New Bounds on the Capacity of Multi-dimensional RLL-Constrained Systems

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Abstract. We examine the well-known problem of determining the capacity of multi-dimensional run-length-limited constrained systems. By recasting the problem, which is essentially a combinatorial counting problem, into a probabilistic setting, we are able to derive new lower and upper bounds on the capacity of $(0, k)$-RLL systems. These bounds are better than all previously-known bounds for $k \geqslant 2$, and are even tight asymptotically. Thus, we settle the open question: what is the rate at which the capacity of $(0, k)$-RLL systems converges to 1 as $k \to \infty$? While doing so, we also provide the first ever non-trivial upper bound on the capacity of general (d, k)-RLL systems.

1 Introduction

A (d, k)-RLL constrained system is the set of all binary sequences in which every two adjacent 1's are separated by at least d zeroes, and no more than k 0's appear consecutively. The study of these systems was initiated by Shannon [10, 11] who defined the capacity of a constrained system S as

$$\mathsf{cap}(S) = \lim_{n \to \infty} \frac{\log_2 |S(n)|}{n} \ ,$$

where $S(n)$ denotes the number of sequences of S of length exactly n.

Constrained systems are widely used today in all manners of storage systems [7, 8]. However, the emergence of two-dimensional recording systems brought to light the need for two-dimensional and even multi-dimensional constrained systems. A two-dimensional (d, k)-RLL constrained system is the set of all binary arrays in which every row and every column obeys the one-dimensional (d, k)-RLL constraint. The generalization to the D-dimensional case is obvious, and we denote such a system as $S^D_{d,k}$. Though we consider in this paper only symmetrical constrains, i.e., the same d and k along every dimension, the results generalize easily to asymmetrical RLL constraints as well.

In the one-dimensional case it is well known that $\mathsf{cap}(S^1_{d,k})$, for $0 \leqslant d \leqslant k$, is the logarithm in base 2 of the largest positive root of the polynomial

$$x^{k+1} - x^{k-d} - x^{k-d-1} - \cdots - x - 1 \ .$$

However, unlike the one-dimensional case, almost nothing is known about the two-dimensional case, and even less in the multi-dimensional case. In [1], Calkin

M. Fossorier et al. (Eds.): AAECC 2006, LNCS 3857, pp. 225–234, 2006.

and Wilf gave a numerical estimation method for the capacity of the two-dimensional $(0,1)$-RLL constraint which gives,

$$0.5878911617 \leqslant \mathsf{cap}(S^2_{0,1}) \leqslant 0.5878911618 \ .$$

Their method ingeniously uses the fact that the transfer matrix is symmetric, but unfortunately, this happens only for the case of $(0,1)$-RLL (and by inverting all the bits, the equivalent $(1,\infty)$-RLL case). Using the same method in the three-dimensional case, it was shown in [9] that

$$0.522501741838 \leqslant \mathsf{cap}(S^3_{0,1}) \leqslant 0.526880847825 \ .$$

Some general bounds on the capacity were given in [5]. Using bit-stuffing encoders, the best known lower bounds on two-dimensional (d,∞)-RLL were shown in [2]. Amazingly, we still do not know the exact capacity of the multi-dimensional RLL-constraint except when it is zero [3].

The bounds we improve upon in this work are those of two-dimensional $(0,k)$-RLL, $k \geqslant 2$. These are given in the following three theorems:

Theorem 1 (Theorem 3, [5]). *For every positive integer k,*

$$\mathsf{cap}(S^2_{0,k}) \geqslant 1 - \frac{1-\mathsf{cap}(S^2_{0,1})}{\lceil k/2 \rceil} \ .$$

Theorem 2 ([12]). *For all integers $k \geqslant 8$,*

$$\mathsf{cap}(S^2_{0,k}) \geqslant 1 + \frac{\log_2(1-(\lfloor k/2 \rfloor+1)2^{-(\lfloor k/2 \rfloor -1)})}{(\lfloor k/2 \rfloor+1)^2} \ .$$

Theorem 3 (Theorem 7, [5]). *For every positive integer k,*

$$\mathsf{cap}(S^2_{0,k}) \leqslant 1 - \frac{1}{k+1}\log_2\left(\frac{1}{1-2^{-(k+1)}}\right) \ .$$

Our new bounds are given in Theorem 6 and Theorem 13. A numerical comparison with the previously-best bounds for $2 \leqslant k \leqslant 10$ is given in Table 1. Furthermore, our lower and upper bounds agree asymptotically, thus settling the open question of the rate of convergence to 1 of $\mathsf{cap}(S^D_{0,k})$ as $k \to \infty$ by showing it to be $\frac{D \log_2 e}{4 \cdot 2^k}$.

Our approach to the problem of bounding the capacity is to recast the problem from a combinatorial counting problem to a probability bounding problem. Suppose we randomly select a sequence of length n with uniform distribution. Let A^S_n denote the event that this sequence is in the constrained system S. Then the total number of sequences in S of length n may be easily written as

$$|S(n)| = \Pr[A^S_n] \cdot 2^n \ .$$

It follows that

$$\mathsf{cap}(S) = \lim_{n\to\infty} \frac{\log_2 |S(n)|}{n} = \lim_{n\to\infty} \frac{\log_2(\Pr[A^S_n]2^n)}{n} = \lim_{n\to\infty} \frac{\log_2 \Pr[A^S_n]}{n} + 1 \ .$$

Table 1. Comparison of lower bounds (LB) and upper bounds (UB) on $\mathsf{cap}(S^2_{0,k})$, for $2 \leqslant k \leqslant 10$. Lower and upper bounds are rounded down and up, respectively, to six decimal digits.

k	LB by [5]	LB by [12]	LB by Theorem 6	UB by Theorem 13	UB by [5]
2	0.587891		0.758292	0.904373	0.935785
3	0.793945		0.893554	0.947949	0.976723
4	0.793945		0.950450	0.970467	0.990840
5	0.862630		0.976217	0.983338	0.996214
6	0.862630		0.988383	0.990816	0.998384
7	0.896972		0.994268	0.995068	0.999295
8	0.896972	0.943398	0.997155	0.997410	0.999687
9	0.917578	0.943398	0.998583	0.998663	0.999860
10	0.917578	0.981164	0.999293	0.999318	0.999936

This translates in a straightforward manner to higher dimensions as well. By calculating or bounding $\Pr[A^S_n]$, we may get the exact capacity or bounds on it, which is the basis for what is to follow.

The work is organized as follows. In Section 2 we use monotone families to achieve lower bounds on $\mathsf{cap}(S^D_{0,k})$ and an upper bound on $\mathsf{cap}(S^D_{d,k})$. While this method may also be used to lower bound $\mathsf{cap}(S^D_{d,\infty})$, the resulting bound is extremely weak. We continue in Section 3 by deriving an upper bound on $\mathsf{cap}(S^D_{0,k})$ using a large-deviation bound for sums of nearly-independent random variables. We conclude in Section 4 by discussing the asymptotics of our new bounds and comparing them with the case of (d,∞)-RLL.

2 Bounds from Monotone Families

We can use monotone increasing and decreasing families to find new lower bounds on the capacity of $(0,k)$-RLL, and a new upper bound on the capacity of (d,k)-RLL, $d \geqslant 1$. We start with the definition of these families.

Definition 4. *Let $n > 0$ be some integer, and $[n]$ denote the set $\{1,2,\dots,n\}$. A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be* monotone increasing *if when $A \in F$ and $A \subseteq A' \subseteq [n]$, then $A' \in F$. It is said to be* monotone decreasing *if when $A \in F$ and $A' \subseteq A$, then $A' \in F$.*

The following theorem is due to Kleitman [6]:

Theorem 5. *Let $\mathcal{A}$, $\mathcal{B}$ be monotone increasing families, and $\mathcal{C}$, $\mathcal{D}$ be monotone decreasing families. Let X be a random variable describing a uniformly-distributed random choice of subset of $[n]$ out of the 2^n possible subsets. Then,*

$$\Pr[X \in \mathcal{A} \cap \mathcal{B}] \geqslant \Pr[X \in \mathcal{A}] \cdot \Pr[X \in \mathcal{B}] \ , \quad (1)$$

$$\Pr[X \in \mathcal{C} \cap \mathcal{D}] \geqslant \Pr[X \in \mathcal{C}] \cdot \Pr[X \in \mathcal{D}] \ , \quad (2)$$

$$\Pr[X \in \mathcal{A} \cap \mathcal{C}] \leqslant \Pr[X \in \mathcal{A}] \cdot \Pr[X \in \mathcal{C}] \ . \quad (3)$$

We can now apply Kleitman's theorem to $(0,k)$-RLL constrained systems:

Theorem 6. *For all integers* $k \geqslant 0$, $\mathsf{cap}(S^2_{0,k}) \geqslant 2\mathsf{cap}(S^1_{0,k}) - 1$.

Proof. The constrained system we examine is $S = S^2_{0,k}$, and with our notation, A^S_n denotes the event that a randomly chosen $n \times n$ array is $(0,k)$-RLL.

We now define two closely related constraints. Let R denote the set of two-dimensional arrays in which every **row** is $(0,k)$-RLL, and C denote the set of two-dimensional arrays in which every **column** is $(0,k)$-RLL. Similarly we define the events A^R_n and A^C_n. By definition,

$$A^S_n = A^R_n \cap A^C_n \ .$$

It is easy to verify that both constraints R and C are monotone increasing families. Hence, by Theorem 5,

$$\Pr[A^S_n] = \Pr[A^R_n \cap A^C_n] \geqslant \Pr[A^R_n] \Pr[A^C_n] \ .$$

It follows that,

$$\mathsf{cap}(S) = \lim_{n\to\infty} \frac{\log_2 \Pr[A^S_n]}{n^2} + 1 \geqslant \lim_{n\to\infty} \frac{\log_2(\Pr[A^R_n] \Pr[A^C_n])}{n^2} + 1 \ . \qquad (4)$$

Now, both $\Pr[A^R_n]$ and $\Pr[A^C_n]$ may be easily expressed in terms of one-dimensional constrained systems. An $n \times n$ binary array chosen randomly with uniform distribution is equivalent to a set of n^2 i.i.d. random variables for each of the array's bits, each having a "1" with probability 1/2. Thus,

$$\Pr[A^R_n] = \Pr[A^C_n] = \left(\Pr[A^{S'}_n]\right)^n \ ,$$

where $S' = S^1_{0,k}$ is the one-dimensional $(0,k)$-RLL constraint. Plugging this back into (4) we get

$$\mathsf{cap}(S^2_{0,k}) \geqslant \lim_{n\to\infty} \frac{2\log_2 \Pr[A^{S'}_n]}{n} + 1 = 2\mathsf{cap}(S^1_{0,k}) - 1 \ .$$

□

This is generalized to higher dimensions in the following corollary.

Corollary 7. *Let* $D_1, D_2 \geqslant 1$ *be integers, then*

$$\mathsf{cap}(S^{D_1+D_2}_{0,k}) \geqslant \mathsf{cap}(S^{D_1}_{0,k}) + \mathsf{cap}(S^{D_2}_{0,k}) - 1 \ .$$

We note that similar lower bounds may be given for the (d,∞)-RLL constraint, since such arrays form a monotone decreasing family. However, the resulting bounds are very weak. We can also mix monotone increasing and decreasing families to get the following result.

Theorem 8. *Let* $D \geqslant 1$ *be some integer, and* $k \geqslant d$ *also integers, then*

$$\mathsf{cap}(S^D_{d,k}) \leqslant \mathsf{cap}(S^D_{d,\infty}) + \mathsf{cap}(S^D_{0,k}) - 1 \ .$$

Proof. Omitted. □

3 New Upper Bounds

In this section we present upper bounds on the capacity of $(0,k)$-RLL. Unlike the previous section, these bounds are explicit. For this purpose we introduce a new probability bound. It is derived from the bound by Janson [4], but by requiring some symmetry, which applies in our case, we can make the bound stronger.

Suppose that ξ_i, $i \in [n]$, is a family of independent 0–1 random variables. Let $\mathcal{S} \subseteq [n]^{\leqslant k}$, where $[n]^{\leqslant k}$ denotes the set of all subsets of $[n]$ of size at most k. We then define the following indicator random variables,

$$I_A = \begin{cases} \prod_{i \in A} \xi_i & A \in \mathcal{S} \ , \\ 0 & A \notin \mathcal{S} \ . \end{cases}$$

For $A, B \in \mathcal{S}$, we denote $A \sim B$ if $A \neq B$ and $A \cap B \neq \emptyset$. Let $X = \sum_{A \in \mathcal{S}} I_A$, and define

$$\Delta = \sum_{A} \sum_{B \sim A} \Pr[I_A = 1 \wedge I_B = 1] \ .$$

Janson [4] gave the following bound:

Theorem 9. *With the setting as defined above, let* $\mu = E(X) = \sum_A E(I_A)$, *then*

$$\Pr[X = 0] \leqslant e^{-\frac{\mu^2}{\mu + \Delta}} \ .$$

Our goal is to use Theorem 9 to show an upper bound on the capacity of two-dimensional $(0,k)$-RLL systems. If $S(n,m)$ denotes the number of two-dimensional $(0,k)$-RLL arrays of size $n \times m$ then by definition,

$$\mathsf{cap}(S^2_{0,k}) = \lim_{n,m \to \infty} \frac{\log_2 |S(n,m)|}{nm} \ .$$

However, it would be more convenient to work in a more symmetric setting. In a sense, positions which are close enough to the edge of the array are "less constrained" than others lying within the array. We overcome this difficulty by considering cyclic $(0,k)$-RLL arrays.

We say that a binary $n \times m$ array $\mathcal{A}$ is *cyclic* $(0,k)$*-RLL* if there does not exist $0 \leqslant i \leqslant n-1$, $0 \leqslant j \leqslant m-1$ such that $\mathcal{A}_{i,j} = \mathcal{A}_{i+1,j} = \cdots = \mathcal{A}_{i+k,j} = 0$ or $\mathcal{A}_{i,j} = \mathcal{A}_{i,j+1} = \cdots = \mathcal{A}_{i,j+k} = 0$, where the indices are taken modulo n and m respectively. We denote the set of all such $n \times m$ arrays as $S_{\mathrm{c}}(n,m)$. The next lemma shows that by restricting ourselves to cyclic $(0,k)$-RLL arrays, we do not change the capacity.

Lemma 10. *For all positive integers* k*,*

$$\mathsf{cap}(S^2_{0,k}) = \lim_{n,m \to \infty} \frac{\log_2 |S_{\mathrm{c}}(n,m)|}{nm} \ .$$

Proof. Omitted. □

We start by considering a random $n \times n$ binary array, chosen with uniform distribution, which is equivalent to saying that we have an array of n^2 i.i.d. 0–1 random variables $\xi_{i,j}$, $0 \leqslant i,j \leqslant n-1$, with $\xi_{i,j} \sim Be(1/2)$.

For the remainder of this section, we invert the bits of the array, or equivalently, we say that an array is $(0,k)$-RLL if it does not contain $k+1$ consecutive 1's along any row or column. Furthermore, by Lemma 10, we consider only cyclic $(0,k)$-RLL arrays. Suppose we define the following subsets of coordinates of the arrays:

$$\mathcal{S}_{\mathrm{V}} = \{\{(i,j),(i+1,j),\dots,(i+k,j)\} \mid 0 \leqslant i,j \leqslant n-1\} \ ,$$
$$\mathcal{S}_{\mathrm{H}} = \{\{(i,j),(i,j+1),\dots,(i,j+k)\} \mid 0 \leqslant i,j \leqslant n-1\} \ ,$$
$$\mathcal{S} = \mathcal{S}_{\mathrm{V}} \cup \mathcal{S}_{\mathrm{H}} \ ,$$

where all the coordinates are taken modulo n. We now define the following indicator random variables

$$I_A = \prod_{(i,j)\in A} \xi_{i,j} \qquad\qquad \text{for all } A \in \mathcal{S} \ .$$

If $I_A = 1$ for some $A \in \mathcal{S}$, we have a forbidden event of $k+1$ consecutive 1's along a row or a column. Finally, we count the number of forbidden events in the random array by defining $X = \sum_{A\in\mathcal{S}} I_A$. It is now clear that the probability that this random array is $(0,k)$-RLL is simply

$$\Pr[A_n^{S^2_{0,k}}] = \Pr[X=0] \ .$$

It is easy to be convinced that this setting agrees with the requirements of Theorem 9. All we have to do now to upper bound $\Pr[X=0]$, is to calculate μ and Δ. We note that X is the sum of $2n^2$ indicator random variables, so by linearity of expectation,

$$\mu = E(X) = \frac{1}{2^{k+1}} \cdot 2n^2 = \frac{n^2}{2^k} \ ,$$

since each of the indicator random variables has probability exactly $1/2^{k+1}$ of being 1. Calculating Δ is equally easy,

$$\begin{aligned}\Delta &= \sum_A \sum_{B\sim A} \Pr[I_A = 1 \wedge I_B = 1] = 2n^2 \left((k+1)^2 \frac{1}{2^{2k+1}} + 2\sum_{i=1}^{k} \frac{1}{2^{k+1+i}} \right) \\ &= n^2 \left(\frac{(k+1)^2}{2^{2k}} + \frac{2}{2^k}\left(1 - \frac{1}{2^k}\right)\right) \ .\end{aligned}$$

By Theorem 9,

$$\Pr[X=0] \leqslant e^{-\frac{\mu^2}{\mu+\Delta}} = e^{-\frac{n^2}{3\cdot 2^k + (k+1)^2 - 2}} \ ,$$

which immediately gives us:

$$\mathsf{cap}(S^2_{0,k}) \leqslant 1 - \frac{\log_2 e}{3\cdot 2^k + (k+1)^2 - 2} \ . \qquad (5)$$

The bound of (5) is already better than the best known bounds for $k \geqslant 2$ given in [5]. But we can do even better by improving the bound of Theorem 9. This is achieved by assuming some more symmetry than the general setting of the theorem. Given some $A \in \mathcal{S} \subseteq [n]^{\leqslant k}$, let $X_A = I_A + \sum_{B \sim A} I_B$. We define

$$\Gamma_A = \sum_i \frac{\Pr[X_A = i \mid I_A = 1]}{i} \ .$$

If Γ_A does not depend on the choice of $A \in \mathcal{S}$, we simply denote it as Γ.

Theorem 11. *With the setting as defined above, let $\mu = E(X) = \sum_A E(I_A)$. If the distribution of X_A given $I_A = 1$ does not depend on the choice of A, then*

$$\Pr[X = 0] \leqslant e^{-\mu\Gamma} \ .$$

Proof. Omitted. □

It is obvious that the symmetry requirements of Theorem 11 hold in our case. So now, in order to apply Theorem 11 we have to calculate Γ, which is a little more difficult than calculating Δ. Since Γ does not depend on the choice of A, we arbitrarily choose the horizontal set of coordinates

$$A = \{(0,0), (0,1), \ldots, (0,k)\} \ .$$

We now have to calculate $\Pr[X_A = i \mid I_A = 1]$. We note that we can partition the set $\{B \mid B \sim A\}$ into the following disjoint subsets:

$$\{B \mid B \sim A\} = \mathcal{S}_{\mathrm{HL}} \cup \mathcal{S}_{\mathrm{HR}} \cup \mathcal{S}_{\mathrm{V},0} \cup \mathcal{S}_{\mathrm{V},1} \cup \cdots \cup \mathcal{S}_{\mathrm{V},k} \ ,$$

where

$$\begin{aligned}
\mathcal{S}_{\mathrm{HL}} &= \{B \in \mathcal{S}_{\mathrm{H}} - \{A\} \mid (0,0) \in B\} \ , \\
\mathcal{S}_{\mathrm{HR}} &= \{B \in \mathcal{S}_{\mathrm{H}} - \{A\} \mid (0,k) \in B\} \ , \\
\mathcal{S}_{\mathrm{V},j} &= \{B \in \mathcal{S}_{\mathrm{V}} \mid (0,j) \in B\} \ , \qquad\qquad \text{for all } 0 \leqslant j \leqslant k \ .
\end{aligned}$$

We define $X_{\mathrm{HL}} = \sum_{B \in \mathcal{S}_{\mathrm{HL}}} I_B$, and in a similar fashion, X_{HR} and $X_{\mathrm{V},j}$ for all $0 \leqslant j \leqslant k$. Since the indicators for elements from different subsets are independent given $I_A = 1$ because their intersection contains only coordinates from A, it follows that X_{HL}, X_{HR} and $X_{\mathrm{V},j}$, $0 \leqslant j \leqslant k$, are independent given $I_A = 1$.

The distribution of X_{HL} and X_{HR} given $I_A = 1$ is easily seen to be

$$\Pr[X_{\mathrm{HL}} = i \mid I_A = 1] = \Pr[X_{\mathrm{HR}} = i \mid I_A = 1] = \begin{cases} \frac{1}{2^{i+1}} & 0 \leqslant i \leqslant k-1 \\ \frac{1}{2^k} & i = k \end{cases}$$

since the 0 closest to A determines the number of runs of 1's of length $k+1$. We denote

$$f_k^{\|}(i) = 2^k \Pr[X_{\mathrm{HL}} = i \mid I_A = 1] = 2^k \Pr[X_{\mathrm{HR}} = i \mid I_A = 1] \ .$$

For the distribution of $X_{\mathrm{V},j}$ we need the following lemma.

Lemma 12. *Let $f_k^{\perp}(i)$ denote the number of binary strings of length $2k+1$ with their middle position a 1, and which contain exactly $0 \leqslant i \leqslant k+1$ runs of $k+1$ 1's. Then,*

$$f_k^{\perp}(i) = \begin{cases} 2^{2k} - (k+2)2^{k-1} & i = 0 \\ (k-i+4)2^{k-i-1} & 1 \leqslant i \leqslant k \\ 1 & i = k+1 \end{cases} .$$

Proof. Omitted. □

Using this lemma, we can now say that

$$\Pr[X_{\mathrm{V},j} = i \mid I_A = 1] = \frac{f_k^{\perp}(i)}{2^{2k}} .$$

Since $X_A = X_{\mathrm{HL}} + X_{\mathrm{HR}} + \sum_{j=0}^{k} X_{\mathrm{V},j} + I_A$, we have that

$$\begin{aligned} &\Pr[X_A = i \mid I_A = 1] \\ &= \sum_{\substack{i_L+i_R+i_0+\ldots+i_k=i-1 \\ 0\leqslant i_L, i_R \leqslant k \\ 0 \leqslant i_0,\ldots,i_k \leqslant k+1}} \Pr[X_{\mathrm{HL}} = i_L \mid I_A = 1] \Pr[X_{\mathrm{HR}} = i_R \mid I_A = 1] \\ &\quad \cdot \prod_{j=0}^{k} \Pr[X_{\mathrm{V},j} = i_j \mid I_A = 1] . \end{aligned}$$

It follows that

$$\Gamma = \sum_{i \geqslant 1} \frac{1}{i} \sum_{\substack{i_L+i_R+i_0+\ldots+i_k=i-1 \\ 0\leqslant i_L, i_R \leqslant k \\ 0 \leqslant i_0,\ldots,i_k \leqslant k+1}} \frac{f_k^{\|}(i_L) f_k^{\|}(i_R)}{2^{2k}} \prod_{j=0}^{k} \frac{f_k^{\perp}(i_j)}{2^{2k}} . \quad (6)$$

We can now apply Theorem 11 and get that

$$\Pr[X = 0] \leqslant e^{-n^2 \Gamma / 2^k} ,$$

where Γ is given by (6). This immediately gives us the following theorem.

Theorem 13. *Let $k \geqslant 1$ be some integer, then*

$$\mathsf{cap}(S_{0,k}^2) \leqslant 1 - \frac{\log_2 e}{2^k} \Gamma ,$$

where Γ is given by (6)

We can make the bound of Theorem 13 weaker for small values of k, but more analytically appealing for an asymptotic analysis. This is achieved by noting that $f_k^{\perp}(0)/2^{2k}$ is almost 1 for large values of k.

Theorem 14. *Let $k \geqslant 1$ be some integer, then*

$$\mathsf{cap}(S^2_{0,k}) \leqslant 1 - \frac{\log_2 e}{2^k}\left(\frac{1}{2} - \frac{1}{2^{k+1}}\right)(1-(k+2)2^{-(k+1)})^{k+1} \ .$$

Proof. Omitted. □

We can generalize both Theorem 13 and Theorem 14, and for simplicity, show just the latter in the following theorem.

Theorem 15. *Let $D \geqslant 2$ and $k \geqslant 1$ be some integers, then*

$$\mathsf{cap}(S^D_{0,k}) \leqslant 1 - \frac{D\log_2 e}{2 \cdot 2^k}\left(\frac{1}{2} - \frac{1}{2^{k+1}}\right)(1-(k+2)2^{-(k+1)})^{(D-1)(k+1)} \ .$$

4 Conclusion

In this work we showed new lower and upper bounds on the multi-dimensional capacity of $(0,k)$-RLL systems, as well as a new upper bound on the capacity of (d,k)-RLL systems. We conclude with an interesting comparison of the asymptotes of our new bounds with those of the best previously known bounds. We examine the rate of convergence to 1 of $\mathsf{cap}(S^2_{0,k})$ as $k \to \infty$. The best asymptotic bounds were given in [5]:

$$\frac{\log_2 e}{2(k+1)2^k} < 1 - \mathsf{cap}(S^2_{0,k}) \leqslant \frac{4\sqrt{2}\log_2 e}{(k+1)2^{k/2}} + \frac{8}{2^k} \ ,$$

for sufficiently large k. Our bounds, given in Theorem 6 and Theorem 14, show:

$$\frac{\log_2 e}{2^k}\left(\frac{1}{2} - \frac{1}{2^{k+1}}\right)(1-(k+2)2^{-(k+1)})^{k+1} \leqslant 1 - \mathsf{cap}(S^2_{0,k}) \leqslant 2(1 - \mathsf{cap}(S^1_{0,k}))$$

for all integers $k \geqslant 1$. As mentioned in [5], the one-dimensional capacity of $(0,k)$-RLL converges to 1 when $k \to \infty$ as $\frac{\log_2 e}{4 \cdot 2^k}$. Hence, our lower and upper bounds agree asymptotically and the rate of convergence to 1 of $\mathsf{cap}(S^2_{0,k})$ as $k \to \infty$ is $\frac{\log_2 e}{2 \cdot 2^k}$. In the D-dimensional case this rate becomes $\frac{D\log_2 e}{4 \cdot 2^k}$.

It is also interesting to make a comparison with (d,∞)-RLL. While $\mathsf{cap}(S^2_{d,\infty})$ converges to 0 as $\frac{\log_2 d}{d}$, just as it does in one dimension, for D-dimensional $(0,k)$-RLL the capacity converges to 1 slower than the one-dimensional case by a factor of D.

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