An Improved Bound for Optimal Locally Repairable Codes

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***Abstract*—The Singleton-type bound that provides an upper limit on the minimum distance of locally repairable codes is studied. An improved bound is presented by carefully analyzing the combinatorial structure of the repair sets. Thus, we show the previous bound is unachievable for certain parameters. Additionally, as a byproduct, some previously known codes are shown to attain the new bound and are thus proved to be optimal.**

I. Introduction

Due to the ever-growing need for more efficient and scalable systems for cloud storage and data storage in general, *distributed storage systems* (DSSs) (such as the Google data centers and Amazon Clouds) have become increasingly important. In a distributed storage system, a data file is stored at a distributed collection of storage devices/nodes in a network. Since any storage device is individually unreliable and subject to failure, redundancy must be introduced to provide the much-needed system-level protection against data loss due to device/node failure.

In today's large distributed storage systems, where node failures are the norm rather than the exception, designing codes that have good distributed repair properties has become a central problem. Several cost metrics and related tradeoffs have been studied in the literature, for example *repair bandwidth* [6], [7], *disk-I/O* [27], and *repair locality* [6], [9], [16]. In this paper *repair locality* is the subject of interest.

Motivated by the desire to reduce repair cost in the design of erasure codes for distributed storage systems, the notions of *symbol locality* and *locally repairable codes* (LRC) were introduced in [9] and [17], respectively. The ith coded symbol of an $[n, k]$ linear code $\mathcal{C}$ is said to have locality r if it can be recovered by accessing at most r other symbols in $\mathcal{C}$. Alternatively, the ith code symbol with the r other symbols form a 1-erasure correcting code. The concept was further generalized to (r, δ)-locality by Prakash *et al.* [18] to address the situation of multiple device failures. Here, the ith coordinate, together with $r + \delta - 2$ other coordinates, form a code capable of correcting $\delta - 1$ erasures. When $\delta = 2$ this coincides with the definition of locality.

There are two types of linear codes with (r, δ)-locality considered in the literature. The first is *information symbol locality*, pertaining to systematic linear codes whose information symbols all have (r, δ)-locality (denoted by $(r, \delta)_i$-locality for short). The second is of *all-symbol locality* (or $(r, \delta)_a$-locality) pertaining to linear codes all of whose symbols have (r, δ)-locality.

For any $[n, k, d]_q$ linear code with minimum Hamming distance d over the finite field $\mathbb{F}_q$, the Singleton bound [22] is given by

$$d \leqslant n - k + 1, \tag{1}$$

which is one of the most classical theorems in coding theory. This bound was generalized for locally repairable codes in [9] (the case $\delta = 2$) and [18] (general δ) as follows. An $[n, k, d]_q$-linear LRC with $(r, \delta)_i$-locality satisfies

$$d \leqslant n - k - 1 - \left(\left\lceil \frac{k}{r} \right\rceil - 1\right)(\delta - 1). \tag{2}$$

It was also proved that a class of codes known as pyramid codes [11] achieves this bound when the alphabet is sufficiently large, say $q \geqslant n + 1$ and $d \geqslant \delta$ (for a weaker field-size requirement please refer to [4]). Since a linear code with $(r, \delta)_a$-locality is also a linear code with $(r, \delta)_i$-locality, (2) also presents an upper bound for the minimum Hamming distance of $(r, \delta)_a$ codes. Other bounds for linear and nonlinear LRCs can be found in [1], [5], [10], [17], [19], [20], [25], [28], [30]. An LRC is *optimal* if it has the highest minimum Hamming distance of any code of the given parameters n, k, r, and δ. In this paper, we focus on Singleton-type bounds (like (1) and (2) above) and their corresponding optimal codes.

There are different constructions of LRCs that are optimal in the sense that they achieve the Singleton-type bound in (2), *e.g.*, [3], [12], [13], [15], [18], [21], [23], [24], [26]. Tamo *et al.* [26] showed that the r-locality of a linear LRC is a matroid invariant, which was used to prove that the minimum Hamming distance of a class of linear LRCs achieves the Singleton-type bound. In [24], Tamo and Barg introduced an interesting construction that can generate optimal linear codes with $(r,\delta)_a$-locality over an alphabet of size $O(n)$. Under the assumption of a sufficiently large alphabet, Song *et al.* [23] investigated for which parameters (n,k,r,δ) there exists a linear LRC with all-symbol locality and minimum Hamming distance d achieving the Singleton-type bound (2). The parameter set (n,k,r,δ) was divided into eight different cases. In four of these cases it was proved that there are linear LRCs achieving the bound, in two of these cases it was proved that there are no linear LRCs achieving the bound, and the existence of linear LRCs achieving the bound in the remaining two cases remained an open problem. Independently of [23], Wang and Zhang [28] used a linear-programming approach to strengthen these result when $\delta=2$. Ernvall *et al.* [8] presented methods to modify already existing codes, and gave constructions for three infinite classes of optimal vector-linear LRCs with all-symbol locality over an alphabet of small size. Recently, Westerbäck *et al.* [29] provided a link between matroid theory and LRCs that are either linear or more generally almost affine, and derived new existence results for linear LRCs and nonexistence results for almost affine LRCs, which strengthened the results for linear LRCs given in [23].

Thus, in general, the bound in (2) is not tight for LRCs with $(r,\delta)_a$-locality, even under the assumption of having a sufficiently large finite field. In this paper, we further study the Hamming distance of LRCs with $(r,\delta)_a$-locality. The main idea is to take the structure of the repair sets into consideration when we analyze the relationship between the parameters of LRCs. We distinguish between three main cases, where in each case, a bound on the Hamming distance is derived using structural properties of repair sets, independently. Combining all these cases, we derive a bound on the minimum Hamming distance, that improves upon (2). As a consequence, the improved bound shows that some previously undecided cases are in fact unachievable for the bound in (2). The improved bound can also prove some LRCs based on matroids in [29] are indeed optimal. In Fig. 1, we extend and refine the summary appearing in [23], and show the known and new results concerning the tightness of the Singleton-type bound for LRCs under the assumption that the alphabet is sufficiently large.

Due to space limitations we omit all proofs, which are available in a full version of this work [2].

II. Preliminaries

Let $\mathcal{C}$ be an $[n,k,d]_q$ linear code over the finite field $\mathbb{F}_q$. Assume $\mathcal{C}$ has a generator matrix $G=(\mathbf{g}_1,\mathbf{g}_2,\dots,\mathbf{g}_n)$, where $\mathbf{g}_i\in\mathbb{F}_q^k$ is a column vector for $i=1,2,\dots,n$. While many different generator matrices exist for $\mathcal{C}$, in what follows, the choice of G is immaterial. Given $\mathcal{C}$ and the matrix G, we introduce some notation and concepts.

For an integer $n\in\mathbb{N}$ we denote $[n]=\{1,2,\dots,n\}$. For any set $N\subseteq[n]$, we denote $\mathcal{G}_N=\{\mathbf{g}_i : i\in N\}$. Then $\mathrm{span}(N)$ denotes the linear space spanned by $\mathcal{G}_N$ over $\mathbb{F}_q$, and $\mathrm{rank}(N)$ denotes the dimension of $\mathrm{span}(N)$. Additionally, $\mathcal{C}_N$ denotes the *punctured code* of $\mathcal{C}$ associated with the coordinate set N. That is, $\mathcal{C}_N$ is obtained from $\mathcal{C}$ by deleting all symbols in the coordinates $[n]\setminus N$.

The following lemma describes a useful fact about $[n,k,d]_q$ linear codes, which plays an important role in our paper.

Lemma 1 ([14])**:** The minimum Hamming distance of any $[n,k,d]_q$ linear codes satisfies

$$d=n-\max\{|N| : N\subseteq[n],\mathrm{rank}(N)<k\}.$$

We now recall the definition of repair sets, and locally repairable codes.

Definition 1 ([18])**:** Let $\mathcal{C}$ be an $[n,k,d]_q$ code. For $1\leqslant r\leqslant k$ and $\delta\geqslant 2$, an (r,δ)-*repair set* of $\mathcal{C}$ is a subset $S\subseteq[n]$ such that

1) $|S|\leqslant r+\delta-1$;
2) For every $l\in S$, $L\subseteq S\setminus\{l\}$ and $|L|=|S|-(\delta-1)$, c_l is a linear function of $\{c_i : i\in L\}$, where $\mathbf{c}=(c_1,\cdots,c_n)\in\mathcal{C}$.

We say that $\mathcal{C}$ is a *locally repairable code (LRC) with all-symbol (r,δ)-locality* (or $\mathcal{C}$ is an LRC with $(r,\delta)_a$-locality) if all the n symbols of the code are contained in at least one (r,δ)-repair set.

Remark 1 ([23], [29])**:** Note that the symbols in an (r,δ)-repair set S can be used to recover up to $\delta-1$ erasures in the same repair set, and therefore each of the following statements are equivalent to Definition 1, item 2):

1) For any $L\subseteq S$ with $|L|=|S|-(\delta-1)$, we have $\mathrm{rank}(L)=\mathrm{rank}(S)$;
2) For any $l\in S$, $L\subseteq S\setminus\{l\}$ and $|L|=|S|-(\delta-1)$, we have $|\mathcal{C}_{L\cup\{l\}}|=|\mathcal{C}_L|$;
3) For any $L\subseteq S$ with $|L|\geqslant|S|-(\delta-1)$, we have $|\mathcal{C}_L|=|\mathcal{C}_S|$;
4) $d(\mathcal{C}_S)\geqslant\delta$, where $d(\mathcal{C}_S)$ is the minimum Hamming distance of $\mathcal{C}_S$.

In what follows, whenever we speak of an LRC with $(r,\delta)_a$-locality, we will by default assume it is an $[n,k,d]_q$ linear code (i.e., its length is n, its dimension is k, its minimum Hamming distance is d, and its alphabet size is q).

III. Properties of LRCs with $(r,\delta)_a$-Locality

The goal of this section is to study the structure of (r,δ)-repair sets induced by $(r,\delta)_a$-locality, and propose some properties which can be used to obtain an upper bound on the minimum Hamming distance in the next section. Generally speaking, we would like to find a set that contains as many code coordinates as possible, under the condition that its rank does not exceed $k-1$. To this end, we distinguish among three cases. The relationship between repair sets, the number of code symbols, and their rank, is easy to determine for the first case (refer to Proposition 2). The remaining two cases are reduced to the first case in Propositions 3-5.

Throughout the paper we assume that $\mathcal{C}$ denotes an $[n,k,d]_q$ LRC with $(r,\delta)_a$-locality. The parameters n and k are written in the following forms:

$$\begin{aligned} n&=w(r+\delta-1)+m, && 0\leqslant m<r+\delta-1,\\ k&=ur+v, && 0<v\leqslant r, \end{aligned} \tag{3}$$

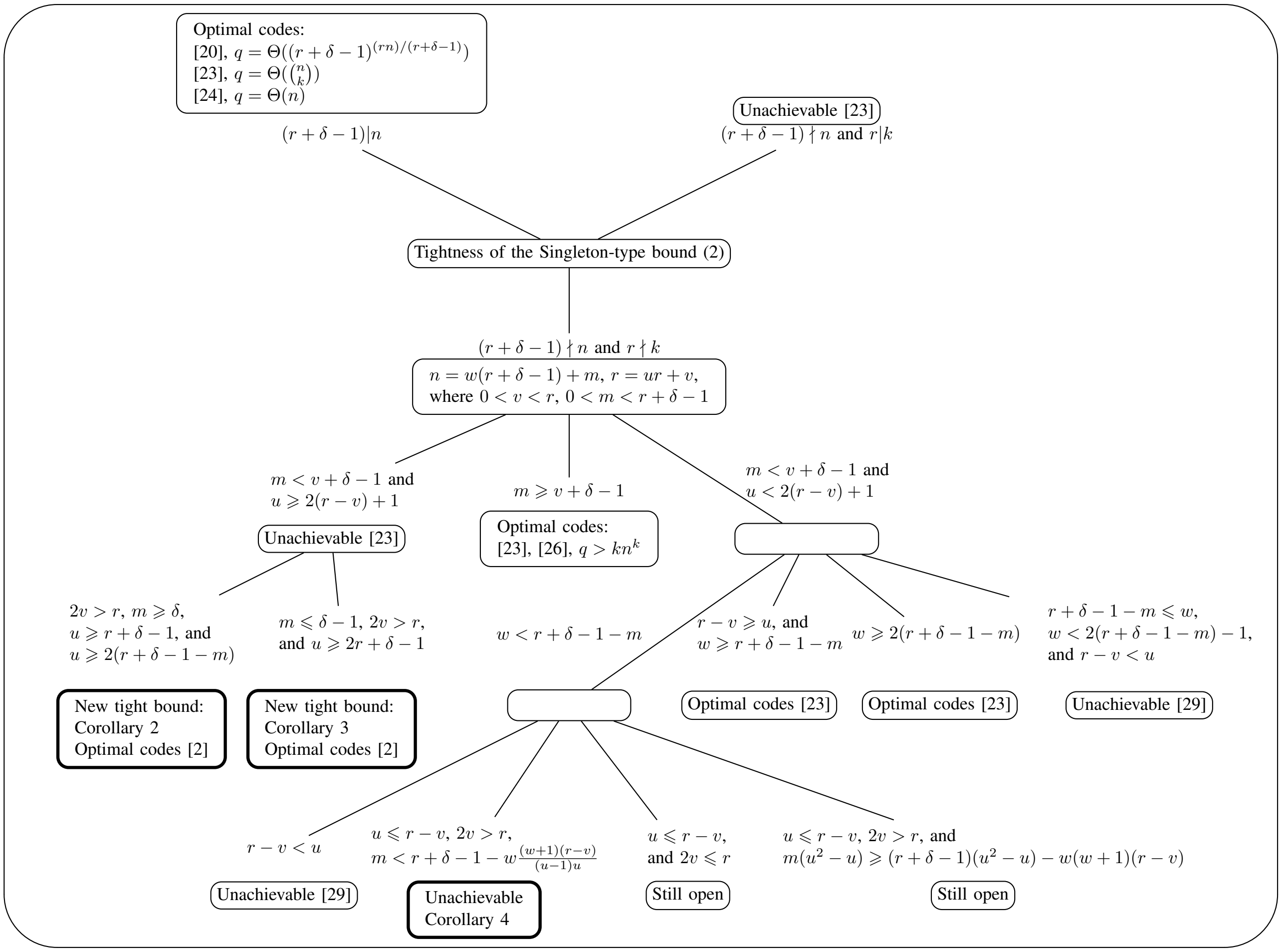

Fig. 1: The tightness of the Singleton-type bound for LRC in (2), where $n=w(r+\delta-1)+m$, $0\leqslant m<r+\delta-1$, $k=ur+v$, and $0<v\leqslant r$. The new contributions of this paper appear in bold frames. We do not consider the case $u=0$, i.e., $k=r$, since this is exactly the case of the classic Singleton bound.

where w,m,u,v are nonnegative integers. Observe that we represent k as $ur+v$ with $0<v\leqslant r$ to make sure that $ur<k$.

Definition 2 ([3])**:** Let $n,T,s\in\mathbb{N}$. Additionally, let $\mathcal{X}$ be a set of cardinality n, whose elements are called *points*. Finally, let $\mathcal{B}=\{B_1,B_2,\dots,B_T\}\subseteq 2^{\mathcal{X}}$ be a set of *blocks* such that $\bigcup_{i\in[T]}B_i=\mathcal{X}$, and for all $i\in[T]$, $|B_i|\leqslant s$ and $\bigcup_{j\in T\setminus\{i\}}B_j\neq\mathcal{X}$. We then say $(\mathcal{X},\mathcal{B})$ is an (n,T,s)-*essential covering family (ECF)*. If all blocks are the same size we say $(\mathcal{X},\mathcal{B})$ is a *uniform* (n,T,s)-ECF.

For an LRC with $(r,\delta)_a$-locality, note that each code symbol may be contained in more than one repair set. Thus, to simplify the discussion, we first use the (r,δ)-repair sets to form an ECF, which can be naturally obtained from Definition 1 and Remark 1, as described in [3].

Lemma 2 ([3])**:** For any $[n,k]_q$ linear code $\mathcal{C}$ with $(r,\delta)_a$-locality, let $\Gamma\subseteq 2^{[n]}$ be the set of all possible (r,δ)-repair sets. Then we can find a subset $\mathcal{S}\subseteq\Gamma$ such that $([n],\mathcal{S})$ is an $(n,|\mathcal{S}|,r+\delta-1)$-ECF with $|\mathcal{S}|\geqslant\lceil\frac{k}{r}\rceil$.

Remark 2: The fact that the components of $\mathcal{S}$ cover all the element of $[n]$ implies that

$$|\mathcal{S}|\geqslant\left\lceil\frac{n}{r+\delta-1}\right\rceil=w+\left\lceil\frac{m}{r+\delta-1}\right\rceil\geqslant w.$$

In particular, $|\mathcal{S}|=w$ if and only if $m=0$, $\mathcal{S}$ is uniform, and the repair sets in $\mathcal{S}$ form a partition of $[n]$.

Let $\mathcal{V}$ be a subset of the set $\mathcal{S}$ that was obtained in Lemma 2. We observe that $\mathcal{V}$ must satisfy at least one of the following three conditions:

- **C1**: $\left|S_i\cap\left(\bigcup_{S_j\in\mathcal{V}\setminus\{S_i\}}S_j\right)\right|<|S_i|-\delta+1$ for any $S_i\in\mathcal{V}$;
- **C2**: $|S_i\cap S_j|<\min\{|S_i|,|S_j|\}-\delta+1$ for any distinct $S_i,S_j\in\mathcal{V}$;
- **C3**: there exist two distinct $S_i,S_j\in\mathcal{V}$, such that $|S_i\cap S_j|\geqslant\min\{|S_i|,|S_j|\}-\delta+1$.

In fact, since Conditions C2 and C3 are complementary, exactly one of them holds, and perhaps Condition C1 holds as well.

The following definitions introduce concepts required in several of our claims.

Definition 3: Assume $r, \delta \geqslant 1$ are fixed. For all integers $a \geqslant r+\delta-1$, $b \geqslant 0$ we define the function $\Phi(a,b)$ as follows:

$$\Phi(a,b) = \begin{cases} \min\{r+\delta-1-c, \max\{\left\lfloor \frac{b}{2} \right\rfloor, \\ \qquad \left\lceil \frac{b(b-1)(r+\delta-1-c)}{(\ell+1)\ell} \right\rceil\}\} & \text{if } c \neq 0, \\ 0 & \text{if } c = 0, \end{cases}$$

and where c denotes the minimum nonnegative integer with $c \equiv a \bmod (r+\delta-1)$, and $\ell = \left\lfloor \frac{a}{r+\delta-1} \right\rfloor$.

Definition 4: Let $\mathcal{S}$ denote the ECF induced by an LRC with $(r,\delta)_a$-locality via Lemma 2, and let $\mathcal{V} \subseteq \mathcal{S}$ be some subset of it. We define

$$\Upsilon(\mathcal{V},\mathcal{S}) = \left(\bigcup_{S_i \in \mathcal{V}} S_i \right) \setminus \left(\bigcup_{S_j \in \mathcal{S} \setminus \mathcal{V}} S_j \right)$$

and denote

$$M(\mathcal{V},\mathcal{S}) = |\Upsilon(\mathcal{V},\mathcal{S})|.$$

We now present a sequence of results on the structure of $\mathcal{S}$, depending at times on which of Conditions C1-C3 it satisfies.

Proposition 1: For any integer $0 \leqslant t \leqslant |\mathcal{S}|$, there exists a t-subset $\mathcal{V}$ of $\mathcal{S}$ such that

$$|\mathcal{V}|(r+\delta-1) - \left| \bigcup_{S_i \in \mathcal{V}} S_i \right| \geqslant \Phi(n,t).$$

Proposition 2 ([3, Lemma 7])**:** Let $\mathcal{V}$ be a subset of $\mathcal{S}$ such that $\mathcal{V}$ satisfies Condition C1. Then

$$\operatorname{rank}\left(\bigcup_{S_i \in \mathcal{V}} S_i \right) \leqslant \left| \bigcup_{S_i \in \mathcal{V}} S_i \right| - |\mathcal{V}|(\delta-1).$$

Proposition 3: Let $\mathcal{V}$ be a subset of $\mathcal{S}$ such that $\mathcal{V}$ satisfies Condition C2, but not Condition C1. Then there exists a subset $\mathcal{V}^* \subseteq \mathcal{V}$, such that

1) $\mathcal{V}^*$ satisfies Condition C1;
2) $|\mathcal{V}^*|(r+\delta-1) - \left|\bigcup_{S_i \in \mathcal{V}^*} S_i\right| \geqslant \lceil r/2 \rceil$.

Proposition 4: Let $\mathcal{V}$ be a subset of $\mathcal{S}$ such that $\mathcal{V}$ satisfies Condition C3. Then there exists a pair of subsets $\mathcal{V}_1^* \subseteq \mathcal{V}_1 \subseteq \mathcal{S}$ such that:

1) $\mathcal{V}_1 \setminus \mathcal{V}_1^*$ satisfies Condition C1;
2) For any $S_j \in \mathcal{V}_1 \setminus \mathcal{V}_1^*$, there exists $S_i \in \mathcal{V}_1^*$, such that $\operatorname{span}(S_i) \subseteq \operatorname{span}(S_j)$;
3) $\mathcal{S} \setminus \mathcal{V}_1^*$ satisfies Condition C2.

Proposition 5: Assume the same setting as in Proposition 4, and let $\mathcal{V}_1^* \subseteq \mathcal{V}_1 \subseteq \mathcal{S}$ be the subsets guaranteed there. Denote $\Upsilon = \Upsilon(\mathcal{V}_1^*,\mathcal{S})$ and $M = M(\mathcal{V}_1^*,\mathcal{S})$. Then

1) $\mathcal{G}_\Upsilon \subseteq \operatorname{span}(\bigcup_{S_i \in \mathcal{V}_1 \setminus \mathcal{V}_1^*} S_i)$;
2) $|\mathcal{G}_\Upsilon \cap \operatorname{span}(\bigcup_{S_i \in \mathcal{U}} S_i)| \geqslant |\mathcal{U}|$, for any subset $\mathcal{U} \subseteq \mathcal{V}_1 \setminus \mathcal{V}_1^*$;
3) $|\mathcal{V}_1^*| \leqslant M$, $|\mathcal{V}_1 \setminus \mathcal{V}_1^*| \leqslant M$, and $|\mathcal{V}_1| \leqslant 2M$.

IV. An Improved Bound

Having laid the foundation in the previous section, we now use the structure of the repair sets, together with Lemma 1, to obtain an upper bound on the minimum Hamming distance of an LRC with $(r,\delta)_a$-locality. Thus, we aim to find a subset $S \subseteq [n]$ with $\operatorname{rank}(S) = k-1$, whose size is as large as possible. We find such a set for the general case in Proposition 6 below. We then describe our main bound in Theorem 1.

Throughout this section, we still assume that $\mathcal{C}$ is an $[n,k,d]_q$ linear code with $(r,\delta)_a$-locality, and $\mathcal{S}$ is the ECF given by Lemma 2. The parameters n and k are written as in (3).

Proposition 6: If the requirements of Proposition 4 hold, let $\mathcal{V}_1^* \subseteq \mathcal{V}_1 \subseteq \mathcal{S}$ by the two guaranteed sets, and otherwise set $\mathcal{V}_1 = \mathcal{V}_1^* = \emptyset$. Denote $M = M(\mathcal{V}_1^*,\mathcal{S})$. Then there exists a subset $S \subseteq [n]$ with $\operatorname{rank}(S) = k-1$, and

$$\begin{aligned} &|S| - k + 1 \\ &\geqslant \begin{cases} \min\left\{ \left(\left\lceil \frac{k + \lceil \frac{r}{2} \rceil}{r} \right\rceil - 1 \right)(\delta-1), \right. \\ \quad \left. M + \left(\left\lceil \frac{k + \Phi(n-M,u-M)}{r} \right\rceil - 1 \right)(\delta-1) \right\}, & \text{if } u > M, \\ u + \left(\left\lceil \frac{k}{r} \right\rceil - 1 \right)(\delta-1), & \text{if } u \leqslant M, \end{cases} \end{aligned}$$

where $\Phi(\cdot,\cdot)$ is from Definition 3.

Now we are ready to obtain an upper bound on the minimum Hamming distance.

Theorem 1: Let $\mathcal{C}$ be an LRC with $(r,\delta)_a$-locality, and let $\mathcal{S}$ be the ECF given by Lemma 2. If the requirements of Proposition 4 hold, let $\mathcal{V}_1^* \subseteq \mathcal{V}_1 \subseteq \mathcal{S}$ by the two guaranteed sets, and otherwise set $\mathcal{V}_1 = \mathcal{V}_1^* = \emptyset$. Denote $M = M(\mathcal{V}_1^*,\mathcal{S})$. Then

$$\begin{aligned} &n - k + 1 - d \\ &\geqslant \begin{cases} \min\left\{ \left(\left\lceil \frac{k + \lceil \frac{r}{2} \rceil}{r} \right\rceil - 1 \right)(\delta-1), \right. \\ \quad \left. M + \left(\left\lceil \frac{k + \Phi(n-M,u-M)}{r} \right\rceil - 1 \right)(\delta-1) \right\}, & \text{if } u > M, \\ \left(u + \left(\left\lceil \frac{k}{r} \right\rceil - 1 \right)(\delta-1)\right), & \text{if } u \leqslant M, \end{cases} \end{aligned}$$

where $\Phi(\cdot,\cdot)$ is from Definition 3.

Remark 3: We point out that the subsets $\mathcal{V}_1^* \subseteq \mathcal{V}_1 \subseteq \mathcal{S}$, whose existence is guaranteed in Proposition 4, are not necessarily unique. Thus, the value of M used in Theorem 1 is not unique as well. Of the (possibly many) choices for M, it is unclear which one results in the best bound.

Remark 4: We make the following observations:

1) If $M = 0$, the bound in Theorem 1 becomes

$$\begin{aligned} &n - k + 1 - d \\ &\geqslant \left(\left\lceil \frac{k + \min\left\{ \left\lceil \frac{r}{2} \right\rceil, \Phi(n,u) \right\}}{r} \right\rceil - 1 \right)(\delta-1), \end{aligned}$$

which is tighter than the one given by (2) (see, [9], [18]) if and only if

$$\min\left\{ \left\lceil \frac{r}{2} \right\rceil, \Phi(n,u) \right\} > r - v.$$

In particular, the bound is exactly the one in (2) when $m = 0$, and it is tighter than the one in (2) when $m \neq 0$ and $v = r$.

2) If $M \neq 0$ and $k > r$, the bound in Theorem 1 is tighter than the bound in (2) if and only if

$$\left\lceil \frac{r}{2} \right\rceil > r - v.$$

In particular, the bound is tighter than the one in (2) when $v = r$, i.e., $r \mid k$ and $k > r$.

V. Case Analysis of the Improved Bound

The new bound of Theorem 1 depends on many parameters. In this section we highlight interesting cases of parameters for this bound. Generally, we should consider all possible M in Theorem 1 to determine the upper bound on d, where M depends on the structure of the (r, δ)-repair sets, i.e., $\mathcal{S}$. However, for some special cases the expression for the bound can be further simplified.

We again assume that $\mathcal{C}$ is an $[n, k, d]_q$ linear code with $(r, \delta)_a$-locality, and $\mathcal{S}$ is the ECF given by Lemma 2. The parameters n and k are written as in (3).

Corollary 1: If an $[n, k, d]_q$ LRC with $(r, \delta)_a$-locality satisfies that the repair sets in $\mathcal{S}$ are pairwise disjoint, then

$$d \leqslant n - k + 1 - \left(\left\lceil \frac{k + \Phi(n, u)}{r} - 1 \right\rceil \right) (\delta - 1).$$

In [29], Westerbäck *et al.* studied locally repairable codes via matroid theory, and obtained the following bound for $d_{\max}$, where $d_{\max}$ is the largest d such that there exists a linear $[n, k, d]_q$ code with $(r, \delta)_a$-locality.

Theorem 2 ([29, Theorem 36-(ii)])**:** Assume $r+\delta-1 \nmid n$ and $r \nmid k$, namely, $m > 0$ and $v < r$. If $0 < r < k \leqslant n - \left\lceil \frac{k}{r} \right\rceil (\delta - 1)$ and $v > m - \delta + 1$, then

$$d_{\max} \geqslant n-k+1-\left\lceil \frac{k}{r} \right\rceil (\delta-1)+\begin{cases} 0, & \text{if } m \geqslant \delta, \\ \delta - 1 - m, & \text{if } m \leqslant \delta - 1, \end{cases}$$

where $d_{\max}$ denotes the maximal integer d such that there exists a linear $[n, k, d]_q$ code with $(r, \delta)_a$-locality.

By applying the bound obtained in Theorem 1, we may now determine the exact value of $d_{\max}$ for certain classes of parameters.

Corollary 2: Under the setting of Theorem 2, if $m \geqslant \delta$, $r > v > \max\left\{m - \delta + 1, \left\lfloor \frac{r}{2} \right\rfloor\right\}$, and $u \geqslant \max\{2(r+\delta-1-m), r + \delta - 1\}$, we have

$$d \leqslant n - k + 1 - \left\lceil \frac{k}{r} \right\rceil (\delta - 1).$$

Corollary 3: Under the setting of Theorem 2, if $m \leqslant \delta - 1$, $r > v > \left\lfloor \frac{r}{2} \right\rfloor$, and $u \geqslant 2r + \delta - 1$, we have

$$d \leqslant n - k + 1 - \left\lceil \frac{k}{r} \right\rceil (\delta - 1) + (\delta - 1 - m).$$

We can now strengthen Theorem 2 by applying Corollaries 2 and 3.

Theorem 3: Assume $r + \delta - 1 \nmid n$ and $r \nmid k$, namely, $m > 0$ and $v < r$. If $0 < r < k \leqslant n - \left\lceil \frac{k}{r} \right\rceil (\delta - 1)$ and $v > \max\left\{m - \delta + 1, \left\lfloor \frac{r}{2} \right\rfloor\right\}$, then

$$d_{\max} - n + k - 1 + \left\lceil \frac{k}{r} \right\rceil (\delta - 1)$$
$$= \begin{cases} 0, \\ \quad \text{if } m \geqslant \delta \text{ and } u \geqslant \max\{2(r+\delta-1-m), r+\delta-1\}, \\ \delta - 1 - m, \\ \quad \text{if } m \leqslant \delta - 1 \text{ and } u \geqslant 2r + \delta - 1, \end{cases}$$

where $d_{\max}$ denotes the maximal integer d such that there exists a linear $[n, k, d]_q$ code with $(r, \delta)_a$-locality.

Corollaries 2 and 3 can be viewed as explicit examples that the bound in Theorem 1 out-performs the known one in (2).

Based on the results in [23], [29], the remaining open cases for the tightness of the bound in (2) are summarized in the following:

Open Problem [23]: Do there exist optimal $[n, k, d]_q$ codes with $(r, \delta)_a$-locality that achieve the minimum Hamming distance bound in (2), under the conditions that $v \neq 0$, $0 < m < v + \delta - 1$, $0 < u \leqslant r - v$, and $w < r + \delta - 1 - m$? (using the notation of (3))

We can answer this open question in part.

Corollary 4: No $[n, k, d]_q$ code with $(r, \delta)_a$-locality achieves the bound in (2) under the conditions of $0 < m < v + \delta - 1$, and $u > 1$, if

$$\min\left\{ \left\lceil \frac{r}{2} \right\rceil, \frac{u(u-1)(r+\delta-1-m)}{(w+1)w} \right\} > r - v.$$

In particular, when $v > \frac{r}{2}$, $u > 1$, and $0 < m < r + \delta - 1 - w\frac{(w+1)(r-v)}{u(u-1)}$, the bound in (2) is unachievable.

Remark 5: By Corollary 4, the remaining open cases can be listed as:

1) $0 < v \leqslant \frac{r}{2}$, $0 < m < v + \delta - 1$, $1 \leqslant u \leqslant r - v$, and $w < r + \delta - 1 - m$.
2) $v > \frac{r}{2}$, $(r + \delta - 1)(u(u-1)) - w(w+1)(r-v) \leqslant mu(u-1)$, $0 < m < v + \delta - 1$, $1 \leqslant u \leqslant r - v$, and $w < r + \delta - 1 - m$.

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