On the Non-existence of Lattice Tilings by Quasi-crosses

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***Abstract*—We study necessary conditions for the existence of lattice tilings of $\mathbb{R}^n$ by quasi-crosses. We prove general non-existence results using a variety of number-theoretic tools. We then apply these results to the two smallest unclassified shapes, the $(3,1,n)$-quasi-cross and the $(3,2,n)$-quasi-cross. We show that for dimensions $n \leqslant 250$, apart from the known constructions, there are no lattice tilings of $\mathbb{R}^n$ by $(3,1,n)$-quasi-crosses except for ten remaining unresolved cases, and no lattice tilings of $\mathbb{R}^n$ by $(3,2,n)$-quasi-crosses except for eleven remaining unresolved cases.**

I. Introduction

Flash memory cells use floating gate technology to store information using trapped charge. By measuring the charge level in a single flash memory cell and comparing it with a predetermined set of threshold levels, the charge level is quantized to one of q values, conveniently chosen to be $\mathbb{Z}_q$. The stored charge levels in flash cells suffer from noise which may affect the information retrieved from the cells. The *unbalanced limited-magnitude error model* was recently suggested in [5] as an appropriate error model. In this model, a transmitted vector $c \in \mathbb{Z}^n$ is now received with error as the vector $y = c + e \in \mathbb{Z}^n$, where we say that t unbalanced limited-magnitude errors occurred if the error vector $e = (e_1, \dots, e_n) \in \mathbb{Z}^n$ satisfies $-k_- \leqslant e_i \leqslant k_+$ for all i, and there are exactly t non-zero entries in e. Both k_+ and k_- are non-negative integers, where we call k_+ the positive-error magnitude limit, and k_- the negative-error magnitude limit. This error model is a natural extension to the asymmetric limited-magnitude error model appearing in [1].

This work studies perfect linear 1-error-correcting codes for the unbalanced limited-magnitude error model, for which the error sphere forms a shape we call a (k_+, k_-, n)-*quasi-cross* (see Figure 1). This is a generalization of the asymmetric semi-cross of [2], [3] which we get when choosing $k_- = 0$, and the full cross of [6] which we get when choosing $k_+ = k_-$. Thus, a 1-error-correcting code is a packing of the space by disjoint translates of the error sphere. By a *linear* code we mean the packing forms a lattice, and by *perfect* we mean the disjoint error spheres cover the entire space, i.e., form a *tiling*.

A few constructions were given in [5] for lattice tilings of $\mathbb{R}^n$ by quasi-crosses, and in particular, a full classification was provided of the dimensions in which there exist lattice tilings by $(2,1,n)$-quasi-crosses. Recently, Yari, Kløve, and Bose [7], gave other constructions for lattice packings and tilings by quasi-crosses, and in particular, new constructions for tilings by $(3,1,n)$-quasi-crosses.

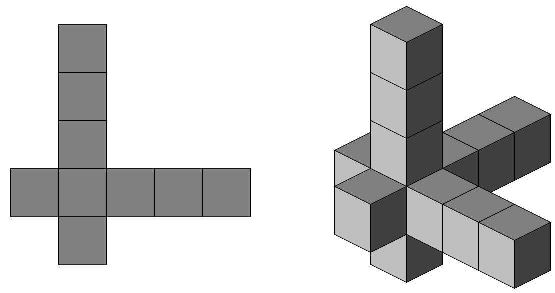

Figure 1. A $(3,1,2)$-quasi-cross and a $(3,1,3)$-quasi-cross

The goal of this work is to derive new general necessary conditions for the existence of tilings of $\mathbb{R}^n$ by quasi-crosses. To demonstrate these tools we shall apply them to the two smallest unclassified cases of the $(3,1,n)$-quasi-cross and the $(3,2,n)$-quasi-cross.

II. Main Results

The following theorems provide a list of necessary conditions for a tiling of $\mathbb{R}^n$ by (k_+, k_-, n)-quasi-crosses to exist. Various methods are used in the process of proving these theorems, including power characters, accounting for zero divisors, and recursion. In this extended abstract we present the theorems without proofs. The interested reader is referred to [4] for the complete proofs.

Theorem 1. *Let $1 \leqslant k_- < k_+$ be positive integers such that $k_+ + k_-$ is an odd prime. If the (k_+, k_-, n)-quasi-cross lattice tiles $\mathbb{R}^n$, and $n(k_+ + k_-) + 1$ is a prime, then $k_+ + k_- \mid n$.*

Theorem 2. *There is no lattice tiling of $\mathbb{R}^n$ by $(4k-1, 1, n)$-quasi-crosses for all positive integers k such that $kn \equiv 5, 8 \pmod 9$.*

Theorem 3. *For any $1 < r < k_+ + k_-$, the (k_+, k_-, n)-quasi-cross does not lattice tile $\mathbb{R}^n$ when*

$$(k_+ + k_-)n + 1 \equiv ru \pmod{r \cdot k_+\#}$$

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for all integers u such that $\gcd(u, k_+\#) = 1$, and where $\ell\#$ denotes the primorial of ℓ, i.e., the product of all primes not exceeding ℓ.

Theorem 4. *Let $p > k_+$ be a prime, $p \not\equiv 1 \pmod{k_+ + k_-}$, and $p \neq k_+ + k_-$. Then the (k_+, k_-, n)-quasi-cross does not lattice tile $\mathbb{R}^n$ for $n \equiv -(k_+ + k_-)^{-1} \pmod{p}$, where $(k_+ + k_-)^{-1}$ is the multiplicative inverse of $k_+ + k_-$ in $\mathbb{Z}_p$.*

Theorem 5. *Let p be a prime, $p \equiv 1 \pmod{k_+ + k_-}$. If the (k_+, k_-, n)-quasi-cross does not lattice tile $\mathbb{R}^n$, then the (k_+, k_-, n')-quasi-cross does not lattice tile $\mathbb{R}^{n'}$,*

$$n' = \frac{((k_+ + k_-)n + 1)\, p^i - 1}{k_+ + k_-},$$

for all positive integers i.

Theorem 6. *Let p be a prime, and let k_+ and k_- be non-negative integers such that $k_- \leqslant k_+$ and $p \leqslant k_+ < p^2$. Then the (k_+, k_-, n)-quasi-cross does not lattice tile $\mathbb{R}^n$ when $(k_+ + k_-)n + 1 \equiv 0 \pmod{p^2}$ unless*

$$n = \frac{p-1}{(k_+ \bmod p) + (k_- \bmod p)}.$$

III. Application to Small Quasi-crosses

In this concluding section we apply the general methods described in the previous sections to the problem of finding necessary conditions for the existence of lattice tilings of $\mathbb{R}^n$ by $(3,1,n)$-quasi-crosses and by $(3,2,n)$-quasi-crosses. These are the smallest unclassified cases, since lattice tilings by the smaller $(2,1,n)$-quasi-cross have been completely classified in [5].

The following two theorems, which are specific to the $(3,1,n)$-quasi-cross, can be derived from the general theorems given above.

Theorem 7. *The $(3,1,n)$-quasi-cross does not lattice tile $\mathbb{R}^n$ when $4n+1$ is a prime, $n \equiv 3 \pmod 6$.*

Theorem 8. *Let $4n+1$ be a prime, with n being an odd integer. If*

$$6^n \not\equiv 1 \pmod{4n+1}$$

then the $(3,1,n)$-quasi-cross does not lattice tile $\mathbb{R}^n$.

In contrast to the non-existence results, we do know of some constructions for lattice tilings by quasi-crosses. For the first shape, the $(3,1,n)$-quasi-cross, we recall there exists a construction of lattice tilings from [5] for dimensions $n = (5^i - 1)/4$, $i \geqslant 1$. In addition, certain primes were shown in [7] to induce lattice tilings, as well as a recursive construction, though a closed analytic form for the dimension appears to be hard to obtain. Using a computer to verify the requirements for the construction from [7], for $n \leqslant 250$ we also have lattice tilings of $\mathbb{R}^n$ by $(3,1,n)$-quasi-crosses for dimensions

$$n = 37, 43, 97, 102, 115, 139, 163, 169, 186, 199, 216.$$

On the other hand, combining the non-existence results with a nice analytic form we achieve the following:

Theorem 9. *If the $(3,1,n)$-quasi-cross lattice tiles $\mathbb{R}^n$ then $n \not\equiv 2 \pmod 3$.*

However, especially for the $(3,1,n)$-quasi-cross, numerous other non-existence results lacking a nice analytic form ensue from the previous section. Aggregating the entire set of necessary conditions, for $n \leqslant 250$, apart from the dimensions mentioned above that allow a lattice tiling, no other lattice tiling of $\mathbb{R}^n$ by $(3,1,n)$-quasi-crosses exists except perhaps in the remaining unclassified cases of

$$n = 22, 24, 60, 111, 114, 121, 144, 220, 234, 235.$$

For the second shape, the $(3,2,n)$-quasi-cross, no lattice tiling is known except for the trivial tiling of $\mathbb{R}^1$. The combined non-existence results we obtained in this work, with a nice analytic form, are much stronger in this case:

Theorem 10. *If the $(3,2,n)$-quasi-cross lattice tiles $\mathbb{R}^n$ then $n \equiv 1, 13 \pmod{36}$.*

Aggregating this result with the other recursive necessary conditions, for $2 \leqslant n \leqslant 250$, no lattice tiling of $\mathbb{R}^n$ by $(3,2,n)$-quasi-crosses exists except perhaps in the remaining unclassified cases of

$$n = 13, 37, 49, 73, 85, 121, 145, 157, 181, 217, 229.$$

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