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Optimal permutation anticodes with the infinity norm via permanents of (0, 1)-matrices

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ABSTRACT

Motivated by the set–antiset method for codes over permutations under the infinity norm, we study anticodes under this metric. For half of the parameter range we classify all the optimal anticodes, which is equivalent to finding the maximum permanent of certain $(0, 1)$-matrices. For the rest of the cases we show constraints on the structure of optimal anticodes.

1. Introduction

Perhaps the first to pioneer the study of permutation theory from a coding perspective were Chadwick and Kurz [7], and Deza and Frankl [11]. The object under study in their work, called a *code* or a *permutation array*, is a set of permutations for which any two distinct members are at distance at least d apart. Codes over permutations have attracted recent interest due to power line communication [8,16,20,28] and storage schemes for flash memories [5,17,19,27]. Moreover, in [4,27] permutation groups were considered as error-correcting codes.

Since a distance measure is involved in the definition of a code, a proper choice of metric is important. There are many well-known metrics over the symmetric group S_n (see [10]), of which the Hamming metric is by far the most studied. However, the infinity metric induced by the infinity norm has received recent interest due to its applications [19,27], and will serve as the metric of choice in this work. We also assume throughout, that the symmetric group S_n is acting on $\{1, 2, \dots, n\}$ (though this is usually a matter of convenience, it is important for the definition of the ℓ_∞-distance). The ℓ_∞-distance between two permutations $f, g \in S_n$ is defined as

$$d(f, g) = \max_{1 \leqslant i \leqslant n} \left| f(i) - g(i) \right|.$$

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In analogy to the definition of a code, a subset $A \subseteq S_n$ is an *anticode* with maximal distance d, if any two of its members are at distance at most d apart. In the context of coding theory, the first use of anticodes was in [14,26] (see also [21] for an overview). The anticodes were in fact multisets (allowing repeated words), and were used to construct codes that attain the Griesmer bound with equality. As a purely combinatorial question, anticodes (though not under this name) appear in earlier works, such as the celebrated Erdős–Ko–Rado Theorem on t-intersecting families [12]. Many other variations on the ambient space and distance measure have created a wealth of anticodes, see for example [2,13, 15,22,25].

The following theorem, which is sometimes referred to as the *set–antiset theorem*, motivates us to explore anticodes of maximum size. This theorem was also proved before over different spaces and distance measures (see [1,9,11,27]), the most general case of which is perhaps Delsarte's treatment of association schemes [9] with the well-known special case of the clique–coclique bound for vertex-transitive graphs.

Theorem 1. *(See [27, Theorem 13].) Let $C, A \subseteq S_n$ be a code and an anticode under the ℓ_∞-metric, with minimal distance d and maximal distance $d-1$, respectively. Then*

$$|C| \cdot |A| \leqslant |S_n| = n!.$$

It should be noted that balls are just a special case of anticodes, since a ball of radius r is an anticode with maximal distance $2r$. The size of balls in S_n under the ℓ_∞-metric has been studied in [18,24].

It is well known (see [10]) that the ℓ_∞-metric over S_n is right invariant, i.e., for any $f, g, h \in S_n$, $d(f, g) = d(fh, gh)$. Hence, w.l.o.g., one can assume that any code or anticode contains the identity permutation simply by taking a translation, and we shall assume so throughout the paper.

Any anticode $A \subseteq S_n$ of maximum distance $d-1$ defines a $(0,1)$-matrix $A^* = (a_{i,j})$ of order n, for which $a_{i,j} = 1$ iff there exists $f \in A$ such that $f(i) = j$. We note that A^* has the property that if $a_{i,j} = 1$ then $|i - j| \leqslant d-1$. Moreover, the $(0,1)$-matrix $A^* = (a_{i,j})$ defines an anticode B with maximum distance $d-1$ by $B = \{f \in S_n\colon\ a_{i,f(i)} = 1 \text{ for all } i \in [n]\}$. Note that $A \subseteq B$ and that the size of B is the permanent of the matrix A^*, which is defined by

$$\operatorname{per}(A^*) = \sum_{f \in S_n} \prod_{i=1}^{n} a_{i,f(i)} = |B|.$$

Let Γ_n^d denote the set of $(0,1)$-matrices of order n with exactly d non-zero entries in each row which form a contiguous block. Let A^* be a $(0,1)$-matrix defined by an anticode A, then by the previous observation, the set of non-zero entries in A^* is a subset of the non-zero entries of some matrix in Γ_n^d. Thus, every anticode A with maximum permanent is equivalent to a matrix $A^* \in \Gamma_n^d$.

The goal of this paper is to study the structure of matrices that attain the maximum permanent, i.e., the set of matrices

$$M_n^d = \{A \in \Gamma_n^d\colon\ \operatorname{per}(A) \geqslant \operatorname{per}(B) \text{ for all } B \in \Gamma_n^d\},$$

and to calculate the value of the maximum permanent.

Similar questions regarding the value of the maximum permanent and the matrices that attain it, have been studied for other sets of matrices. Perhaps the most related is the study of constant line-sum $(0,1)$-matrices, in which the number of non-zero entries in each row and each column is equal. This is still an open problem, first stated by Minc [23], and more recently studied by Wanless [29].

The problem was partly solved, both for constant line-sum $(0,1)$-matrices, and for the matrices studied in this paper, Γ_n^d, by Brégman [6], who showed that if A is a $(0,1)$-matrix of order n and row sums $d_1, d_2, \dots, d_n$ then

$$\operatorname{per}(A) \leqslant \prod_{i=1}^{n} (d_i!)^{\frac{1}{d_i}}. \tag{1}$$

Moreover, equality holds iff $d_1 = d_2 = \cdots = d_n = d$ and A is a direct sum of $1_{d\times d}$ matrices, where $1_{d\times d}$ is the all-ones square matrix of order d. Our results focus on the case where d does not divide n. The main results of this paper are:

- For $d > \frac{n}{2}$ let $A \in M_n^d$, then up to row and column permutations, in the first $\lfloor \frac{n}{2} \rfloor$ rows, the non-zero blocks are flushed to the left, and in the last $\lfloor \frac{n}{2} \rfloor$ rows, they are flushed to the right. For n even this looks like

$$A = \left(\begin{array}{c|c} 1_{(n/2)\times d} & 0_{(n/2)\times(n-d)} \\ \hline 0_{(n/2)\times(n-d)} & 1_{(n/2)\times d} \end{array} \right),$$

  where $1_{i\times j}$ (respectively, $0_{i\times j}$) denotes the all-ones (respectively, all-zeros) matrix of size $i \times j$. When n is odd, note that the position of the non-zero block in the middle row is unconstrained. Thus, for any $A \in M_n^d$,

$$\operatorname{per}(A) = \binom{2d-n}{\lfloor \frac{2d-n}{2} \rfloor} \left(\left\lfloor \frac{n}{2} \right\rfloor \right)! \left(\left\lceil \frac{n}{2} \right\rceil \right)!.$$

  The corresponding optimal anticode of maximal distance $d-1$ is

$$\left\{ f \in S_n\colon\ f(i) \in \begin{cases} \{1, 2, \ldots, d\} & \text{if } 1 \leqslant i \leqslant \frac{n}{2} \\ \{n-d+1, n-d+2, \ldots, n\} & \text{otherwise} \end{cases} \right\}.$$

  As before, when n is odd the unconstrained middle row of the matrix A allows the construction of several optimal anticodes with expressions similar to the one above.
- For $d < \frac{n}{2}$ we give some results based on results of Wanless [29], adjusted to our case. We show that any $A \in M_n^d$ has a certain structure, and for sufficiently large n, A satisfies some periodic property.

The rest of the paper is organized as follows. In Section 2 we focus on the case of $d > \frac{n}{2}$, classify precisely all the optimal anticodes up to isomorphism, and calculate their size. We proceed in Section 3 to the case of $d < \frac{n}{2}$ and present some asymptotic results regarding the structure of the optimal anticodes. We conclude in Section 4 with a summary of the results and short concluding remarks.

2. Full classification for $d > \frac{n}{2}$

We consider the case of $n = d + r$, where $0 < r < d$. Let $A \in \Gamma_{d+r}^d$, $A = (a_{i,j})_{i,j\in[n]}$, and for any $i \in [d+r]$, we define $x_i = \min\{j\colon\ a_{i,j} = 1\}$, i.e., the left-most column of the non-zero block in row i. Since the permanent is preserved under column and row permutations we can assume w.l.o.g. that $x_i \leqslant x_j$ for all $i \leqslant j$.

It can be seen that A is defined uniquely by the vector $(x_1, x_2, \ldots, x_{d+r})$, so by abuse of notation we will sometimes write

$$A = (x_1, x_2, \ldots, x_{d+r})$$

and also write

$$\operatorname{per}(A) = \operatorname{per}(x_1, x_2, \ldots, x_{d+r}) = \operatorname{per}\big(\{x_i\}\big).$$

Also, for any $i, j \in [d+r]$ we define $\operatorname{per}(A_{i,j})$ to be the permanent of A after deleting row i and column j.

In addition, for each $i \in [d+r]$, we define

$$i^* = \max\{k\colon\ x_k = x_i\}, \qquad i_* = \min\{k\colon\ x_k = x_i\}.$$

For all $i \in [d+r]$ such that $x_i \leqslant r$ we define the following operator

$$\mathrm{per}_i^+\big(\{x_k\}\big) = \mathrm{per}(x_1, \ldots, x_{i^*-1}, x_{i^*} + 1, x_{i^*+1}, \ldots, x_{d+r}).$$

If $x_i < x_{i+1} \leqslant r$ we define

$$\mathrm{per}_i^{++}\big(\{x_k\}\big) = \mathrm{per}_i^+\big(\mathrm{per}_i^+\big(\{x_k\}\big)\big).$$

Finally, in the same manner, for all $i \in [d+r]$ such that $2 \leqslant x_i$,

$$\mathrm{per}_i^-\big(\{x_k\}\big) = \mathrm{per}(x_1, \ldots, x_{i_*-1}, x_{i_*} - 1, x_{i_*+1}, \ldots, x_{d+r}).$$

If $2 \leqslant x_{i-1} < x_i$, we define

$$\mathrm{per}_i^{--}\big(\{x_k\}\big) = \mathrm{per}_i^-\big(\mathrm{per}_i^-\big(\{x_k\}\big)\big).$$

Lemma 2. *For any $1 \leqslant m \leqslant n \leqslant r$,*

$$\{i\colon\ a_{i,m} = 1\} \subseteq \{i\colon\ a_{i,n} = 1\},$$

and for any $d+1 \leqslant m \leqslant n \leqslant d+r$,

$$\{i\colon\ a_{i,n} = 1\} \subseteq \{i\colon\ a_{i,m} = 1\}.$$

Proof. Let $1 \leqslant m \leqslant n \leqslant r$, and let $i \in \{i\colon\ a_{i,m} = 1\}$. Since the length of the non-zero block is d, for any $m \leqslant k \leqslant d$ we have $a_{i,k} = 1$, and in particular, for $k = n$ we get $a_{i,n} = 1$. Therefore, $i \in \{i\colon\ a_{i,n} = 1\}$, and this proves the first part of the lemma. The second part of the lemma follows easily from the symmetry of the problem, i.e., rotating A and using the first part of the proof. □

Corollary 3. *If $1 \leqslant m \leqslant n \leqslant r$, then for any $i \in [d+r]$ we get*

$$\mathrm{per}(A_{i,m}) \geqslant \mathrm{per}(A_{i,n}).$$

If $d+1 \leqslant m \leqslant n \leqslant d+r$, then for any $i \in [d+r]$ we get

$$\mathrm{per}(A_{i,m}) \leqslant \mathrm{per}(A_{i,n}).$$

Proof. For the first claim of the corollary, by Lemma 2 we have $\{k\colon\ a_{k,m} = 1\} \subseteq \{k\colon\ a_{k,n} = 1\}$. Therefore, $A_{i,n}$ has the same columns as $A_{i,m}$ except for one column in $A_{i,m}$ that has a superset of the 1's of the corresponding column in $A_{i,n}$. Then we conclude that $\mathrm{per}(A_{i,m}) \geqslant \mathrm{per}(A_{i,n})$. The second claim of the corollary is again proved by rotating A and applying the first part of the proof. □

Lemma 4. *Let i be such that $x_i \leqslant r$. Then*

$$\mathrm{per}_i^+\big(\{x_k\}\big) = \mathrm{per}\big(\{x_k\}\big) + \mathrm{per}(A_{i^*, x_{i^*}+d}) - \mathrm{per}(A_{i^*, x_{i^*}}).$$

Let i be such that $2 \leqslant x_i$. Then

$$\mathrm{per}_i^-\big(\{x_k\}\big) = \mathrm{per}\big(\{x_k\}\big) + \mathrm{per}(A_{i_*, x_{i_*}-1}) - \mathrm{per}(A_{i_*, x_{i_*}+d-1}).$$

Proof. The proof follows by developing the permanent along row i. □

Lemma 5. *Let i be such that $2 \leqslant x_i \leqslant r$. Then*

$$\mathrm{per}\big(\{x_k\}\big) \leqslant \max\big\{\mathrm{per}_i^+\big(\{x_k\}\big), \mathrm{per}_i^-\big(\{x_k\}\big)\big\}.$$

Proof. Assume the contrary, i.e., $\mathrm{per}(\{x_k\}) > \mathrm{per}_i^+(\{x_k\})$ and $\mathrm{per}(\{x_k\}) > \mathrm{per}_i^-(\{x_k\})$. From Lemma 4 we get the following inequalities:

$$\mathrm{per}(A_{i^*,x_{i^*}+d}) < \mathrm{per}(A_{i^*,x_i^*}),$$
$$\mathrm{per}(A_{i_*,x_{i_*}-1}) < \mathrm{per}(A_{i_*,x_{i_*}+d-1}).$$

By Corollary 3 we get that

$$\mathrm{per}(A_{i^*,x_{i^*}}) \leqslant \mathrm{per}(A_{i^*,x_{i^*}-1}),$$
$$\mathrm{per}(A_{i_*,x_{i_*}+d-1}) \leqslant \mathrm{per}(A_{i_*,x_{i_*}+d}).$$

Combining the four inequalities, it now follows that

$$\begin{aligned}\mathrm{per}(A_{i_*,x_{i_*}-1}) &< \mathrm{per}(A_{i_*,x_{i_*}+d-1}) \leqslant \mathrm{per}(A_{i_*,x_{i_*}+d}) \\ &= \mathrm{per}(A_{i^*,x_{i^*}+d}) \qquad (2) \\ &< \mathrm{per}(A_{i^*,x_{i^*}}) \leqslant \mathrm{per}(A_{i^*,x_{i^*}-1}) \\ &= \mathrm{per}(A_{i_*,x_{i_*}-1}) \qquad (3)\end{aligned}$$

where equalities (2) and (3) follow from the fact that $x_i = x_{i^*} = x_{i_*}$. Thus, we get a contradiction, and the claim follows. □

Lemma 6. *When i is such that $x_i < x_{i+1} \leqslant r$ and $\mathrm{per}(A) \leqslant \mathrm{per}_i^+(A)$, then*

$$\mathrm{per}(A) \leqslant \mathrm{per}_i^+(A) \leqslant \mathrm{per}_i^{++}(A).$$

When i is such that $2 \leqslant x_{i-1} < x_i$ and $\mathrm{per}(A) \leqslant \mathrm{per}_i^-(A)$, then

$$\mathrm{per}(A) \leqslant \mathrm{per}_i^-(A) \leqslant \mathrm{per}_i^{--}(A).$$

Proof. We start by proving the first claim. Define the $(0,1)$-matrix B to be

$$B = (x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_{d+r}),$$

and denote $j = (i+1)^* = \max\{k\colon\ x_k = x_{i+1}\}$.

Case 1. Assume that $x_{i+1} - x_i = 1$, then by the definition of the operators

$$\mathrm{per}_i^+\big(\{x_k\}\big) = \mathrm{per}(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_{d+r}) = \mathrm{per}(B),$$

and also

$$\begin{aligned}\mathrm{per}_i^{++}\big(\{x_k\}\big) &= \mathrm{per}(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_{(i+1)^*-1}, x_{(i+1)^*} + 1, x_{(i+1)^*+1}, \dots, x_{d+r}) \\ &= \mathrm{per}_j^+(B).\end{aligned}$$

By Lemma 4, in order to prove the claim, i.e., $\mathrm{per}(B) \leqslant \mathrm{per}_j^+(B)$, it suffices to show that $\mathrm{per}(B_{j,x_j}) \leqslant \mathrm{per}(B_{j,x_j+d})$. Since $\mathrm{per}(A) \leqslant \mathrm{per}_i^+(A)$, we conclude

$$\mathrm{per}(A_{i,x_i}) \leqslant \mathrm{per}(A_{i,x_i+d}). \qquad (4)$$

It is easy to see that

$$\operatorname{per}(A_{i,x_i}) = \operatorname{per}(B_{(i+1)^*,x_{(i+1)^*}-1}) = \operatorname{per}(B_{j,x_j-1}),$$
$$\operatorname{per}(A_{i,x_i+d}) = \operatorname{per}(B_{(i+1)^*,x_{(i+1)^*}-1+d}) = \operatorname{per}(B_{j,x_j-1+d}).$$

Therefore, (4) turns to

$$\operatorname{per}(B_{j,x_j-1}) \leqslant \operatorname{per}(B_{j,x_j-1+d}). \tag{5}$$

By Corollary 3,

$$\operatorname{per}(B_{j,x_j}) \leqslant \operatorname{per}(B_{j,x_j-1}), \tag{6}$$
$$\operatorname{per}(B_{j,x_j-1+d}) \leqslant \operatorname{per}(B_{j,x_j+d}). \tag{7}$$

Combining inequalities (5), (6), and (7), we get

$$\operatorname{per}(B_{j,x_j}) \leqslant \operatorname{per}(B_{j,x_j-1}) \leqslant \operatorname{per}(B_{j,x_j-1+d}) \leqslant \operatorname{per}(B_{j,x_j+d}),$$

which proves the claim.

Case 2. Assume that $x_{i+1} - x_i \geqslant 2$. The proof in this case is nearly identical. By definition,

$$\operatorname{per}_i^+(A) = \operatorname{per}(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_{d+r}),$$
$$\operatorname{per}_i^{++}(A) = \operatorname{per}(x_1, \ldots, x_{i-1}, x_i + 2, x_{i+1}, \ldots, x_{d+r}).$$

Therefore, by Lemma 4, in order to prove the claim it suffices to show that

$$\operatorname{per}(A_{i,x_i+1}) \leqslant \operatorname{per}(A_{i,x_i+d+1}).$$

From the fact that $\operatorname{per}(A) \leqslant \operatorname{per}_i^+(A)$, we conclude that

$$\operatorname{per}(A_{i,x_i}) \leqslant \operatorname{per}(A_{i,x_i+d}). \tag{8}$$

From Corollary 3,

$$\operatorname{per}(A_{i,x_i+1}) \leqslant \operatorname{per}(A_{i,x_i}), \tag{9}$$
$$\operatorname{per}(A_{i,x_i+d}) \leqslant \operatorname{per}(A_{i,x_i+d+1}). \tag{10}$$

Combining inequalities (8), (9), and (10), we get

$$\operatorname{per}(A_{i,x_i+1}) \leqslant \operatorname{per}(A_{i,x_i}) \leqslant \operatorname{per}(A_{i,x_i+d}) \leqslant \operatorname{per}(A_{i,x_i+d+1}),$$

and that completes the proof for the first claim.

The second claim of the lemma, again, easily follows from the symmetry of the problem by rotating A. $\square$

We are now in a position to prove the two main claims of the section. We first calculate the value of the maximum permanent.

Theorem 7. *Let $A \in M_{d+r}^d$, $0 < r \leqslant d$. Then*

$$\operatorname{per}(A) = \binom{d-r}{\lfloor \frac{d-r}{2} \rfloor} \left(\left\lfloor \frac{d+r}{2} \right\rfloor \right)! \left(\left\lceil \frac{d+r}{2} \right\rceil \right)!.$$

Furthermore, the matrix $A = (\{x_i\})$ given by

$$x_i = \begin{cases} 1, & 1 \leqslant i \leqslant \lfloor \frac{d+r}{2} \rfloor, \\ r+1, & \textit{otherwise}, \end{cases} \tag{11}$$

is a member of M_{d+r}^d.

Proof. Let A be a matrix that achieves the maximum permanent. If A is not of the required form there is at least one row, i, that is not flushed to the right or left, i.e., $1 < x_i < r+1$. By Lemma 5 we know that either $\mathrm{per}(A) \leqslant \mathrm{per}_i^+(A)$ or $\mathrm{per}(A) \leqslant \mathrm{per}_i^-(A)$.

Assume that $\mathrm{per}(A) \leqslant \mathrm{per}_i^+(A)$ (the proof for the other case is symmetric). Since $\mathrm{per}(A)$ is the maximum achievable permanent, necessarily $\mathrm{per}(A) = \mathrm{per}_i^+(A)$. It now follows by Lemma 6 that $\mathrm{per}(A) \leqslant \mathrm{per}_i^{++}(A)$.

By repeatedly using Lemma 6 on the last block of 1's that was moved we can continue to push blocks one step at a time, all in the same direction. This procedure is terminated when we can no longer push the block, i.e., when we have reached one of the matrix's edges. Thus, we have reduced by one the number of blocks that are not flushed to the right or left edges. We can therefore push all the blocks that are not flushed to the edges until reaching some edge.

We conclude that the maximum permanent is also attained when all the blocks are flushed to the edges. Let $\lfloor \frac{d+r}{2} \rfloor + x$, $x \geqslant 0$, be the number of blocks that are flushed to the left edge, and thus, $\lceil \frac{d+r}{2} \rceil - x$ are flushed to the right. We note that if $\lceil \frac{d+r}{2} \rceil - x < r$ then the permanent is 0, and so we can safely assume $\lceil \frac{d+r}{2} \rceil - x \geqslant r$. Let us also denote by 1_{odd} the indicator function for $d+r$ being odd. Thus, the permanent of this configuration is

$$\begin{aligned}
&\left(\left\lfloor \frac{d+r}{2} \right\rfloor + x\right)!\left(\left\lceil \frac{d+r}{2} \right\rceil - x\right)!\binom{d-r}{\lfloor \frac{d+r}{2} \rfloor + x - r} \\
&= (d-r)! \prod_{k=1}^{r} \left(\left\lfloor \frac{d-r}{2} \right\rfloor + x + k\right)\left(\left\lceil \frac{d-r}{2} \right\rceil - x + k\right) \\
&= (d-r)! \prod_{k=1}^{r} \left(\frac{d-r-1_{\mathrm{odd}}}{2} + x + k\right)\left(\frac{d-r+1_{\mathrm{odd}}}{2} - x + k\right) \\
&= (d-r)! \prod_{k=1}^{r} \left[\left(\frac{d-r}{2} + k\right)^2 - \left(x - \frac{1_{\mathrm{odd}}}{2}\right)^2\right].
\end{aligned}$$

Hence, for $n = d+r$ even, the maximum is achieved when $x = 0$, and for $n = d+r$ odd, the maximum is achieved when $x = 0, 1$. In either case, when all the blocks are flushed to the edges of the matrix, the maximum is achieved only when $\lfloor \frac{d+r}{2} \rfloor$ of the blocks are flushed to one edge, and all the rest are flushed to the other edge, and this completes the proof. $\square$

Having proved the upper bound we want to know which matrix configurations achieve the bound with equality.

Theorem 8. *Let $A \in M_{d+r}^d$, $0 \leqslant r \leqslant d$. Then the only possible configurations of $A = (x_1, x_2, \dots, x_{d+r})$, up to a permutation of the rows and columns, are*

$$x_i = \begin{cases} 1, & 1 \leqslant i \leqslant \lfloor \frac{d+r}{2} \rfloor, \\ r+1, & \lceil \frac{d+r}{2} \rceil < i \leqslant d+r. \end{cases} \tag{12}$$

Note that for $n = d+r$ odd, the value of $x_{\lceil \frac{d+r}{2} \rceil}$ is unconstrained.

Proof. For the case of $n = d+r$ even, assume to the contrary that there exists $A \in M_{d+r}^d$ with a different configuration than the claimed. By Theorem 7, we know that we can push the non-zero blocks of A along the rows without reducing the permanent to achieve a matrix A' with configuration as in (11). Let us denote the matrix before the last block push as A''. W.l.o.g., the configuration of $A'' = (\{x_i''\})$ is given by

$$x_i'' = \begin{cases} 1, & 1 \leqslant i < \frac{d+r}{2}, \\ 2, & i = \frac{d+r}{2}, \\ r+1, & \text{otherwise.} \end{cases}$$

By our assumption, $\mathrm{per}(A) = \mathrm{per}(A') = \mathrm{per}(A'')$. However,

$$\begin{aligned} \mathrm{per}(A'') - \mathrm{per}(A') &= \mathrm{per}\big(A''_{\frac{d+r}{2},d+1}\big) - \mathrm{per}\big(A''_{\frac{d+r}{2},1}\big) \\ &= \left(\frac{d+r}{2} - 1\right)! \left(\frac{d+r}{2}\right)! \left[\binom{d-r}{\frac{d-r}{2}+1} - \binom{d-r}{\frac{d-r}{2}}\right] \\ &< 0, \end{aligned}$$

a contradiction. For the case of $n = d + r$ odd, the proof follows the same logical steps but is more tedious as it has to consider more cases, and is therefore given in Appendix A. □

3. Asymptotic results for $d < \frac{n}{2}$

We now turn to show some asymptotic results for the case of $d < \frac{n}{2}$. We follow the notation of Wanless [29]. With $A \in \Gamma_n^d$, $A = (a_{i,j})$, we associate a bipartite graph $G(A)$ with two vertex sets, $V = \{v_1, v_2, \dots, v_n\}$, which represents the rows of A, and $U = \{u_1, u_2, \dots, u_n\}$, which represents the columns of A. There is an edge (v_i, u_j) iff $a_{i,j} = 1$. For every vertex $w \in V \cup U$ we denote by $N(w)$ its set of neighbors, and its degree by $D(w) = |N(w)|$. Finally, we denote by $\oplus$ the direct-sum operator, and moreover, as in [29], we use rA as shorthand for $A \oplus A \oplus \dots \oplus A$ (where there are r copies of A).

We are interested only in the structure of matrices in M_n^d up to isomorphism because the permanent function is preserved under permutations of rows and columns. We say that $\{C_i\}_{i=1}^k$ are the components of A if $A \cong C_1 \oplus C_2 \oplus \dots \oplus C_k$ and each C_i is fully indecomposable. Denote the order of a component, C_i, or a matrix, A, by $\mathrm{ord}(C_i)$, and $\mathrm{ord}(A)$, respectively.

Our main results in this section are based heavily on the results of [29]. We first mention a technical result from [29] using the same notation. Define the following functions:

$$\begin{aligned} F(a,b) &= (a!)^{\frac{b}{a}}, \\ D(k) &= \frac{F(k,1)}{F(k-1,1)}, \\ C(k) &= \frac{D(k)}{D(k-1)}, \\ B(k,v) &= C(k)^v \big((k-v)^2 + 2v(k-v)D(k-1) + v(v-1)\big(D(k-1)\big)^2\big). \end{aligned}$$

Lemma 9. *(See [29, Lemma 1].) For every integer $k \geqslant 3$ there exists $\epsilon_k > 0$ such that $B(k,v) < k^2 - \epsilon_k$ for each integer v satisfying $0 < v < k$.*

We will use another technical lemma:

Lemma 10. *(See [3, p. 50].) For every two integers a, b satisfying $b \geqslant a + 2 > 3$, the following inequality holds*

$$(a!)^{\frac{1}{a}} (b!)^{\frac{1}{b}} < \big((a+1)!\big)^{\frac{1}{(a+1)}} \big((b-1)!\big)^{\frac{1}{(b-1)}}.$$

We now turn to our specific setting and prove the following lemma.

Lemma 11. *Let $B \in \Gamma_n^d$ be such that it does not contain $1_{d\times d}$ as a sub-matrix. Let W be a set of $2d$ contiguous column vertices, i.e., $W = \{u_i, u_{i+1}, \dots, u_{i+2d-1}\} \subseteq U$ for some $i \in [n]$, then there is either some vertex $u_j \in W$ such that $D(u_j) \neq d$ or there are two row vertices $v_x, v_y \in V$ such that*

1. $0 < |N(v_x) \cap N(v_y)| < D(v_x)$,
2. $D(u_k) = d$ *for all* $u_k \in N(v_x) \cup N(v_y)$,
3. $N(v_x), N(v_y) \subseteq W$.

Proof. If there is some column vertex $u_j \in W$ such that $D(u_j) \neq d$ then we are done. Otherwise, $D(u_j) = d$ for each $u_j \in W$. Now, we know the column vertex u_{i+d} has degree d, and, by our assumption throughout the paper that the identity permutation is in the anticode, $v_{i+d} \in N(u_{i+d})$. On the other hand B does not contain $1_{d\times d}$ as a sub-matrix, and so there is a row vertex $v_j \in N(u_{i+d})$ such that $N(v_{i+d}) \cap N(v_j) \neq N(v_{i+d})$. Note that $u_{i+d} \in N(v_{i+d}) \cap N(v_j)$, and since the neighbors of v_{i+d} and v_j form a contiguous block of column vertices, we get that

$$N(v_{i+d}), N(v_j) \subseteq W.$$

Now set $v_x = v_{i+d}$, $v_y = v_j$, and the proof is complete. $\square$

Corollary 12. *For any integers d, T, and n, where T is even and $T \leqslant n$, the maximum of the function $\prod_{i=1}^n F(x_i, 1)$ subject to the constraints*

1. $\sum_{i=1}^n x_i = nd$,
2. $x_i \geqslant 1$ *are integers,*
3. $T \leqslant |\{x_i\colon x_i \neq d\}|$,

is obtained exactly when the variables x_i are as equal as possible, i.e.,

$$\bigl|\{x_i\colon x_i = d\}\bigr| = n - T, \qquad \bigl|\{x_i\colon x_i = d+1\}\bigr| = \bigl|\{x_i\colon x_i = d-1\}\bigr| = \frac{T}{2}.$$

Therefore,

$$\prod_{i=1}^n F(x_i, 1) \leqslant F(d, n-T) F\left(d-1, \frac{T}{2}\right) F\left(d+1, \frac{T}{2}\right).$$

Proof. Recall that $F(x,1) = (x!)^{\frac{1}{x}}$. If there are two indices i and j such that $x_i \geqslant x_j + 2$, then by Lemma 10, the value of $\prod_{i=1}^n F(x_i, 1)$ would increase if we add 1 to x_j and subtract 1 from x_i, as long as we do not violate constraint 3. $\square$

Theorem 13. *For each $A \in \Gamma_a^d$ there exists $m(A) \in \mathbb{N}$ such that $\mathrm{per}(A \oplus t1_{d\times d}) > \mathrm{per}(B)$ for every integer t such that $a + td > m(A)$ and every $B \in \Gamma_{a+td}^d$ which does not contain $1_{d\times d}$ as a sub-matrix.*

Proof. The claim is empty for $d = 1, 2$, since for $d = 1$ there is nothing to prove, and for $d = 2$ there is no such $B \in \Gamma_{a+td}^d$ which does not contain $1_{d\times d}$. Set $n = a + td$, then by Lemma 11 we know that for every $l \in [\lfloor \frac{n}{2d} \rfloor]$ there is either a column vertex $u_{i_l} \in \{u_{2d(l-1)+1}, \dots, u_{2ld}\}$ such that $D(u_{i_l}) \neq d$, or there is a pair of row vertices, v_{x_l} and v_{y_l}, such that

1. $0 < |N(v_{x_l}) \cap N(v_{y_l})| < D(v_{x_l})$,
2. $D(u_k) = d$ for all $u_k \in N(v_{x_l}) \cup N(v_{y_l})$,
3. $N(v_{x_l}) \cup N(v_{y_l}) \subseteq \{u_{2d(l-1)+1}, \dots, u_{2ld}\}$.

Let M be the set of all $l \in [\lfloor \frac{n}{2d} \rfloor]$ such that there exists a pair of row vertices, v_{x_l} and v_{y_l}, as above. Set

$$T = \begin{cases} \lfloor \frac{n}{2d} \rfloor - |M|, & \lfloor \frac{n}{2d} \rfloor - |M| \text{ is even}, \\ \lfloor \frac{n}{2d} \rfloor - |M| - 1, & \text{otherwise}. \end{cases}$$

It is easy to see that $T+|M| \geqslant \lfloor \frac{n}{2d} \rfloor - 1$. Note that for any $l,k \in M$, $l \neq k$,

$$\big(N(v_{x_l}) \cup N(v_{y_l})\big) \cap \big(N(v_{x_k}) \cup N(v_{y_k})\big) = \emptyset.$$

For each pair (v_{x_l}, v_{y_l}), $l \in M$, let $a_l = |N(v_{x_l}) \cap N(v_{y_l})|$. We bound the permanent of B from above by using the following steps:

1. Expand $\mathrm{per}(B)$ along the row vertices $\{v_{x_l}, v_{y_l}\}_{l \in M}$.
2. Upper bound the expansion of the columns $\bigcup_{l \in M}(N(v_{x_l}) \cup N(v_{y_l}))$ by using Eq. (1).
3. Upper bound the expansion of the rest of the columns by using Eq. (1), Corollary 12, and the fact that in these columns there are exactly $d(n - |\bigcup_{l \in M}(N(v_{x_l}) \cup N(v_{y_l}))|)$ non-zero entries, with at least T columns vertices with degree not equal to d.

Therefore, for steps 1 and 2, the upper bound is

$$\begin{aligned}
&\prod_{l \in M}\big[(d-a_l)^2 F(d-1, 2d-2a_l-2) F(d-2, a_l) \\
&\quad + 2a_l(d-a_l) F(d-1, 2d-2a_l-1) F(d-2, a_l-1) \\
&\quad + a_l(a_l-1) F(d-1, 2d-2a_l) F(d-2, a_l-2)\big].
\end{aligned} \tag{13}$$

The upper bound for step 3 using Corollary 12 is

$$\begin{aligned}
&f\left(d, n-t-\sum_{l \in m}(2d-a_l)\right)\left((d+1)!^{\frac{1}{d+1}}(d-1)!^{\frac{1}{d-1}}\right)^{\frac{t}{2}} \\
&\quad = f\left(d, n-\sum_{l \in m}(2d-a_l)\right)\frac{((d+1)!^{\frac{1}{d+1}}(d-1)!^{\frac{1}{d-1}})^{\frac{t}{2}}}{d!^{\frac{t}{d}}} \\
&\quad < f\left(d, n-\sum_{l \in m}(2d-a_l)\right) \cdot \delta^{\frac{t}{2}},
\end{aligned} \tag{14}$$

where the last inequality follows from Lemma 10 for some $0 < \delta < 1$. Combining (13) and (14) we get

$$\begin{aligned}
\mathrm{per}(B) \leqslant F(d,n)\delta^{\frac{T}{2}} \prod_{l \in M}&\left[\frac{F(d-1,2d-2)}{F(d,2d)}\frac{F(d,a_l)F(d-2,a_l)}{F(d-1,2a_l)}\right] \\
&\cdot\left[(d-a_l)^2 + 2a_l(d-a_l)\frac{F(d-1,1)}{F(d-2,1)} + a_l(a_l-1)\frac{F(d-1,2)}{F(d-2,2)}\right],
\end{aligned}$$

which in our notation becomes

$$\mathrm{per}(B) \leqslant F(d,n)\delta^{\frac{T}{2}} \prod_{l \in M}\frac{1}{d^2}B(d,a_l).$$

We know that $T+|M| \geqslant \lfloor \frac{n}{2d} \rfloor - 1$, thus, by Lemma 9 and by taking t, and hence n, large enough, we can make $\mathrm{per}(B)$ be less than an arbitrary small fraction of $F(d,n)$.

On the other hand, for $n = a+td$,

$$\mathrm{per}(A \oplus t1_{d\times d}) = (d!)^t \,\mathrm{per}(A) = F(d,n)\frac{\mathrm{per}(A)}{F(d,a)},$$

which is a constant fraction of $F(d,n)$ for any t. Hence, there exists $m(A)$ such that if $a+td > m(A)$ then $\mathrm{per}(B) < \mathrm{per}(A \oplus t1_{d\times d})$ as required. $\square$

Though the set of matrices under study is different from the one studied by Wanless [29], the claim regarding their permanent in Theorem 13 is exactly the same as the claim in Theorem 1 in [29]. Thus, Theorems 3, 5, and 7 in [29], which rely almost entirely on that claim, follow in our setting as well with very slight adjustments. We bring them here for completeness. For adjusted proofs, the reader is referred to Appendix B.

Theorem 14. *For each integer d there exists b_d such that for any n and any $A \in M_n^d$, the largest component in A that does not contain $1_{d\times d}$ as a sub-matrix, is of order at most b_d.*

Let b_d^* denote the smallest integer with the property of b_d from Theorem 14.

Theorem 15. *Let $d \leqslant n$ be positive integers. Every $A \in M_n^d$ is of the form*

$$A \cong a1_{d\times d} \oplus C_1 \oplus C_2 \oplus \cdots \oplus C_h$$

where $a \geqslant 0$ and $0 \leqslant h \leqslant d-1$. Moreover $G(C_i)$ is connected, $C_i \in M^d_{\mathrm{ord}(C_i)}$, and if in addition C_i does not contain $1_{d\times d}$ as a sub-matrix, then $\mathrm{ord}(C_i) \leqslant b_d^*$.

Theorem 16. *(See [29, Theorem 7].) For each positive integer d there exists μ_d such that M_n^d is periodic for $n \geqslant \mu_d$ in the sense that $A \in M_n^d$ if and only if $A \oplus 1_{d\times d} \in M^d_{n+d}$.*

4. Summary and conclusions

Motivated by new applications of error-correcting codes over permutations under the ℓ_∞-norm, we have studied anticodes of maximum size for the infinity metric. The results, together with the set–antiset method, enable us to derive an improved upper bound on the size of optimal codes (see [27]). For $d > \frac{n}{2}$ we classified all the optimal anticodes with maximal distance $d-1$, and showed that their size is

$$\binom{2d-n}{\lfloor\frac{2d-n}{2}\rfloor}\left(\left\lfloor\frac{n}{2}\right\rfloor\right)!\left(\left\lceil\frac{n}{2}\right\rceil\right)!.$$

For $d < \frac{n}{2}$, based on the results of [29], we gave asymptotic results on the structure of optimal anticodes. We showed that for sufficiently large n, all but at most $d-1$ components of any optimal anticode are $1_{d\times d}$. Moreover, some periodic property of the optimal anticodes was shown.

It is tempting to combine all the results to the following conjecture.

Conjecture 1. *Denote $r = n \bmod d$. Then for any n, the structure of the optimal anticode of maximal distance $d-1$ is the set of permutations $M = \{\sigma \in S_n\colon\ a_{i,\sigma(i)} = 1,\ 1 \leqslant i \leqslant n\}$, where*

$$pA = (a_{i,j}) = \begin{pmatrix} \boxed{1_{d\times d}} & & & & \\ & \boxed{1_{d\times d}} & & 0 & \\ & & \ddots & & \\ & 0 & & \boxed{1_{d\times d}} & \\ & & & & \boxed{P} \end{pmatrix},$$

where along the diagonal we have $\lfloor\frac{n}{d}\rfloor - 1$ blocks of $1_{d\times d}$, and P is of the form given in (12). It can then be easily seen that

$$|M| = \mathrm{per}(A) = (d!)^{\lfloor\frac{n}{d}\rfloor-1}\binom{d-r}{\lfloor\frac{d-r}{2}\rfloor}\left(\left\lfloor\frac{d+r}{2}\right\rfloor\right)!\left(\left\lceil\frac{d+r}{2}\right\rceil\right)!.$$

Appendix A. Proof of Theorem 8 – continued

We give here the proof of Theorem 8 for the case of $n = d + r$ odd.

Proof. Let us consider the case of $n = d + r$ odd. First, we note that by developing the permanent along the middle row, all matrices of configuration (12) have the same permanent regardless of the value of $x_{\lceil \frac{d+r}{2} \rceil}$, i.e., the starting column of the non-zero block in the middle row. Since one of these configurations coincides with (11), all matrices of configuration (12) have maximum permanent.

Assume to the contrary that there exists $A \in M^d_{d+r}$ with a different configuration than the claimed. By Theorem 7, we know that we can push the non-zero blocks of A along the rows without reducing the permanent to achieve a matrix A' with configuration as in (12). Let us denote the matrix before the last block push as A''. W.l.o.g., the configuration of $A'' = (\{x''_i\})$ is given by

$$x''_i = \begin{cases} 1, & 1 \leqslant i < \lfloor \frac{d+r}{2} \rfloor, \\ 2, & i = \lfloor \frac{d+r}{2} \rfloor, \\ r+1, & \lceil \frac{d+r}{2} \rceil < i \leqslant d+r. \end{cases}$$

Note again that the value of $x_{\lceil \frac{d+r}{2} \rceil}$ is unconstrained.

By repeatedly using Theorem 7 we can push the non-zero block of row $\lceil \frac{d+r}{2} \rceil$ while maintaining the maximum permanent value, until reaching a matrix $A^* = (\{x^*_i\})$ of one of the two following configurations:

$$x^*_i = \begin{cases} 1, & 1 \leqslant i < \lfloor \frac{d+r}{2} \rfloor, \\ 2, & i = \lfloor \frac{d+r}{2} \rfloor, \\ 2 \quad \text{or} \quad r+1, & i = \lceil \frac{d+r}{2} \rceil, \\ r+1, & \lceil \frac{d+r}{2} \rceil < i \leqslant d+r. \end{cases}$$

Finally, let $(A^{**}) = (\{x^{**}_i\})$ be defined by

$$x^{**}_i = \begin{cases} 1, & i = \lfloor \frac{d+r}{2} \rfloor, \\ x^{**}_i, & \text{otherwise.} \end{cases}$$

We note that A^{**} is of configuration (12), and thus, of maximum permanent. Therefore, by our assumptions,

$$\operatorname{per}(A) = \operatorname{per}(A') = \operatorname{per}(A'') = \operatorname{per}(A^*) = \operatorname{per}(A^{**}).$$

Case 1. Let $x^*_{\lceil \frac{d+r}{2} \rceil} = 2$. One can readily verify that

$$\operatorname{per}\left(A^*_{\lfloor \frac{d+r}{2} \rfloor, 1}\right) = \left(\frac{d+r+1}{2}\right)! \left(\frac{d+r-3}{2}\right)! \binom{d-r}{\lfloor \frac{d-r}{2} \rfloor},$$

$$\operatorname{per}\left(A^*_{\lfloor \frac{d+r}{2} \rfloor, d+1}\right) = \left(\frac{d+r+1}{2}\right)! \left(\frac{d+r-3}{2}\right)! \binom{d-r}{\lfloor \frac{d-r}{2} \rfloor} \frac{d+r-3}{d+r+1}.$$

It follows that

$$\operatorname{per}(A^*) - \operatorname{per}(A^{**}) = \operatorname{per}\left(A^*_{\lfloor \frac{d+r}{2} \rfloor, d+1}\right) - \operatorname{per}\left(A^*_{\lfloor \frac{d+r}{2} \rfloor, 1}\right) < 0,$$

a contradiction.

Case 2. Let $x_{\lceil \frac{d+r}{2} \rceil} = r+1$. Again, it is easily verifiable that

$$\operatorname{per}\left(A^*_{\lfloor \frac{d+r}{2} \rfloor,1}\right) = \left(\frac{d+r+1}{2}\right)!\left(\frac{d+r-3}{2}\right)!\binom{d-r}{\lfloor \frac{d-r}{2} \rfloor},$$

$$\operatorname{per}\left(A^*_{\lfloor \frac{d+r}{2} \rfloor,d+1}\right) = \left(\frac{d+r+1}{2}\right)!\left(\frac{d+r-3}{2}\right)!\binom{d-r}{\lfloor \frac{d-r}{2} \rfloor - 1}.$$

Once again, it follows that

$$\operatorname{per}\left(A^*\right) - \operatorname{per}\left(A^{**}\right) = \operatorname{per}\left(A^*_{\lfloor \frac{d+r}{2} \rfloor,d+1}\right) - \operatorname{per}\left(A^*_{\lfloor \frac{d+r}{2} \rfloor,1}\right) < 0,$$

a contradiction. □

Appendix B. Proofs of Theorems 14, 15, and 16

The following are very slight adjustments to the proofs given by Wanless in [29]. They are brought here for completeness.

Proof of Theorem 14. For every integer $d \leqslant i < 2d$, choose some $A_i \in \Gamma_n^d$. Define

$$b_d = \max\left\{m(A_d), m(A_{d+1}), \ldots, m(A_{2d-1})\right\}$$

where $m(A_i)$ was defined in Theorem 13.

Assume to the contrary that $A \in M_n^d$ contains a component C bigger than b_d that does not contain $1_{d\times d}$ as a sub-matrix. By Theorem 13 we can increase $\operatorname{per}(A)$ by replacing C with $A_i \oplus t1_{d\times d}$, where $d \leqslant i < 2d$, $i \equiv \operatorname{ord}(C) \bmod d$, and $t = (\operatorname{ord}(C) - i)/d$. This contradiction proves the claim. □

Proof of Theorem 15. Assume A has d connected components $C_1, \ldots, C_d$. By Theorem 14, the order of any component of A not having $1_{d\times d}$ as a sub-matrix, is upper bounded by b_d^*. Let us now look at the partial sums $s_i = \sum_{j=1}^{i} \operatorname{ord}(C_i)$. Obviously, either there is some j such that $s_j \equiv 0 \pmod d$, or there are distinct i and j for which $s_i \equiv s_j \pmod d$. In any case, there are surely integers $1 \leqslant a \leqslant b \leqslant d$ for which $\sum_{i=a}^{b} \operatorname{ord}(C_i) = ld$ for some positive integer l.

The permanent is multiplicative components, and therefore, $C_i \in M^d_{\operatorname{ord}(C_i)}$. Furthermore, by Brégman's Theorem, $C_a \oplus \cdots \oplus C_b \cong l1_{d\times d}$. Thus, A has at most $d-1$ connected components which are not isomorphic to $1_{d\times d}$. □

Proof of Theorem 16. For the first direction, assume $A \oplus 1_{d\times d} \in M^d_{n+d}$, then every component maximizes its permanent and so $A \in M_n^d$. For the other direction, assume $A \in M_n^d$ and $B \in M^d_{n+d}$. Further, let us assume $n > (d-1)b_d^*$. We now have one of two cases:

Case 1. Either B contains a connected component which is $1_{d\times d}$, or all the connected components of B do not contain $1_{d\times d}$ as a sub-matrix. Thus, by using Theorems 14 and 15 in the latter case, we are assured that $B \cong 1_{d\times d} \oplus B'$ for some $B' \in \Gamma_n^d$. It now follows that $\operatorname{per}(B') \leqslant \operatorname{per}(A)$ and so $\operatorname{per}(B) = \operatorname{per}(B' \oplus 1_{d\times d}) \leqslant \operatorname{per}(A \oplus 1_{d\times d})$ and then necessarily $\operatorname{per}(A \oplus 1_{d\times d}) = \operatorname{per}(B)$, i.e., $A \oplus 1_{d\times d} \in M^d_{n+d}$.

Case 2. There exists a connected component C of B which contains $1_{d\times d}$ as a sub-matrix. When viewed as a matrix of configuration $C = (x_1, \ldots, x_{\operatorname{ord}(C)})$, $x_{i+1} \geqslant x_i$, let us examine the top left occurrence of $1_{d\times d}$ as a sub-matrix in C. By changing all the 1's above and below the sub-matrix $1_{d\times d}$ to 0's, we get a matrix $C' \in \Gamma^{\leqslant d}_{\operatorname{ord}(C)}$, where $\Gamma_n^{\leqslant d}$ stands for the set of $(0,1)$-matrices with exactly one contiguous non-zero block in each row of size at most d. It is readily verifiable that $\operatorname{per}(C') = \operatorname{per}(C)$.

After the change, the matrix B becomes $B' \in \Gamma_{n+d}^{\leqslant d}$ for which $\mathrm{per}(B) = \mathrm{per}(B')$. In addition, $B' = 1_{d\times d} \oplus B''$, where $B'' \in \Gamma_n^{\leqslant d}$. We can, now, arbitrarily change 0's to 1's in B'' so as to get a matrix $B^* \in \Gamma_n^d$. Obviously, $\mathrm{per}(B'') \leqslant \mathrm{per}(B^*) \leqslant \mathrm{per}(A)$, and so

$$\mathrm{per}(B) = \mathrm{per}\big(B'\big) = \mathrm{per}\big(B'' \oplus 1_{d\times d}\big) \leqslant \mathrm{per}\big(B^* \oplus 1_{d\times d}\big) \leqslant \mathrm{per}(A \oplus 1_{d\times d}).$$

Just like in the previous case, it now follows that $A \oplus 1_{d\times d} \in M_{n+d}^d$. $\square$

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