Quasi-Cross Lattice Tilings
with Applications to Flash Memory

Moshe Schwartz

Electrical and Computer Engineering
Ben-Gurion University of the Negev
Beer Sheva 84105, Israel
schwartz@ee.bgu.ac.il

Abstract—We consider lattice tilings of $\mathbb{R}^n$ by a shape we call a (k_+, k_-, n)-**quasi-cross**. Such lattices form perfect error-correcting codes which correct a single limited-magnitude error with prescribed maximal-magnitudes of positive error and negative error (the ratio of which is called the balance ratio). These codes can be used to correct both disturb and retention errors in flash memories, which are characterized by having limited magnitudes and different signs.

We construct infinite families of perfect codes for any rational balance ratio, and provide a specific construction for $(2, 1, n)$-quasi-cross lattice tiling. The constructions are related to group splitting and modular B_1 sequences. We also study bounds on the parameters of lattice-tilings by quasi-crosses, connecting the arm lengths of the quasi-crosses and the dimension. We also prove constraints on group splitting, a specific case of which shows that the parameters of the lattice tiling by $(2, 1, n)$-quasi-crosses is the only ones possible.

I. INTRODUCTION

Flash memory is perhaps the fastest growing memory technology today. Flash memory cells use floating gate technology to store information using trapped charge. By measuring the charge level in a single flash memory cell and comparing it with a predetermined set of threshold levels, the charge level is quantized to one of q values, conveniently chosen to be $\mathbb{Z}_q$. While originally q was chosen to be 2, and each cell stored a single bit of information, current *multi-level flash* memory technology allows much larger values of q, thus storing $\log_2 q$ bits of information in each cell[1].

As is usually the case, the stored charge levels in flash cells suffer from noise which may affect the information retrieved from the cells. Many off-the-shelf coding solutions exist and have been applied for flash memory, see for example [3], [14]. However, the main problem with this approach is the fact that these codes are not tailored for the specific errors occurring in flash memory and thus are wasteful. A more accurate model of the flash memory channel is therefore required to design better-suited codes.

The most notorious property of flash memory is its inherent asymmetry between cell programming (charge injection into cells), and cell erasure (charge removal from cells). While the former is easy to perform on single cells, the latter works on large blocks of cells and physically damages the cells. Thus, when attempting to reach a target stored value in a cell,

charge is slowly injected into the cell over several iterations. If the desired level has not been reached, another round of charge injection is performed. If, however, the desired charge level has been passed, there is no way to remove the excess charge from the cell without erasing an entire block of cells. In addition, the actions of cell programming and cell reading disturb adjacent cells by injecting extra unwanted charge into them. Because the careful iterative programming procedure employs small charge-injection steps, it follows that over-programming errors, as well as cell disturbs, are likely to have a small magnitude of error.

This motivated the application of the asymmetric limited-magnitude error model to the case of flash memory [2], [10]. In this model, a transmitted vector $c \in \mathbb{Z}^n$ is received with error as $y = c + e \in \mathbb{Z}^n$, where we say that t asymmetric limited-magnitude errors occurred with magnitude at most k if the error vector $e = (e_1, \dots, e_n) \in \mathbb{Z}^n$ satisfies $0 \leqslant e_i \leqslant k$ for all i, and there are exactly t non-zero entries in e. Not in the context of flash memory, it was shown in [1] how to construct optimal asymmetric limited-magnitude errors correcting *all* errors, i.e., t equals the code length. Systematic all-error-correcting codes for asymmetric and symmetric limited magnitude errors were studied in [4]. General code constructions and bounds for arbitrary t were given in [2]. More specifically, for $t = 1$, i.e., correcting a single error, codes were proposed in the context of flash in [10], but were also described as semi-cross packing in [7].

The main drawback of the asymmetric limited-magnitude error model is the fact that not all error types were considered during the model formulation. Another type of common error in flash memories is due to *retention* which is a slow process of charge leakage. Like before, the magnitude of errors created by retention is limited, however, unlike over-programming and cell disturbs, retention errors are in the opposite direction.

We therefore suggest a generalization to the error model we call the *unbalanced limited-magnitude error model*. A transmitted vector $c \in \mathbb{Z}^n$ is now received with error as the vector $y = c + e \in \mathbb{Z}^n$, where we say that t unbalanced limited-magnitude errors occurred if the error vector $e = (e_1, \dots, e_n) \in \mathbb{Z}^n$ satisfies $-k_- \leqslant e_i \leqslant k_+$ for all i, and there are exactly t non-zero entries in e. Both k_+ and k_- are non-negative integers, where we call k_+ the positive-error magnitude limit, and k_- the negative-error magnitude limit.

In this work we consider only single error-correcting codes. In general, assuming at most a single error occurs, the error

[1]It should be noted that other alternatives have been suggested to the conventional multi-level modulation scheme, such as, for example, rank modulation [9] and local rank modulation [5], [11].

sphere containing all possible received words $y = c + e$ forms a shape we call a (k_+, k_-, n)-*quasi-cross* (see Figure 1). This is a generalization of the asymmetric semi-cross of [7], [10] which we get when choosing $k_- = 0$, and the full cross of [12] which we get when choosing $k_+ = k_-$. On a side note, when $k_+ = k_- = 1$, the full cross is also an ℓ_1 sphere of radius 1. Such spheres are known to tile for radius 1 (see [6]), and are conjectured to be impossible to tile for larger radii (see [8] for a recent survey).

To avoid these two studied cases we shall consider only $0 < k_- < k_+$. An error-correcting code is a packing of pair-wise disjoint quasi-crosses. We shall only consider perfect codes, i.e., tilings of the space, which form lattices, since these are easier to analyze, construct, and encode, than non-lattice packings.
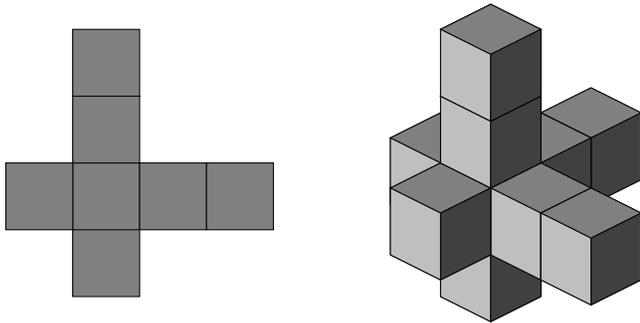

Figure 1. A $(2, 1, 2)$-quasi-cross and a $(2, 1, 3)$-quasi-cross

The paper is organized as follows: In Section II we introduce the notation and definitions used throughout the paper and discuss connections with known results. We continue in Section IV with constructions of such tilings. We follow in Section III with simple bounds on the parameter of lattice tilings by quasi crosses, and conclude in Section V. Most proofs have been omitted due to the page restriction.

II. PRELIMINARIES

A. Quasi-Crosses, Tilings, and Lattices

In the unbalanced limited-magnitude-error channel model, the transmitted (or stored) word is a vector $v \in \mathbb{Z}^n$. A single error is a vector in $e \in \mathbb{Z}^n$ all of whose entries are 0 except for a single entry with value belonging to the set

$$M = \{-k_-, \dots, -2, -1, 1, 2, \dots, k_+\},$$

where the integers $0 < k_- < k_+$ are the negative-error and positive-error magnitudes. For convenience we denote this set as $M = [-k_-, k_+]^*$. We denote $\beta = k_-/k_+$ and call it the *balance ratio*. Obviously, $0 < \beta < 1$.

Given a transmitted vector $v \in \mathbb{Z}^n$, and provided at most a single error occurred, the received word resides in the error sphere centered about v defined by

$$\mathcal{E}(v) = \{v\} \cup \{v + m \cdot e_i \mid i \in [n], m \in M\},$$

where $[n] = \{1, \dots, n\}$, and e_i denotes the all-zero vector except for the i-th position which contains a 1. We call $\mathcal{E}(0)$

a (k_+, k_-, n)-*quasi-cross*. By translation, $\mathcal{E}(v) = v + \mathcal{E}(0)$ for all $v \in \mathbb{Z}^n$.

Following the notation of [12], let

$$Q = \{(x_1, \dots, x_n) \mid 0 \leqslant x_i < 1, x_i \in \mathbb{R}\}$$

denote the *unit cube* centered at the origin. By abuse of terminology, we shall also call the set of unit cubes $Q + \mathcal{E}(v)$, a (k_+, k_-, n)-quasi-cross centered at v for any $v \in \mathbb{Z}^n$. Examples of such quasi-crosses are given in Figure 1. We note that the volume of $Q + \mathcal{E}(v)$ does not depend on the choice of v and is equal to $n(k_+ + k_-) + 1$.

A set $V = \{v_1, v_2, \dots\} \subseteq \mathbb{Z}^n$ defines a set of quasi-crosses by translation: $\{\mathcal{E}(v_1), \mathcal{E}(v_2), \dots\}$. The set V is said to be a *packing* of $\mathbb{R}^n$ by quasi-crosses if the translated quasi-crosses are pairwise disjoint. A packing V is called a *tiling* if the union of the translated quasi-crosses equals $\mathbb{R}^n$. If V happens to be an additive subgroup of $\mathbb{Z}^n$ with a basis $\{b_1, b_2, \dots, b_n\}$, then we call V a *lattice*. The $n \times n$ integer matrix formed by placing the elements of a basis as its rows is called a *generating matrix* of the lattice.

Let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice with a generating matrix $\mathcal{G}(\Lambda) \in \mathbb{Z}^{n \times n}$ whose rows form a basis $\{b_1, b_2, \dots, b_n\} \subseteq \mathbb{Z}^n$. A *fundamental region* of Λ is defined as

$$\left\{ \sum_{i=1}^n \alpha_i b_i \;\middle|\; \alpha_i \in \mathbb{R}, 0 \leqslant \alpha_i < 1 \right\}.$$

It is easily seen, by definition, that Λ tiles $\mathbb{R}^n$ with translates of the fundamental region.

It is well known that the volume of a fundamental region does not depend on the choice of basis for Λ and equals $\det \mathcal{G}(\Lambda)$. The *density* of Λ is defined as $1/\det \mathcal{G}(\Lambda)$ and if Λ forms a packing of (k_+, k_-, n)-quasi-crosses, then the *packing density* of Λ is defined as

$$\rho(\Lambda) = \frac{n(k_+ + k_-) + 1}{\det \mathcal{G}(\Lambda)},$$

which intuitively measures (for a large enough finite area) the ratio of the area covered by (k_+, k_-, n)-quasi-crosses centered at the lattice points, to the total area. It follows that $0 \leqslant \rho(\Lambda) \leqslant 1$, and Λ forms a tiling with (k_+, k_-, n)-quasi-crosses if and only if $\rho(\Lambda) = 1$, i.e., $\det \mathcal{G}(\Lambda) = n(k_+ + k_-) + 1$.

Example 1. *If we take the* $(3, 2, 2)$-*quasi-cross, one can verify that the lattice* Λ *with basis* $b_1 = (4, 1)$, $b_2 = (3, 5)$, *is indeed a lattice packing for this quasi-cross. The resulting packing density is* $\rho(\Lambda) = \frac{11}{17}$. □

B. Lattice Tiling via Group Splitting

An equivalence between lattice packings and group splitting was described in [7], [12], which we describe for completeness. Let G be an Abelian group, where we shall denote the group operation as $+$. Given some $s \in G$ and a non-negative integer $m \in \mathbb{Z}$, we denote by ms the sum $s + s + \dots + s$, where s appears in the sum m times. The definition is extended in the natural way to negative integers m.

A splitting of G is a pair of sets, $M \subseteq \mathbb{Z} \setminus \{0\}$, called the *multiplier set*, and $S = \{s_1, s_2, \dots, s_n\} \subseteq G$, called the

splitter set, such that the elements of the form ms, $m \in M$, $s \in S$, are all distinct and non-zero in G. Next, we define a homomorphism $\phi : \mathbb{Z}^n \to G$ by

$$\phi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i s_i.$$

If the multiplier set is $M = [-k_-, k_+]^*$, then it may be easily verifiable that $\ker \phi$ is a lattice packing of $\mathbb{R}^n$ by (k_+, k_-, n)-quasi-crosses.

A simple representation of the lattice may also be given in matrix form: Let $\mathcal{H} = [s_1, s_2, \ldots, s_n]$ be a $1 \times n$ matrix over G. The lattice Λ is the set of vectors $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $\mathcal{H}x^T = 0$. Thus, $\mathcal{H}$ plays the role of a "parity-check matrix".

Example 2. *Continuing Example 1, let $G = \mathbb{Z}_{17}$ and let $M = \{-2, -1, 1, 2, 3\} = [-2, 3]^*$ stand for the multiplier set of the $(3, 2, n)$-quasi-cross. A possible splitting of G is $S = \{1, 13\}$, which results in a parity-check matrix $\mathcal{H} = [1, 13]$ for the packing described in Example 1.* □

Group splitting as a method for constructing error-correcting codes was also discussed, for example, in the case of shift-correcting codes [15] and integer codes [16].

C. Lattice Packings and Sequences

It was noted in [10] that there is a connection between the codes suggested in [10] (which are equivalent to semi-cross packings) and a certain sub-case of sequences called modular B_h sequences. We detail the relevant connection in our case.

A v-modular $B_h(M)$ sequence, where $M \subseteq \mathbb{Z} \setminus \{0\}$, is a subset $S \subseteq \mathbb{Z}_v \setminus \{0\}$, whose elements $S = \{s_1, \ldots, s_n\}$ satisfy that all sums $\sum_{i=1}^{h} m_i s_{i_j}$, where $1 \leqslant i_1 < i_2 < \cdots < i_h \leqslant n$, and $m_i \in M$, are all distinct.

Thus, a v-modular $B_1(M)$ sequence is a splitting of $\mathbb{Z}_v$ defined by M and S. We note that a specific group is being split, i.e., a cyclic group.

As was also described in [10], when we have a v-modular $B_1(M)$ sequence S, i.e., a splitting of $\mathbb{Z}_v$ by M and S, and therefore a resulting $1 \times n$ parity-check matrix $\mathcal{H} = [s_1, s_2, \ldots, s_n]$, we can construct other packings, provided the elements of M are co-prime to v. This is done by constructing any $k \times n(v^k - 1)/(v - 1)$ parity-check matrix $\mathcal{H}'$ containing all distinct column vectors whose top non-zero element is from S. This is equivalent to a splitting of the non-cyclic group $\mathbb{Z}_v^k$ by M and S being the columns of $\mathcal{H}'$. We note that if $\mathcal{H}$ results in a tiling, then so does $\mathcal{H}'$.

III. Constructions of Tilings by Quasi-Crosses

We shall now consider constructions of lattice tilings by (k_+, k_-, n)-quasi-crosses. We first examine the case of a constant balance ratio $\beta = k_- / k_+$ and show that for any rational ratio there exist infinitely-many tilings by splitting cyclic and non-cyclic groups. We then focus on a particular case of $(2, 1, n)$-quasi-crosses and show an infinite family of tilings for them.

A. Constant Balance-Ratio Quasi-Cross Tilings

Construction 1. *Let $0 < k_- < k_+$ be positive integers such that $k_+ + k_- = p - 1$, where p is a prime. We set the multiplier set $M = [-k_-, k_+]^*$. Consider the cyclic group $G = \mathbb{Z}_{p^\ell}$, $\ell \in \mathbb{N}$. We split G using a splitter set S constructed recursively in the following manner:*

$$S_1 = \{1\}$$
$$S_{i+1} = pS_i \cup \left\{ s \in \mathbb{Z}_{p^{i+1}} \mid s \equiv 1 \pmod{p} \right\}.$$

The requested set is $S = S_\ell$.

Theorem 3. *The sets S and M from Construction 1 split $\mathbb{Z}_{p^\ell}$, forming a tiling by $(k_+, k_-, (p^\ell - 1)/(p - 1))$-quasi-crosses and a p^ℓ-modular $B_1(M)$ sequence.*

Proof: The proof is by a simple induction. Obviously M and $S_1 = \{1\}$ split $\mathbb{Z}_p$. Now assume M and S_i split $\mathbb{Z}_{p^i}$. Let us consider M, S_{i+1}, and $\mathbb{Z}_{p^{i+1}}$. We now show that if $ms = m's'$ in $\mathbb{Z}_{p^{i+1}}$, $m, m' \in M$, $s, s' \in S_{i+1}$, then $m = m'$ and $s = s'$.

In the first case, given any $s \in S_{i+1}$, $p \nmid s$, and given $m, m' \in M$, $m \neq m'$, since $M = [-k_-, k_+]^*$, it follows that $ms \neq m's$ since they leave different residues modulo p. For the second case, let $s, s' \in S$, $s' \neq s$, and let $m, m' \in M$, where m and m' are not necessarily distinct. If $p | s'$ then $ms \neq m's'$ since $p \nmid ms$ but $p | m's'$. We assume then that $s' \equiv 1 \pmod{p}$. Write $s = qp + 1$ and $s' = q'p + 1$, $0 \leqslant q, q' \leqslant p^i - 1$, then $ms = m's'$ implies $m = m'$ (by reduction modulo p). It then follows that $mqp \equiv mq'p \pmod{p^{i+1}}$. But $\gcd(m, p) = 1$ and so $q \equiv q' \pmod{p^i}$, which (due to the range of q and q') implies $q = q'$, i.e., $s = s'$.

For the last case, $s, s' \in pS_i$. We note that the multiples of p in $\mathbb{Z}_{p^{i+1}}$ are isomorphic to $\mathbb{Z}_{p^i}$, and since M and S_i split $\mathbb{Z}_{p^i}$, for all $m, m' \in M$, if $ms = m's'$ then $m = m'$ and $s = s'$.

Finally, $|M| = p - 1$, $|S_\ell| = (p^\ell - 1)/(p - 1)$, and so $|M| \cdot |S_\ell| + 1 = \left|\mathbb{Z}_{p^\ell}\right|$, implying that the splitting induces a tiling. ∎

The following construction splits a non-cyclic group of the same parameters.

Construction 2. *Let $0 < k_- < k_+$ be positive integers such that $k_+ + k_- = p - 1$, where p is a prime. We set the multiplier set $M = [-k_-, k_+]^*$. Consider the additive group of $G = \mathrm{GF}(p^\ell)$, $\ell \in \mathbb{N}$. Let $\alpha \in \mathrm{GF}(p^\ell)$ be a primitive element, and define $S = \left\{ P(\alpha) \mid P \in \mathcal{M}_\ell^p[x] \right\}$ where $\mathcal{M}_\ell^p[x]$ denotes the set of all monic polynomials of degree strictly less than $\ell - 1$ over $\mathrm{GF}(p)$ in the indeterminate x.*

Theorem 4. *The sets S and M from Construction 2 split the additive group of $\mathrm{GF}(p^\ell)$ and form a tiling by $(k_+, k_-, (p^\ell - 1)/(p - 1))$.*

We point out several interesting observations. In Construction 2, if we take $\ell = 1$ we get $S = \{1\}$. For $\ell > 1$, write the elements of $\mathrm{GF}(p^\ell)$ as length-ℓ vectors over $\mathrm{GF}(p)$ (using the basis $1, \alpha, \ldots, \alpha^{\ell-1}$, with α a primitive element of $\mathrm{GF}(p^\ell)$).

The elements of S then become the set of all vectors of length ℓ over $GF(p)$ with the leading non-zero element being 1. We will get the same set by extending the "matrix-extension" method implied in [10] to our quasi-cross case.

Another interesting thing to note is that, using the same vector notation as above, the parity-check matrix for the lattice is simply the parity-check matrix of the $[\frac{p^\ell-1}{p-1}, \frac{p^\ell-1}{p-1} - \ell, 3]$ Hamming code over $GF(p)$.

Yet another observation is that we can mix Constructions 1 and 2, by taking the p^ℓ-modular $B_1(M)$ sequence resulting from Construction 1 and applying the "matrix" method of Construction 2 to form a splitting of $G = \mathbb{Z}_{p^\ell} \times \mathbb{Z}_{p^\ell} \times \cdots \times \mathbb{Z}_{p^\ell}$ which induces a tiling by quasi-crosses. The latter works since the elements of M are all co-prime to p.

Finally, we observe that the lattice tilings resulting from Constructions 1 and 2 are not equivalent. Before we do so we need another definition. A lattice $\Lambda \subseteq \mathbb{Z}^n$ has period $(t_1, \ldots, t_n) \in \mathbb{Z}^n$ if whenever $v \in \Lambda$, then also $v + t_i e_i \in \Lambda$ for all i. Lattices are always periodic, and t_i is the smallest positive integer for which $t_i e_i \in \Lambda$.

Example 5. *Consider six-dimensional lattice tilings by $(3, 1, 6)$-quasi-crosses. Using Construction 1 we construct a lattice Λ_1 by splitting $\mathbb{Z}_{25}$ and getting a splitter set $S = \{1, 5, 6, 11, 16, 21\}$, resulting in a parity-check matrix*

$$\mathcal{H}_1 = \begin{bmatrix} 1 & 5 & 6 & 11 & 16 & 21 \end{bmatrix}$$

over $\mathbb{Z}_{25}$. This produces a generating matrix for Λ_1

$$\mathcal{G}_1 = \begin{bmatrix} 25 & 0 & 0 & 0 & 0 & 0 \\ 20 & 1 & 0 & 0 & 0 & 0 \\ 19 & 0 & 1 & 0 & 0 & 0 \\ 14 & 0 & 0 & 1 & 0 & 0 \\ 9 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We confirm that $\det \mathcal{G}_1 = 25 = 6(3+1) + 1$ making Λ_1 a tiling for $(3, 1, 6)$-quasi-crosses.

If, on the other hand, we choose to use Construction 2 to construct a lattice Λ_2, we split $GF(5^2)$ to get a parity-check matrix

$$\mathcal{H}_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

over $GF(5)$. A corresponding generating matrix is then

$$\mathcal{G}_2 = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 4 & 4 & 1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Again, we confirm $\det \mathcal{G}_2 = 25$.

Finally, to show the lattices are not equivalent, it is readily verified that the period of Λ_1 is $(25, 5, 25, 25, 25, 25)$, while the period of Λ_2 is $(5, 5, 5, 5, 5, 5)$. □

The following shows there are infinitely-many tilings by quasi-crosses of any given rational balance ratio.

Theorem 6. *For any given rational balance ratio $\beta = k_- / k_+$, $0 < \beta < 1$, there exists an infinite sequence of quasi-crosses, $\{(k_+^{(i)}, k_-^{(i)}, n^{(i)})\}_{i=1}^\infty$, such that $n^{(i)} < n^{(i+1)}$, $k_-^{(i)} / k_+^{(i)} = \beta$, and there exists a tiling by $(k_+^{(i)}, k_-^{(i)}, n^{(i)})$-quasi-crosses, for all $i \in \mathbb{N}$.*

B. Construction of $(2, 1, n)$-Quasi-Cross Tilings

We turn to constructing $(2, 1, n)$-quasi-cross tilings and their associated modular $B_1(M)$ sequences. The construction is similar in flavor to Construction 1.

Construction 3. *Let $k_+ = 2, k_- = 1$, and let the multiplier set be $M = \{-1, 1, 2\}$. We split the group $G = \mathbb{Z}_{4^\ell}, \ell \in \mathbb{N}$, using a splitter set S constructed recursively in the following manner:*

$$S_1 = \{1\}$$
$$S_{i+1} = 4S_i \cup \left\{s \in \mathbb{Z}_{4^{i+1}} \mid s \equiv 1 \pmod 4, 2s < 4^{i+1}\right\}$$

The requested set is $S = S_\ell$.

Theorem 7. *The sets S and M from Construction 3 split $\mathbb{Z}_{4^\ell}$, forming a tiling by $(2, 1, (4^\ell - 1)/3)$-quasi-crosses and a 4^ℓ-modular $B_1(M)$ sequence.*

We observe that in this case, since the elements of M are not co-prime to 4, extending the matrix method from [10] does not produce a valid tiling or even packing. For example, if we were to take the trivial 4-modular $B_1(M)$ sequence, $\{1\}$ and attempt to create a parity-check matrix over $\mathbb{Z}_4$

$$\mathcal{H} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 3 \end{bmatrix}$$

we would find that M together with the columns of $\mathcal{H}$ is not a splitting of $\mathbb{Z}_4^2$ since $2 \cdot [1, 0]^T = 2 \cdot [1, 2]^T$ over $\mathbb{Z}_4$. Hence, the lattice formed by the parity-check matrix $\mathcal{H}$ is not a lattice packing of $(2, 1, 5)$-quasi-crosses.

IV. BOUNDS ON THE PARAMETERS OF LATTICE TILINGS BY QUASI-CROSSES

In this section we focus on showing bounds on the parameters of (k_+, k_-, n)-quasi-cross tilings. We first consider the restrictions (k_+, k_-, n)-quasi-cross tilings imply on k_+, k_-, and n. We then continue to study the group G being split to create the tilings, and show restrictions which, in particular, prove that the parameters of the $(2, 1, n)$-quasi-cross tiling of Construction 3 are unique.

A. Dimension and Arm Length Bounds

We first discuss bounds connecting the arm lengths of the quasi-cross and the dimension of the tiling. Some of the theorems to follow may be viewed as extensions to [13].

Theorem 8. *For any $n \geqslant 2$, if $\frac{2k_+(k_-+1)-k_-^2}{k_++k_-} > n$, then there is no lattice tiling of (k_+, k_-, n)-quasi-crosses.*

Proof: Given an integer $n \geqslant 2$, assume a (k_+, k_-, n)-quasi-cross lattice tiling Λ exists. Consider the plane

$\{(x, y, 0, \ldots, 0) \mid x, y \in \mathbb{Z}\}$. Translates of this plane tile $\mathbb{Z}^n$. Within this plane, we look at the subset

$$A = \{(x, y, 0, \ldots, 0) \mid 0 \leqslant x, y < k_+ + 2 \text{ and}$$
$$x < k_- + 2 \text{ or } y < k_- + 2\}.$$

It is easily seen that A cannot contain two points from Λ, or else the arms of two quasi-crosses overlap. Thus, the density of Λ (which we know is exactly $1/(n(k_+ + k_-) + 1)$, since Λ is a tiling) cannot exceed the reciprocal of the volume of A, i.e., $\frac{1}{n(k_+ + k_-) + 1} \leqslant \frac{1}{(k_+ + 1)^2 - (k_+ - k_-)^2}$. Rearranging gives us the desired result. ∎

Corollary 9. *There is no lattice tiling of $\mathbb{R}^2$ by $(k_+, k_-, 2)$-quasi-crosses.*

In the following theorem and corollary we can restrict the arm lengths of quasi-crosses that lattice-tile $\mathbb{R}^n$.

Theorem 10. *For any $n \geqslant 2$, if a lattice tiling of $\mathbb{R}^n$ by (k_+, k_-, n)-quasi-crosses exists, then $k_- \leqslant n - 1$.*

Corollary 11. *For any $n \geqslant 3$, if a lattice tiling of $\mathbb{R}^n$ by (k_+, k_-, n)-quasi-crosses exists and $k_- > \frac{n}{2} - 1$, then $k_+ \leqslant \frac{3n^2}{8}$ for n even, and $k_+ \leqslant \frac{3n^2 - 4n + 1}{4}$ otherwise.*

B. Restrictions on the Split Group

We now turn to examining connections between properties of the Abelian group being split, G, and the multiplier and splitter sets, M and S. We shall eventually show, as a special case of the theorems presented, that the $(2, 1, n)$-quasi-cross tiles $\mathbb{R}^n$ only with the parameters of Construction 3. The following is an adaptation of a theorem from [13].

Theorem 12. *[13, p. 75, Theorem 9] Let $M = [-k_-, k_+]^*$ be the multiplier set of the (k_+, k_-, n)-quasi-cross. If M splits G, then M splits $\mathbb{Z}_{|G|}$.*

Theorem 12 is important since now, to show the existence or nonexistence of a lattice tiling by (k_+, k_-, n)-quasi-crosses, it is sufficient to check splittings of $\mathbb{Z}_n$. We shall now do exactly that, and reach the conclusion that $(2, 1, n)$-quasi-crosses lattice-tile $\mathbb{R}^n$ only with the parameters of Construction 3.

Theorem 13. *Let $M = [-(k-1), k]^*$ be the multiplier set of the $(k, k - 1, n)$-quasi-cross, $k \geqslant 2$. If M splits a finite Abelian group G, $|G| > 1$, then $\gcd(k, |G|) \neq 1$.*

Theorem 14. *Let $M = [-2^w + 1, 2^w]^*$ be the multiplier set of the $(2^w, 2^w - 1, n)$-quasi-cross, $w \in \mathbb{N}$. If M splits $\mathbb{Z}_q$ then $q = 2^{r(w+1)}$ for some $r \in \mathbb{N}$.*

Corollary 15. *The $(2, 1, n)$-quasi-cross lattice-tiles $\mathbb{R}^n$ only with the parameters of Construction 3.*

V. Conclusion

We considered lattice tilings of $\mathbb{R}^n$ by (k_+, k_-, n)-quasi-crosses. These lattices form perfect codes correcting a single error with limited magnitudes k_+ and k_- for positive and negative errors, respectively. We have seen how these lattice tilings are equivalent to certain group splittings, and in certain cases (when the group is cyclic), to modular B_1 sequences.

We provided two constructions which may be used recursively to build infinite families of such lattice tilings for any given rational balance ration $\beta = k_-/k_+$. We also specifically constructed an infinite family of lattice tilings for the $(2, 1, n)$-quasi-cross.

We followed by studying bounds on the parameters of such lattice tilings, showing bounds connecting k_+, k_-, and n. We also examined restrictions on group splitting, and concluded through a special case of the theorems presented, that $(2, 1, n)$-quasi-crosses lattice-tile $\mathbb{R}^n$ only with the parameters of the construction presented earlier.

We conclude with a computer search looking for lattice tilings by (k_+, k_-, n)-quasi-crosses. It was found that for all $0 < k_- < k_+ \leqslant 10$ and split group $G = \mathbb{Z}_q$ of order $q \leqslant 100$, that only lattice tilings with the parameters of the constructions provided in this paper exist.

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