $d_3(\Lambda') = 8k+2$, then by Theorem 1, $V(\Lambda') \geq 12k^2 + 6k + 1$. Either way, we have a contradiction. $\square$

Theorem 13. *Let Λ be a sublattice of $\mathbb{Z}^2$ with $d_3^*(\Lambda) = 4k+3$ and $k \geq 1$, then $V(\Lambda) \geq 6k^2 + 9k + 2$.*

Proof. Let us assume the contrary, i.e., that $d_3^*(\Lambda) = 4k+3$, and $V(\Lambda) < 6k^2 + 9k + 2$. Let $\Lambda' = \varphi(\Lambda)$, so then $d_3(\Lambda')$ is either $8k+5$ or $8k+6$, and $V(\Lambda') < 12k^2 + 18k + 4$.

Note that Λ' is a sublattice of D_2. Therefore, if $d_3(\Lambda') = 8k+5$, then by Theorem 11, $V(\Lambda') \geq 12k^2 + 18k + 4$. If $d_3(\Lambda') = 8k+6$, then by Theorem 1, $V(\Lambda') \geq 12k^2 + 18k + 7$. Either way, we have a contradiction. $\square$

Corollary 4. *The lattices Λ^*_{4k+1} and Λ^*_{4k+3} are optimal.*

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