On Tilings of Asymmetric Limited-Magnitude Balls

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***Abstract*—We study whether an asymmetric limited-magnitude ball may tile $\mathbb{Z}^n$. This ball generalizes previously studied shapes: crosses, semi-crosses, and quasi-crosses. Such tilings act as perfect error-correcting codes in a channel which changes a transmitted integer vector in a bounded number of entries by limited-magnitude errors.**

A construction of lattice tilings based on perfect codes in the Hamming metric is given. Several non-existence results are proved, both for general tilings, and lattice tilings. A complete classification of lattice tilings for two certain cases is proved.

I. Introduction

In some applications, information is encoded as a vector of integers, $\mathbf{x} \in \mathbb{Z}^n$, most notably, flash memories (e.g., see [2]). Additionally, a common noise affecting these applications is a limited-magnitude error affecting some of the entries. Namely, at most t entries are increased by as much as k_+ or decreased by as much as k_-. Thus, for integers $n \geqslant t \geqslant 1$, and $k_+ \geqslant k_- \geqslant 0$, we define the (n,t,k_+,k_-)*-error-ball* as

$$\mathcal{B}(n,t,k_+,k_-) \triangleq \{\mathbf{x} \in \mathbb{Z}^n \mid x_i \in [-k_-,k_+] \text{ and } \mathrm{wt}(\mathbf{x}) \leqslant t\},$$

where $\mathbf{x} = (x_1,x_2,\dots,x_n)$, and $\mathrm{wt}(\mathbf{x})$ denotes the Hamming weight of $\mathbf{x}$. It now follows that an error-correcting code in this setting is equivalent to a packing of $\mathbb{Z}^n$ by $\mathcal{B}(n,t,k_+,k_-)$, and the subject of interest for this paper, a perfect code is equivalent to a tiling of $\mathbb{Z}^n$ by $\mathcal{B}(n,t,k_+,k_-)$. An example of $\mathcal{B}(3,2,2,1)$ is shown in Fig. 1.

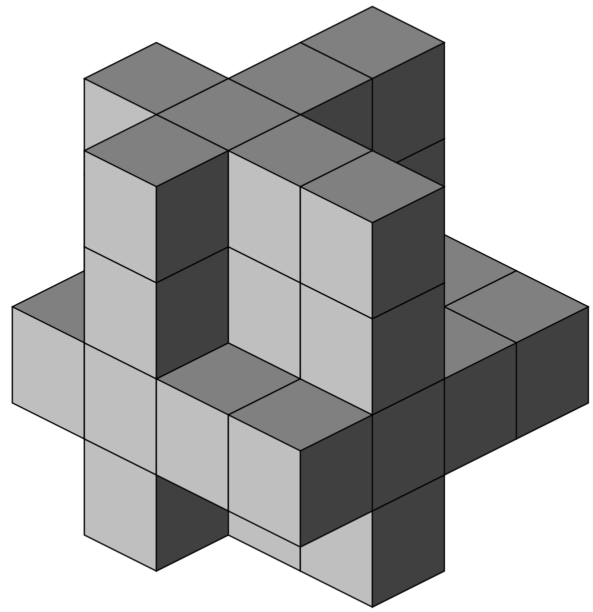

Fig. 1. A depiction of $\mathcal{B}(3,2,2,1)$ where each point in $\mathcal{B}(3,2,2,1)$ is shown as a unit cube.

Previous works on tiling these shapes almost exclusively studied the case of $t=1$. The *cross*, $\mathcal{B}(n,1,k,k)$, and semi-cross, $\mathcal{B}(n,1,k,0)$ have been extensively researched, e.g., see [3]–[5], [10], [12] and the many references therein. This was recently extended to *quasi-crosses*, $\mathcal{B}(n,1,k_+,k_-)$, in [7], creating a flurry of activity on the subject [8], [16]–[20]. To the best of our knowledge, [11] and later [1], are the only works to consider $t \geqslant 2$, by considering a notched cube (or a "chair"), which for certain parameters becomes $\mathcal{B}(n,n-1,k,0)$. Tilings of these shapes have been constructed in [1], [11]. Additionally, [1] showed that $\mathcal{B}(n,n-2,k,0)$, $n \geqslant 4$, $k \geqslant 1$, can never lattice-tile $\mathbb{Z}^n$.

The goal of this paper is to study tilings of $\mathcal{B}(n,t,k_+,k_-)$ for $t \geqslant 2$. Our main contributions are a construction of lattice tilings from perfect codes in the Hamming metric, and a sequence of non-existence results, both for lattice tilings and for general non-lattice tilings. We use both algebraic techniques and geometric ones. In particular, we provide a complete classification of lattice tilings with $\mathcal{B}(n,2,1,0)$ and $\mathcal{B}(n,2,2,0)$.

The paper is organized as follows: In Section II we provide the notation used throughout the paper, as well as definitions and basic results concerning lattice tilings and group splittings. We construct lattice tilings in Section III, and prove non-existence results in Section IV. A short discussion and open questions are given in Section V. Due to space limitation proofs are mostly omitted, except for a sampling to showcase the various techniques. The complete version may be found in [15].

II. Preliminaries

Throughout the paper we let n and t be integers such that $n \geqslant t \geqslant 1$. We further assume k_+ and k_- are non-negative integers such that $k_+ \geqslant k_- \geqslant 0$. For integers $a \leqslant b$ we define $[a,b] \triangleq \{a,a+1,\dots,b\}$ and $[a,b]^* \triangleq [a,b] \setminus \{0\}$. We use $\mathbb{Z}_m$ to denote the cyclic group of integers with addition modulo m, and $\mathbb{F}_q$ to denote the finite field of size q. Since we shall almost always use just the additive group of the finite field, when p is a prime we shall sometimes write $\mathbb{F}_p$ and sometimes $\mathbb{Z}_p$.

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A *lattice* $\Lambda \subseteq \mathbb{Z}^n$ is an additive subgroup of $\mathbb{Z}^n$. A lattice Λ may be represented by a matrix $\mathcal{G}(\Lambda) \in \mathbb{Z}^{n\times n}$, the span of whose rows (with integer coefficients) is Λ. A *fundamental region* of Λ is defined as

$$\left\{ \sum_{i=1}^{n} c_i \mathbf{v}_i \,\middle|\, c_i \in \mathbb{R}, 0 \leqslant c_i < 1 \right\},$$

where $\mathbf{v}_i$ is the i-th row of $\mathcal{G}(\Lambda)$. It is well known that the volume of the fundamental region is $|\det(\mathcal{G}(\Lambda))|$, and is independent of the choice of $\mathcal{G}(\Lambda)$.

We say $\mathcal{B} \subseteq \mathbb{Z}^n$ *packs* $\mathbb{Z}^n$ by $\Lambda \subseteq \mathbb{Z}^n$, if the translates of $\mathcal{B}$ by elements from Λ do not intersect, namely, for all $\mathbf{v}, \mathbf{v}' \in \Lambda$, $\mathbf{v} \neq \mathbf{v}'$,

$$(\mathbf{v} + \mathcal{B}) \cap (\mathbf{v}' + \mathcal{B}) = \varnothing.$$

We say $\mathcal{B}$ *covers* $\mathbb{Z}^n$ by Λ if

$$\bigcup_{\mathbf{v}\in\Lambda} (\mathbf{v} + \mathcal{B}) = \mathbb{Z}^n.$$

If $\mathcal{B}$ both packs and covers $\mathbb{Z}^n$ by Λ, then we say $\mathcal{B}$ *tiles* $\mathbb{Z}^n$ by Λ. It is well known that if $\mathcal{B}$ packs $\mathbb{Z}^n$ by Λ, and $|\mathcal{B}| = |\det(\mathcal{G}(\Lambda))|$, then $\mathcal{B}$ tiles $\mathbb{Z}^n$ by Λ.

A. Lattice Tiling and Group Splitting

Lattice tiling of $\mathbb{Z}^n$ with $\mathcal{B}(n,t,k_+,k_-)$, in connection with group splitting, has a long history when $t = 1$ (e.g., see [9]), called lattice tiling by crosses if $k_+ = k_-$ (e.g., [10]), semi-crosses when $k_- = 0$ (e.g., [3], [4], [10]), and quasi-crosses when $k_+ \geqslant k_- \geqslant 0$ (e.g., [7], [8]). For an excellent treatment and history, the reader is referred to [12] and the many references therein. Other variations, keeping $t = 1$ include [13], [14]. More recent results may be found in [17] and the references therein.

Since we are interested in codes that correct more than one error, namely, $t \geqslant 2$, an extended definition of group splitting is required.

Definition 1. Let G be a finite Abelian group, where $+$ denotes the group operation. For $m \in \mathbb{Z}$ and $g \in G$, let mg denote $g + g + \cdots + g$ (with m copies of g) when $m > 0$, which is extended in the natural way to $m \leqslant 0$. Let $M \subseteq \mathbb{Z} \setminus \{0\}$ be a finite set, and $S = \{s_1, s_2, \ldots, s_n\} \subseteq G$. We say the set M t-splits G with splitter set S, denoted

$$G = M \diamond_t S$$

if the following two conditions hold:

1) The elements $\mathbf{e} \cdot (s_1, \ldots, s_n)$, where $\mathbf{e} \in (M \cup \{0\})^n$ and $1 \leqslant \mathrm{wt}(\mathbf{e}) \leqslant t$, are all distinct and non-zero in G.
2) For every $g \in G$ there exists a vector $\mathbf{e} \in (M \cup \{0\})^n$, $\mathrm{wt}(\mathbf{e}) \leqslant t$, such that $g = \mathbf{e} \cdot (s_1, \ldots, s_n)$.

Intuitively, $G = M \diamond_t S$ means that the non-trivial linear combinations of elements from S, with at most t non-zero coefficients from M, are distinct and give all the non-zero elements of G exactly once. We note that when $t = 1$, this definition coincides with the definition of splitting used in previous papers.

The following two theorems show the equivalence of t-splittings and lattice tilings, summarizing Lemma 3, Lemma 4, and Corollary 1 in [1]. They generalize the treatment for $t = 1$ in previous works (e.g., see [12]).

Theorem 2 (Lemma 4 and Corollary 1 in [1])**.** *Let G be a finite Abelian group, $M \triangleq [-k_-, k_+]^*$, and $S = \{s_1, \ldots, s_n\} \subseteq G$, such that $G = M \diamond_t S$. Define $\phi : \mathbb{Z}^n \to G$ as $\phi(\mathbf{x}) \triangleq \mathbf{x} \cdot (s_1, \ldots, s_n)$ and let $\Lambda \triangleq \ker \phi$ be a lattice. Then $\mathcal{B}(n,t,k_+,k_-)$ tiles $\mathbb{Z}^n$ by Λ.*

Theorem 3 (Lemma 3 and Corollary 1 in [1])**.** *Let $\Lambda \subseteq \mathbb{Z}^n$ be a lattice, and assume $\mathcal{B}(n,t,k_+,k_-)$ tiles $\mathbb{Z}^n$ by Λ. Then there exists a finite Abelian group G and $S = \{s_1, s_2, \ldots, s_n\} \subseteq G$ such that $G = M \diamond_t S$, where $M \triangleq [-k_-, k_+]^*$.*

III. Construction of Lattice Tilings

In this section we describe a construction for tilings with $\mathcal{B}(n,t,k_+,k_-)$. The method described here takes a linear perfect code in the well known and extensively studied Hamming metric, and uses it to construct the tiling. The obvious downside to this method is the fact that very few perfect codes exist in the Hamming metric (see [6] for more on perfect codes).

Theorem 4. *In the Hamming metric space, let C be a perfect linear $[n, k, 2t+1]$ code over $\mathbb{F}_p$, with p a prime. If $k_+ + k_- + 1 = p$, then*

$$\Lambda \triangleq \{\mathbf{x} \in \mathbb{Z}^n \mid (\mathbf{x} \bmod p) \in C\}$$

is a lattice, and $\mathcal{B}(n,t,k_+,k_-)$ lattice-tiles $\mathbb{Z}^n$ by Λ.

Proof: Directly from its definition, Λ is closed under addition and under multiplication by integers. Thus, Λ is a lattice. Denote $\mathcal{B} \triangleq \mathcal{B}(n,t,k_+,k_-)$, and we now prove $\mathcal{B}$ tiles $\mathbb{Z}^n$ by Λ.

To show packing, assume $\mathbf{v} + \mathbf{e} = \mathbf{v}' + \mathbf{e}'$, for some $\mathbf{v}, \mathbf{v}' \in \Lambda$ and $\mathbf{e}, \mathbf{e}' \in \mathcal{B}$. But then $\mathbf{e} - \mathbf{e}' = \mathbf{v}' - \mathbf{v} \in \Lambda$, and by the definition of Λ, also $\mathbf{e}'' \triangleq ((\mathbf{e} - \mathbf{e}') \bmod p) \in C$. We note that $\mathrm{wt}(\mathbf{e}) \leqslant t$ and $\mathrm{wt}(\mathbf{e}') \leqslant t$, hence $\mathrm{wt}(\mathbf{e}'') \leqslant 2t$. By the minimum distance of C this implies that $\mathbf{e}'' = \mathbf{0}$. Now, since each entry of $\mathbf{e} - \mathbf{e}'$ is in the range $[-(k_+ + k_-), k_+ + k_-]$, and since $k_+ + k_- + 1 = p$, we necessarily have that $\mathbf{e} - \mathbf{e}' = \mathbf{0}$, which in turn implies $\mathbf{v} - \mathbf{v}' = \mathbf{0}$. It follows that translates of $\mathcal{B}$ by Λ pack $\mathbb{Z}^n$.

To show covering, let $\mathbf{x} \in \mathbb{Z}^n$ be any integer vector. Then $\mathbf{x}' \triangleq (\mathbf{x} \bmod p) \in \mathbb{F}_p^n$. Since C is a perfect code, there exists $\mathbf{v}' \in C$ and $\mathbf{e}' \in \mathbb{F}_p^n$, $\mathrm{wt}(\mathbf{e}') \leqslant t$, such that $\mathbf{x}' \equiv \mathbf{v}' + \mathbf{e}' \pmod p$. Since $k_+ + k_- + 1 = p$, there exists $\mathbf{e} \in \mathcal{B}$ such that $\mathbf{e} \bmod p = \mathbf{e}'$. But then $\mathbf{x} - \mathbf{e} \equiv \mathbf{v}' \pmod p$ and by definition $\mathbf{x} - \mathbf{e} \in \Lambda$. Hence, the translates of $\mathcal{B}$ by Λ cover $\mathbb{Z}^n$. ∎

Example 5. Take the $[\frac{p^m-1}{p-1}, \frac{p^m-1}{p-1} - m, 3]$ p-ary Hamming code (p a prime), together with Theorem 4, to obtain a tiling of $\mathbb{Z}^{(p^m-1)/(p-1)}$ by $\mathcal{B}(\frac{p^m-1}{p-1}, 1, k_+, k_-)$, where $k_+ + k_- + 1 = p$. This particular tiling was already described in [7] together with the lattice generator matrix and equivalent splitting.

Example 6. If we use Theorem 4 with the perfect binary linear $[2t+1, 1, 2t+1]$ repetition code, we obtain a lattice tiling of $\mathbb{Z}^{2t+1}$ by $\mathcal{B}(2t+1, t, 1, 0)$. The lattice is spanned by

$$\mathcal{G} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ & 2 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & 2 \end{pmatrix}.$$

When viewed as a splitting, the additive group $\mathbb{F}_2^{2t}$ is t-split as $\mathbb{F}_2^{2t} = \{1\} \diamond_t S$, where $S = \{\mathbf{e}_i \,|\, 1 \leqslant i \leqslant 2t\} \cup \{\mathbf{1}\}$, and where $\mathbf{e}_i$ is the i-th unit vector of length $2t$.

Example 7. Again using Theorem 4 with the $[23, 12, 7]$ binary Golay code, we obtain a lattice tiling of $\mathbb{Z}^{23}$ by $\mathcal{B}(23, 3, 1, 0)$. The lattice Λ is spanned by

$$\mathcal{G} = \begin{pmatrix} I_{12} & G_b \\ \mathbf{0} & 2I_{11} \end{pmatrix},$$

where $\begin{pmatrix} I_{12} & G_b \end{pmatrix}$ is a generator matrix of the $[23, 12, 7]$ binary Golay code, and $2I_{11}$ is an 11×11 matrix with entries on the diagonal being 2 and all the others being 0. Now, we look at the corresponding group splitting. Since $\mathbb{Z}^{23}$ can be spanned by the matrix

$$\begin{pmatrix} I_{12} & G_b \\ \mathbf{0} & I_{11} \end{pmatrix},$$

the quotient group $\mathbb{Z}^{23}/\Lambda$ is isomorphic to the additive group $\mathbb{F}_2^{11}$. Note that

$$\begin{pmatrix} I_{12} & G_b \\ \mathbf{0} & 2I_{11} \end{pmatrix} \begin{pmatrix} G_b \\ I_{11} \end{pmatrix}$$

is a 23×11 all-zero matrix over $\mathbb{F}_2$. The natural homomorphism $\phi : \mathbb{Z}^{23} \to \mathbb{F}_2^{11}$ sends the standard basis to the rows of $\begin{pmatrix} G_b \\ I_{11} \end{pmatrix}$. It follows that $\mathbb{F}_2^{11} = \{1\} \diamond_3 S$, where $S = \{\mathbf{e}_i \,|\, 1 \leqslant i \leqslant 11\} \cup \{\mathbf{r} \,|\, \mathbf{r} \text{ is a row of } G_b\}$.

Example 8. Finally, using Theorem 4 with the $[11, 6, 5]$ ternary Golay code, we obtain a lattice tiling of $\mathbb{Z}^{11}$ by $\mathcal{B}(11, 2, 2, 0)$ or $\mathcal{B}(11, 2, 1, 1)$. The lattice is spanned by

$$\mathcal{G} = \begin{pmatrix} I_6 & G_t \\ \mathbf{0} & 3I_5 \end{pmatrix},$$

where $\begin{pmatrix} I_6 & G_t \end{pmatrix}$ is a generator matrix of the $[11, 6, 5]$ ternary Golay code, and $3I_5$ is a 5×5 matrix with entries on the diagonal being 3 and all the others being 0. When viewed as a splitting, the additive group $\mathbb{F}_3^5$ is 2-split as $\mathbb{F}_3^5 = \{1, 2\} \diamond_2 S$, where $S = \{\mathbf{e}_i \,|\, 1 \leqslant i \leqslant 5\} \cup \{\mathbf{r} \,|\, \mathbf{r} \text{ is a row of } G_t\}$.

Theorem 4 has its dual as well, as shown in the following theorem.

Theorem 9. *Assume $\mathcal{B}(n, t, k_+, k_-)$ lattice-tiles $\mathbb{Z}^n$ by the lattice Λ, with an equivalent t-splitting $\mathbb{F}_p^m = M \diamond_t S$, where $M \triangleq [-k_-, k_+]^*$, p is a prime, and $p = k_+ + k_- + 1$. Then $\Lambda \cap \mathbb{F}_p^n$ is a perfect linear $[n, k, 2t+1]$ code over $\mathbb{F}_p$ in the Hamming metric space.*

IV. Nonexistence Results

The nonexistence results we present in this section are divided into results on general tilings, and results on lattice tilings. The former use mainly geometric arguments, whereas the latter employ algebraic ones.

A. Nonexistence of General Tilings

The first result we present uses a comparison between the density of a tiling of $\mathcal{B}(n, t, k_+, k_-)$ with that of a tiling of a certain notched cube of a lower dimension.

Theorem 10. *For any $n \geqslant t+1$, and $k_+ \geqslant k_- \geqslant 0$ not both 0, if*

$$\sum_{i=0}^{t} \binom{n}{i} (k_+ + k_-)^i < (k_+ + 1)^{t+1} - (k_+ - k_-)^{t+1}$$

then $\mathbb{Z}^n$ cannot be tiled by translates of $\mathcal{B}(n, t, k_+, k_-)$.

Proof: Given integers $n \geqslant t+1$, assume that there is a set $T \subseteq \mathbb{Z}^n$ such that $\mathcal{B} \triangleq \mathcal{B}(n, t, k_+, k_-)$ tiles $\mathbb{Z}^n$ by T. Consider the set

$$A = \Big\{ (x_1, x_2, \dots, x_{t+1}, 0, \dots, 0) \,\Big|\, (x_1, \dots, x_{t+1}) \in [0, k_+]^{t+1} \setminus [k_- + 1, k_+]^{t+1} \Big\}.$$

Hence, if we remove the last $n - t - 1$ zero coordinates, the elements of A are exactly a notched cube, as defined in [1], [11]. Thus, by [1], [11], translates of A tile the space[1]

$$\{(x_1, x_2, \dots, x_{t+1}, 0, \dots, 0) \,|\, x_i \in \mathbb{Z} \text{ for all } 1 \leqslant i \leqslant t+1\}.$$

Trivially, it follows that translates of A can tile the space $\mathbb{Z}^n$.

We now claim that any translate of A contains at most one point from T. Suppose to the contrary that both $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ belong to the intersection $(\mathbf{v} + A) \cap T$, where $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{Z}^n$, and $\mathbf{x} \neq \mathbf{y}$. Then $v_i \leqslant x_i, y_i \leqslant v_i + k_+$ for $1 \leqslant i \leqslant t+1$, $x_i = y_i = v_i$ for $t+2 \leqslant i \leqslant n$, and there are indices $1 \leqslant j_x, j_y \leqslant t+1$ such that $x_{j_x} \leqslant v_{j_x} + k_-$ and $y_{j_y} \leqslant v_{j_y} + k_-$. W.l.o.g., assume that $x_1 \leqslant v_1 + k_-$. We proceed in two cases.

1) If $y_1 \leqslant v_1 + k_-$, let $\mathbf{z} = (z_1, z_2, \dots, z_{t+1}, v_{t+2}, v_{t+3} \dots, v_n)$, where

$$z_1 = \begin{cases} x_1, & \text{if } x_i \leqslant y_i \text{ for all } i = 2, 3, \dots, t+1, \\ y_1, & \text{otherwise,} \end{cases}$$

and

$$z_i = \max\{x_i, y_i\} \text{ for } i = 2, 3, \dots, t+1.$$

Then it is easy to see that

$$\mathbf{z} \in (\mathbf{x} + \mathcal{B}) \cap (\mathbf{y} + \mathcal{B}),$$

a contradiction.

[1] While [1], [11] discuss a tiling of $\mathbb{R}^n$, it is easily seen that the tiling constructed there is in fact a tiling of $\mathbb{Z}^n$ as in our setting.

2) If $y_1 > v_1 + k_-$, then there is $2 \leqslant j \leqslant t+1$ such that $y_j \leqslant v_j + k_-$. W.l.o.g., assume that $y_2 \leqslant v_2 + k_-$ and let $\mathbf{z} = (y_1, z_2, z_3, \dots, z_{t+1}, v_{t+2}, v_{t+3} \dots, v_n)$, where
$$z_2 = \begin{cases} x_2, & \text{if } x_i \leqslant y_i \text{ for all } i = 2,3,\dots,t+1, \\ \max\{x_2, y_2\}, & \text{otherwise,} \end{cases}$$
and $z_i = \max\{x_i, y_i\}$ for $i = 3,4,\dots,t+1$. Again,
$$\mathbf{z} \in (\mathbf{x} + \mathcal{B}) \cap (\mathbf{y} + \mathcal{B}),$$
a contradiction.

We have shown that any translate of A contains at most one point from T, and so the tiling by A is denser than the tiling by $\mathcal{B}$. It follows that the reciprocal of the volume of $\mathcal{B}$ cannot exceed the reciprocal of the volume of A, i.e.,
$$\frac{1}{\sum_{i=0}^{t} \binom{n}{i} (k_+ + k_-)^i} \leqslant \frac{1}{(k_+ + 1)^{t+1} - (k_+ - k_-)^{t+1}}.$$
Rearranging gives us the desired result. ■

Remark 11. If $k_- = k_+(1 - o(1))$, while n and t are fixed, then according to Theorem 10, there is an upper bound on k_+ for which $\mathcal{B}(n,t,k_+,k_-)$ can tile $\mathbb{Z}^n$.

The geometric approach is also used to prove the following two theorems. The first is analogous to that of proper quasi-crosses when $t = 1$, namely, the case when $k_+ > k_- > 0$. The second concerns equal arm length, $k_+ = k_-$. The method used is an elaboration of the one used in the proof of Theorem 12.

Theorem 12. *Let $2t \geqslant n \geqslant t+1$ and $k_+ > k_- > 0$. Then $\mathbb{Z}^n$ cannot be tiled by $\mathcal{B}(n,t,k_+,k_-)$.*

Theorem 13. *Let $k_+ = k_- \geqslant 2$ and $n > t \geqslant (4n-2)/5$. Then for any $n \geqslant 3$, $\mathbb{Z}^n$ cannot be tiled by $\mathcal{B}(n,t,k_+,k_-)$.*

B. Nonexistence of Lattice Tilings

We now turn to the more specific case of lattice tilings. Some of the nonexistence results presented in this section are stated as necessary conditions. The main tool used is Theorem 3, and the algebraic study of the t-splitting. We begin with the lattice-tiling equivalents of Theorem 10.

Theorem 14. *For any $n \geqslant t+1$, and $k_+ \geqslant k_- \geqslant 0$ not both 0, if $\mathcal{B}(n,t,k_+,k_-)$ lattice-tiles $\mathbb{Z}^n$ then*
$$\sum_{i=1}^{t} \binom{n}{i} (k_+ + k_-)^{i-1} \geqslant (k_- + 1)^t.$$

Theorem 15. *Let $n \geqslant 2t$, and $k_+ \geqslant k_- \geqslant 0$. If $\mathcal{B}(n,t,k_+,k_-)$ lattice-tiles $\mathbb{Z}^n$ then*
$$\frac{(k_- + 1)^2}{k_+ + k_- + 1} < \binom{n}{t}^{1/t},$$

Theorem 15 is particularly useful in an asymptotic regime where $t = \Theta(n)$, as shown in the following corollary.

Corollary 16. *If $\alpha \leqslant \frac{t}{n} \leqslant \frac{1}{2}$, $k_+ \geqslant k_- \geqslant 0$, and*
$$\frac{(k_- + 1)^2}{k_+ + k_- + 1} \geqslant \frac{e}{\alpha},$$
then $\mathcal{B}(n,t,k_+,k_-)$ does not lattice-tile $\mathbb{Z}^n$.

We continue on to a few more specific cases. The next two theorems deal with the analogue of semi-crosses when $t = 1$, namely, the case of $k_- = 0$.

Theorem 17. *Let $2 \leqslant t < n/4$ and $k_+ > k_- = 0$. Then $\mathcal{B}(n,t,k_+,0)$ cannot lattice-tile $\mathbb{Z}^n$ when*
$$k_+ \geqslant 2\binom{n}{t} - 2.$$

Unlike the other proofs in this section, the next one uses a geometric argument.

Theorem 18. *Let $\frac{2}{3}(n-1) \leqslant t \leqslant n-3$. Then $\mathcal{B}(n,t,k_+,0)$ cannot lattice-tile $\mathbb{Z}^n$ when $k_+ \geqslant 2$.*

Continuing our specialization, we turn to tackle the case of $t = 2$, and present a strong restriction on the dimension n.

Theorem 19. *For any $k_+ \geqslant k_- \geqslant 0$, if $\mathcal{B}(n,2,k_+,k_-)$ lattice-tiles $\mathbb{Z}^n$ and also $|\mathcal{B}(n,2,k_+,k_-)|$ is even, then*
$$n = \frac{4\ell^2 - (k_+ + k_- - 3)^2 + 8}{4(k_+ + k_-)},$$
for some $\ell \in \mathbb{Z}$.

Proof: By Theorem 3 there exists an Abelian group G whose size is $|G| = |\mathcal{B}(n,2,k_+,k_-)|$ such that $G = M \diamond_2 S$ for some $S \subseteq G$, $|S| = n$, where $M \triangleq [-k_-, k_+]^*$. Since G is Abelian and of even order, necessarily $G = \mathbb{Z}_{2^r} \times G'$, for some $r \geqslant 1$. We may therefore write any element $g \in G$ as a pair (a, b) where $a \in \mathbb{Z}_{2^\ell}$ and $b \in G'$, and we say g is *even* if $a \equiv 0 \pmod 2$, and *odd* otherwise.

Denote by n_1 the number of odd elements in S. Additionally, denote by $m_0 \triangleq \lfloor k_+/2 \rfloor + \lfloor k_-/2 \rfloor$ (respectively, $m_1 \triangleq \lceil k_+/2 \rceil + \lceil k_-/2 \rceil$) the number of even (respectively, odd) numbers in M.

Let us examine how the $\frac{1}{2}(\binom{n}{2}(k_+ + k_-)^2 + n(k_+ + k_-) + 1)$ odd elements of G are obtained via the 2-splitting. There are three possible ways:

1) An odd element in S times an odd number in M.
2) An odd element in S times an odd number in M, plus an even element in S times any number from M.
3) An odd element in S times an odd number in M, plus a different odd element in S times an even number from M.

Thus,
$$n_1 m_1 + n_1 m_1 (n - n_1)(m_0 + m_1) + n_1 m_1 (n_1 - 1) m_0 = \frac{1}{2}\left(\binom{n}{2}(m_0 + m_1)^2 + n(m_0 + m_1) + 1\right).$$
Solving for n_1 we obtain
$$n_1 = \frac{n(m_0 + m_1) - m_0 + 1 \pm \sqrt{n(m_1^2 - m_0^2) + m_0^2 - 2m_0 - 1}}{2m_1}.$$

We recall that $m_0 + m_1 = k_+ + k_-$. Additionally,

$$|\mathcal{B}(n,2,k_+,k_-)| = \binom{n}{2}(k_+ + k_-)^2 + n(k_+ + k_-) + 1$$

is even, which implies that $k_+ + k_-$ is odd, and then $m_1 - m_0 = 1$. It follows that $m_1^2 - m_0^2 = (m_1 - m_0)(m_1 + m_0) = m_1 + m_0 = k_+ + k_-$. Substituting back, we use the fact that the square root must be an integer $\ell \in \mathbb{Z}$ to obtain the desired claim after some simple rearranging. ∎

Finally, we focus on the smallest case not studied before – tiling $\mathcal{B}(n,2,1,0)$. In this case, by a careful study of the possible group splittings we obtain a full classification of possible tilings.

Theorem 20. *Let $n \geqslant 3$. Then $\mathcal{B}(n,2,1,0)$ lattice-tiles $\mathbb{Z}^n$ only when $n \in \{3,5\}$, and only by 2-splitting $\mathbb{Z}_7$ and 2-splitting $\mathbb{F}_2^4$, respectively.*

Using a similar method, we now direct our attention to the case of $\mathcal{B}(n,2,2,0)$.

Theorem 21. *Let $n \geqslant 3$, then $\mathcal{B}(n,2,2,0)$ lattice-tiles $\mathbb{Z}^n$ only when $n \in \{3,11\}$, and only by 2-splitting $\mathbb{Z}_{19}$ and 2-splitting $\mathbb{F}_3^5$, respectively.*

V. Conclusion

In this paper we studied general tilings as well as lattice tilings of $\mathbb{Z}^n$ with $\mathcal{B}(n,t,k_+,k_-)$. These may act as perfect error-correcting codes over a channel with at most t limited-magnitude errors. We constructed such lattice tilings from perfect codes in the Hamming metric, and provided several non-existence results. We summarize some of our non-existence results for lattice tilings below, where it is interesting to note the difference between the cases of $\frac{t}{n} < \frac{1}{2}$ and $\frac{t}{n} \geqslant \frac{1}{2}$.

Corollary 22. *Let $2 \leqslant t < n/2$, and $k_+ \geqslant k_- \geqslant 0$ not both 0. Then $\mathcal{B}(n,t,k_+,k_-)$ cannot lattice-tile $\mathbb{Z}^n$ when one of the following holds:*

1) $\frac{(k_-+1)^2}{k_++k_-+1} \geqslant \binom{n}{t}^{1/t}$.
2) $2 \leqslant t < n/4$, $k_- = 0$ *and* $k_+ \geqslant 2\binom{n}{t} - 2$.
3) $t = 2$, $k_- = 0$, $k_+ = 1$ *and* $n \neq 5$.
4) $t = 2$, $k_- = 0$, $k_+ = 2$ *and* $n \neq 11$.

Corollary 23. *Let $2 \leqslant t < n \leqslant 2t$, and $k_+ \geqslant k_- \geqslant 0$ not both 0. If $\mathcal{B}(n,t,k_+,k_-)$ lattice-tiles $\mathbb{Z}^n$, then one of the following holds:*

1) $k_- = 0$ *and one of the following holds:*
   a) $t = n - 1$ *(such tilings have been constructed in [1], [11]);*
   b) $(2n-2)/3 \leqslant t \leqslant n-3$ *and* $k_+ = 1$*;*
   c) $n/2 \leqslant t < (2n-2)/3$*;*
2) $k_+ = k_-$ *and one of the following holds:*
   a) $(4n-2)/5 \leqslant t \leqslant n-1$ *and* $k_+ = k_- = 1$*;*
   b) $n/2 \leqslant t < (4n-2)/5$ *and* $\sum_{i=1}^{t} \binom{n}{i}(2k_+)^{i-1} \geqslant (k_+ + 1)^t$.

It is also interesting to compare the results here, when $t \geqslant 2$, with the known results for $t = 1$. The non-existence results we have here rely heavily on geometric arguments, or general algebraic arguments. The notable exceptions are Theorem 20 and Theorem 21, which carefully study the structure of the group being split. This is in contrast with the strong non-existence results when $t = 1$, due to the fact that when $t = 1$, if G is split then so is the cyclic group of the same size, $\mathbb{Z}_{|G|}$. This does not hold when $t \geqslant 2$, as evident, for example, during the proof of Theorem 20, where $\mathbb{F}_2^4$ is 2-split but $\mathbb{Z}_{16}$ is not.

Whether some strong statement may be said about the structure of the group being split, remains as an open question for further research. It is also interesting to ask whether more t-splittings exist, namely, whether t-splittings exist which are not derived from perfect codes in the Hamming metric. Finally, it remains open whether any other non-lattice tilings of $\mathcal{B}(n,t,k_+,k_-)$ exist.

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